Optimal Debt Management in a Liquidity Trap

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November 2016

Abstract

We study optimal debt management in the face of shocks that can drive the economy into a liquidity trap and call for an increase in public spending in order to mitigate the resulting recession. Our approach follows the literature of macroeconomic models of debt management, which we extend to the case where the zero lower bound on the short-term interest rate may bind. We wish to identify the conditions under which removing long-maturity government debt from the secondary market can be an optimal policy outcome. We show that the optimal debt-management strategy is to issue short-term debt if the government faces a sizable exogenous increase in public spending and if its initial liability is not very large. In this case, our results run against the standard prescription of the debt-management literature. In contrast, if the initial debt level is high, then issuing long term government bonds is optimal.

Finding the portfolios requires to solve the model using global numerical approximation methods. As a methodological contribution, we propose numerical procedures within the class of parameterized expectations algorithms (PEA) to solve the nonlinear model subject the zero lower bound.

JEL Classification : E43, E62, H63

Keywords: Debt Management, Debt Maturity, Fiscal Policy, Liquidity Trap, Monetary Policy, Tax Smoothing.

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1 Introduction

A prominent characteristic of the latest global recession was the sharp decline in short-term nominal interest rates, which reached their zero lower bound (ZLB), a situation referred to as the liquidity trap (LT). One of the main policy responses during this episode has been the buy-backs of long maturity debt from the secondary market, widely known as unconventional monetary policy. This intervention has received considerable attention in recent academic and policy work. On the empirical side, a large body of literature has shown that removing long debt from the hands of the private sector, reduces long interest rates, potentially leading to higher demand for consumption and investment goods. On the theoretical side, Vayanos and Vila (2009), Chen, Curdia and Ferrero (2012), Gibaud, Nosbusch and Vayanos (2013) (among many others) propose microfoundations that can explain the effects of buy-backs on bond prices.

In this paper, we approach public debt management in a LT from a different perspective than these recent papers. We follow the trail of a literature that analyzes optimal government portfolios in macroeconomic models (initiated by the work of Angeletos (2002) and Buera and Nicolini (2004), hereafter ABN) and which shows that appropriately choosing the maturity structure of public debt can help the government 'hedge' its intertemporal budget. Existing macroeconomic models of debt management have, however, completely abstracted from disturbances that may drive the economy into a LT. Whereas LTs have been quite rare since the Great Depression, there is a growing concern that such episodes may become recurrent events in the future.[1] The purpose of this paper is to offer an alternative benchmark to evaluate the recent changes in the maturity structure in industrialized economies. We ask: is reducing the amounts of long-maturity debt held by private agents optimal during LTs?

A well known conclusion of the 'fiscal-hedging' approach to debt management is that governments should issue only long-term debt.[2] Under this presumption, the recent policy interventions in the secondary market have driven the maturity structure of public debt further from its optimum. We wish to evaluate whether this conclusion persists when economic shocks can drive interest rates to their ZLB, or whether there are conditions under which removing long-maturity government debt from the secondary market can be an optimal policy outcome in such circumstances.

The model we construct is broadly similar to Schmitt-Grohé and Uribe (2004, SGU), and Faraglia, Marcet, Oikonomou and Scott (2013, FMOS): It is an economy with monopolistic competition and sticky prices and with monetary and fiscal policies which are coordinated; a benevolent planner with full commitment controls inflation and distortionary taxes. Moreover, following several papers (e.g. Eggertsson and Woodford (2003, 2006, EW) and Christiano, Eichenbaum and Rebelo (2011, CER) among others), we assume that LT episodes occur after shocks to preferences raise the discount factor and induce agents to postpone consumption. We study separately cases where these shocks are the only source of uncertainty in the economy, and cases where they are accompanied by a simultaneous increase

[1] Indeed, many observers believe that the global economy has entered an era of *secular stagnation,* in which growth will be anemic and interest rates persistently low. See for example Baldwin and Teulings (2014).

[2] ABN were the first to show that governments want to issue long bonds to hedge against exogenous spending and productivity shocks. Faraglia, Marcet and Scott (2010) extend ABN's model by introducing habit preferences and capital accumulation and show that their result remains. Nosbusch (2008) and Lustig, Sleet and Yeltekin (2008) assume incomplete markets; their conclusion is (again) that long-term debt is optimal.

in public spending to mitigate the effects of the recession. With the latter scenario, we seek to capture another well known characteristic of the recent downturn, namely the large public deficits and mounting debt levels in industrialized economies.

Our key finding is that issuing only long term debt is not always the optimal policy. Instead, depending on the magnitude of the spending shocks and the initial debt level of the government, it may be that short term financing is optimal. In particular, when the initial liability of the government is low and the increase in public spending that occurs when the economy enters the LT is substantial, the government prefers to issue short bonds. However, if at the beginning of the planning horizon, the government has to refinance a large stock of debt, then issuing only long bonds may be optimal.

To understand these findings, note that when preference shocks hit the economy long bond prices increase. Therefore, a government that issues long term debt experiences an increase in its outstanding liability. Whether or not is optimal to incur the loss in the value of the portfolio depends on the response of current and future primary surpluses to preference and spending shocks. There are two effects in the model. First, sizable increases in spending levels tend to put the intertemporal budget of the government into deficit. Second, since preference shocks tend to lower future interest rates, they also tend to increase the present value of the future surpluses to which the government must commit in order to repay a positive (and high) initial debt level. If the first effect prevails, then short term debt is optimal. In contrast, if the second force is the dominant one, financing with long bonds is the optimal strategy.

We show that these results hold under a variety of different parameterizations of the model, when we make alternative assumptions over the specification of preferences, the degree of price stickiness in the economy and the duration of the LT episodes. Moreover, our results hold ex ante (prior to the realization of the shocks) but also ex post, when the economy has entered the LT, and uncertainty evolves around the duration of the episode. This is also a key finding, that the (qualitative) features of the optimal portfolio, do not vary across a range of parameters which maybe hard to measure in the data. The optimal policy is a function whose most crucial arguments are the initial debt level and the level of spending during the LT.

As ABN, this paper studies optimal debt management assuming that the government can complete the market through a portfolio of long and short bonds. It is well known (see for example Lucas and Stockey (1983), Marcet and Scott (2009)) that when markets are complete, the optimal allocation is history independent in the sense that only the current values of the disturbances influence consumption, hours and taxes. However, history independence does not hold in our context; this is a standard implication of optimal policy problems under the ZLB constraint (e.g., EW (2003, 2006)). Finding the optimal portfolio requires to solve the model using global numerical approximation methods. In Section 3 of the paper, we discuss extensively the challenges of doing so when the history of shocks matters for allocations. We show that, in a special case, when the utility is linear in consumption (e.g. Aiyagari, Marcet, Sargent and Seppälä (2002, AMSS) and FMOS (2016)) the difficulty is not severe; the multiplier associated with the ZLB constraint, which summarizes the history, remains constant through time. In this case, we can solve the model using a procedure very similar to the algorithms proposed by FMS (2010) and FMOS (2014) for standard complete market models (with history independence). In the more genereal case (where utility is not quasi-linear), however, the numerical approximation is not trivial because the value of the multiplier shifts over time. The problem then needs to be dealt with methods suitable to solve models of government debt and incomplete financial markets (for instance AMSS, FMOS (2015)); these models also feature time varying multipliers, which summarize histories and policy commitments. We propose an algorithm based on parameterizing expectations and which extends the algorithms of FMOS (2014, 2015) to the binding ZLB. This is a separate contribution of our paper.

There is a long stream of papers which characterize optimal fiscal and monetary policies under the ZLB. Prominent examples are Eggerston and Woodford (2003, 2006), Adam and Billi (2006), Werning (2011), Schmidt (2013), Jung et al (2013), and Nakata (2013). Whereas these papers typically study policy choices during LTs using linearized versions of the new-Keynesian model (allowing only for the nonlinear effects exerted by the ZLB constraint), we solve a fully non-linear new-Keynesian model and propose stochastic simulation algorithms to do so.[3] Moreover, in Section 3 of our paper we consider several model versions which enable us to derive the optimal tax schedule analytically. We use these expressions to build our numerical algorithms but also to provide insights on the effects of current and lagged shocks on the sequence of taxes and inflation. Some of the analytical expressions and results contained in Section 3 are new and complement previous findings in the literature.

Our analysis is also complementary to the theoretical models that link asset purchases and bond yields, which were previously mentioned (e.g. Vayanos and Vila (2009), Chen, Curdia and Ferrero (2012), Gibaud, Nosbusch and Vayanos (2013)). These papers construct models with 'bond clienteles', that is, groups of investors with preferences over specific maturities, and study the effects of shocks to the demand or the supply of a particular maturity on the yield curve.[4] Relative to these papers, we build a standard macro model with one infinitely lived household, and therefore we do not delve in issues related to the market microstructure. Moreover, our model brings together fiscal and monetary policies under a consolidated intertemporal budget, meaning that purchases of long term debt by either the 'Fed' or the 'Treasury' have identical impacts on economic outcomes.[5] On the other hand, we emphasize a different and crucial role of debt management during LTs: If the government is to depart from the optimal maturity that is uniquely pinned down in the model, markets become incomplete and this in

[3]Our algorithms are suitable to deal with large scale versions of the new-Keynesian model. These applications are not pursued here but we refer the reader to Judd et al (2011) for a discussion on this issue. Previous papers that study optimal policies under the ZLB use methods which suffer from the 'curse of dimensionality' as the state space expands. For example, Adam and Billi (2006) derive a Bellman equation following the arguments of Marcet and Marimon (2009). Schmidt (2013) solves his model using a collocation method which approximates the policy rules as functions of the state variables. Both approaches become impractical quickly when state variables are added to the models. This problem however, will be less felt in our model which assumes a simple structure of shocks and complete markets. It becomes severe in models with commitment, incomplete markets and long term government bonds (e.g. FMOS (2015)).

Finally, Nakata (2015) solves a nonlinear model with optimal government spending, using a shooting algorithm. His is basically a non-stochastic environment whereby the preference parameter drops in the initial period and subsequently reverts gradually to the long run value. Ours is a stochastic model with uncertainty over the duration of the LT episode.

[4]Vayanos and Vila (2009) assume exogenously the clienteles and add risk averse arbitrageurs in the bond market. In Gibaud, Nosbusch and Vayanos (2013) bond clienteles arise endogenously in an overlapping generations model. Chen, Curdia and Ferrero (2012) build a DSGE model in which a fraction of agents holds only long term debt and the remaining agents hold both short and long bonds subject to transaction costs.

[5]Put differently, in our model transactions between the central bank and the Treasury have no direct effect on the private sector's behavior.

turn implies that the distortionary impacts of taxes, debt and inflation are exacerbated in both the long and the short run. The interplay between the forces we identify and the recent literature on government debt buybacks remains to be explored.

This paper proceeds as follows: Section 2 presents the model and the planning program. Section 3 discusses the properties of equilibria, with analytical results and numerical algorithms. Section 4 discusses the results on optimal debt management based on the numerical solution of the model. Section 5 extends the baseline model and presents several robustness exercises. Section 6 estimates the welfare impact of optimal debt management in the model. A final section concludes.

2 Model

2.1 Agents

2.1.1 Preferences

We consider an infinite horizon economy, populated by a representative household with preferences defined by

$$E_0\Big[\sum_{t=0}^{\infty}\beta^t\Big(\widetilde{u}(c_t,\xi_t)+\widetilde{v}(h_t,\xi_t)\Big)\Big], \tag{1}$$

where c_t denotes consumption, h_t denotes hours, and β is the discount factor. The term ξ_t represents a shock to preferences that we will model as in EW. We assume that preferences are such that $\widetilde{u}(c_t,\xi_t)=\xi_t u(c_t)$ and $\widetilde{v}(h_t,\xi_t)=\xi_t v(h_t)$. A drop in ξ_t relative to ξ_{t+j}, $j=1,2...$ implies that the household wants to postpone consumption (and leisure) to the future.

2.1.2 Firms

The consumption good is produced by a representative, perfectly competitive, final-good producer using a Dixit-Stigltiz aggregator of a continuum of differentiated intermediate products. The production of a generic intermediate product i is carried out by a monopolistically competitive firm using a linear technology $y_{i,t}=h_{i,t}$. The demand for product i is given by $Y_t d(P_{i,t}/P_t)$, where $P_{i,t}$ is the price of intermediate product i, P_t is the price of the composite final good, and Y_t is output in the final-good sector. The demand function, d, satisfies additional assumptions that guarantee the existence of a symmetric equilibrium, namely, $d(1)=1$ and $d'(1)\equiv\eta<-1$.

We assume that the prices of intermediate goods are costly to adjust. Following Rotemberg (1982), adjustment costs are given by $\frac{\theta}{2}(\frac{P_{i,t}}{P_{i,t-1}}-1)^2$, where $\theta\geq 0$ governs the degree of price stickiness. When $\theta=0$, prices are fully flexible. When $\theta\to\infty$, prices will remain constant through time.

Intermediate-good producers seek to maximize

$$E_t\sum_{j=0}^{\infty}\beta^j\frac{u_c(c_{t+j})\xi_{t+j}}{u_c(c_t)\xi_t}\left\{\frac{P_{i,t+j}}{P_{t+j}}Y_{t+j}d\left(\frac{P_{i,t+j}}{P_{t+j}}\right)-w_{t+j}h_{i,t+j}-\frac{\theta}{2}\left(\frac{P_{i,t+j}}{P_{i,t+j-1}}-1\right)^2\right\}, \tag{2}$$

subject to the constraint $h_{i,t+j} = Y_{t+j} d(P_{i,t+j}/P_{t+j})$. The first-order condition with respect to $P_{i,t}$ is given by

$$\frac{1}{P_t} Y_t d\left(\frac{P_{i,t}}{P_t}\right) + \frac{P_{i,t}}{P_t^2} Y_t d'\left(\frac{P_{i,t}}{P_t}\right) - w_t Y_t d'\left(\frac{P_{i,t}}{P_t}\right)\frac{1}{P_t} - \theta\left(\frac{P_{i,t}}{P_{i,t-1}} - 1\right)\frac{1}{P_{i,t-1}}$$
$$+\beta E_t \frac{u_{c,t+1}\xi_{t+1}}{u_{c,t}\xi_t}\theta\left(\frac{P_{i,t+1}}{P_{i,t}} - 1\right)\frac{P_{i,t+1}}{P_{i,t}^2} = 0,$$

where $u_{c,t} \equiv u'(c_t)$.

This equation forms the Phillips curve, describing the inflation output trade-off in our model. Imposing a symmetric equilibrium (all firms set the same price) gives

$$(\pi_t - 1)\pi_t = \frac{\eta}{\theta}\left(\frac{1+\eta}{\eta} - w_t\right) Y_t + \beta E_t \frac{u_{c,t+1}\xi_{t+1}}{u_{c,t}\xi_t}(\pi_{t+1} - 1)\pi_{t+1}, \tag{3}$$

where $\pi_t \equiv \frac{P_t}{P_{t-1}}$ denotes gross inflation.

2.1.3 Government and markets

The government engages in two activities — it levies taxes on the household's labor income and trades with the household in bond markets to finance a spending process $\{g_t\}_0^\infty$. We denote labor-income taxes by τ_t. Moreover, let B_t^j be the quantity of a bond issued in period t that promises to pay a unit of income in $t + j$. Let the price of such a bond be q_t^j. For simplicity, denote by $\mathcal{J}$ the set of available maturities.[6]

Following ABN, we study an economy in which government bonds can complete the market. That is, there exists a portfolio of bonds that perfectly insures the government's budget at any period and for every realization of the shocks, given q_t^j and τ_t. Under complete markets, it becomes immaterial whether the government redeems the outstanding bonds at maturity or whether it buys back debt in every period and then reissues. The latter assumption is followed by many papers in the literature (see, for example, ABN, FMOS (2015), among others). For simplicity, we will also follow this setup. The government budget constraint can be written as

$$\sum_{j\in\mathcal{J}} q_t^j B_t^j = \sum_{j\in\mathcal{J}} q_t^{j-1} B_{t-1}^j + P_t(g_t - \tau_t w_t h_t). \tag{4}$$

The initial debt level of the government will be denoted by $\overline{B}_{-1}$.

[6]Notice that even though we can have $\mathcal{J} \equiv \{1, 2, ..., \overline{j}\}$, that is, allow the government to trade with any maturity ranging from one period to some maximum length $\overline{j}$, in practice, we will not need all of these trades to be realized. Since there will be only two states of shocks, it suffices to have $\mathcal{J} = \{1, N\}$ (a one-period bond and an N-period bond). This structure is standard in the macro debt-management literature.

2.1.4 Household optimization

Given the policies described above, the household maximizes utility (1) subject to the sequence of budget constraints

$$\sum_{j\in\mathcal{J}} q_t^j B_t^{j,H} = \sum_{j\in\mathcal{J}} q_t^{j-1} B_{t-1}^{j,H} + P_t(1-\tau_t)w_t h_t - P_t c_t + \widetilde{\Pi}_t, \tag{5}$$

where $B_t^{j,H}$ represents the quantity of debt of maturity j demanded by the household in t and $\widetilde{\Pi}_t$ denotes profits of the firms operated by the household. From the household's optimization, we can derive the following optimality conditions:

$$(1-\tau_t)w_t = -\frac{v_{h,t}}{u_{c,t}},$$

which gives the tax rate as one minus the marginal utility ratio and

$$q_t^j = \beta^j E_t \frac{u_{c,t+j}\xi_{t+j}P_t}{u_{c,t}\xi_t P_{t+j}},$$

which equates the bond price with the marginal utility growth divided by the price ratio. Moreover, it holds that $q_t^0 = 1$. Notice further that since we assume a representative household, the prices q_t^j are also the prices of bonds in the *secondary market*. When the government buys back outstanding debt in the market it has to pay q_t^{j-1} for the $j-1$ maturity as is evident from equations (4) and (5).

2.2 Uncertainty

Our goal is to determine debt-management policies both before and during the LT. We therefore assume that, in period 1, the economy can be hit by a shock that lowers ξ_1 to a value $\underline{\xi} < \overline{\xi} = \xi_0$. This shock occurs with probability equal to ω. If the shock is realized then the value of the preference parameter remains equal to $\underline{\xi}$ with probability ϕ in each period until another shock arrives (at rate $1-\phi$) and thereafter $\xi_t = \overline{\xi}$ for all t. In other words, following a shock that lowers the value of ξ_1 the preference parameter is a first order Markov process with the transition matrix

$$P_\xi \equiv \begin{bmatrix} 1 & 0 \\ (1-\phi) & \phi \end{bmatrix}.$$

In the case where the shock to preferences does not occur in period 1 (probability $1-\omega$) we set $\xi_t = \overline{\xi}$ for all t.

Allowing for the value of ξ_t to drop in period 1 is not sufficient for the economy to fall in a LT. It must be that the drop is large enough so that the zero lower bound is violated and policy reacts to satisfy the constraint. Because we will use a variety of different setups, it is not possible to find sufficient conditions under which the difference between $\underline{\xi}$ and $\overline{\xi}$ is such that the constraint binds. For each of the versions of the model we will consider, we will report the required value $\underline{\xi}$ that gives persistent LTs according to the process defined previously.

Turning to the process of government spending, which is assumed exogenous to the model, we let $g_t = \overline{g}$ (the steady-state value) in periods where the ZLB is not binding. When the economy is in the LT, we will set $g_t = \underline{g} \geq \overline{g}$. Preference shocks will be, in some parameterizations of the model, accompanied by increases in spending levels that will last for as long as the economy remains in the LT.

2.3 The Ramsey problem

We assume that the government maximizes the household's utility under full commitment as in ABN, SGU, FMOS. We follow the primal approach; we eliminate the tax rate and the bond prices from the program using the equilibrium expressions for these objects.

Let $\mathbf{c}$ denote the sequence of consumptions $[c_0, c_1, ...]$ and similarly for the other variables of the model. A competitive equilibrium is a feasible allocation $\mathbf{c}, \mathbf{h}, \mathbf{g}$, a price system $\mathbf{w}, \mathbf{q}, \pi$ and a government policy $\mathbf{g}, \tau, \pi, \mathbf{b}$ such that given prices and the policy, $\mathbf{c}, \mathbf{h}, \pi$ solve the firms' and household's maximization problem and satisfy the sequence of government budget constraints.

The Ramsey program chooses $\tau, \pi, \mathbf{b}$ selecting the competitive equilibrium that maximizes (1). As is well known, under complete markets, this is equivalent to choosing $\mathbf{c}, \mathbf{h}, \pi, \mathbf{w}$ to the maximize the household's utility subject to the following date 0 implementability constraint (e.g. Chari and Kehoe (1999), FMS (2010)):[7]

$$(6) \qquad E_0 \sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}\xi_t}{u_{c,0}\xi_0} \left[-g_t + \left(1 + \frac{v_{h,t}}{u_{c,t} w_t}\right) w_t h_t \right] = \overline{B}_{-1},$$

together with the Phillips curve (3), the resource constraint,

$$(7) \qquad c_t + g_t + \frac{\theta}{2}(\pi_t - 1)^2 = h_t.$$

Notice, however, that in our economy maximizing welfare subject to (3), (6) and (7) does not suffice to find an equilibrium policy that also satisfies the ZLB on the nominal interest rate. Therefore, the optimal allocation also needs to satisfy

$$(8) \qquad q_t^1 \equiv \beta E_t \frac{u_{c,t+1}\xi_{t+1}}{u_{c,t}\xi_t \pi_{t+1}} \leq 1,$$

for all t, to ensure that the competitive equilibrium selected by the planner gives a sequence of short term rates $i_t = \frac{1}{q_t^1} - 1$ bounded from below by zero.

[7] Notice that (6) is the intertemporal budget of the government and be easily derived by iterating forward on equation (4).

Let $(\lambda_s, \lambda_{p,t}, \lambda_{f,t}, \lambda_{ZLB,t})$ be the vector of Lagrange multipliers; we formulate the Lagrangian as:

$$\mathcal{L} = E_0 \sum_t \beta^t \Big\{ u(c_t)\xi_t + v(h_t)\xi_t + \lambda_{f,t}\left(h_t - c_t - g_t - \frac{\theta}{2}(\pi_t - 1)^2\right) - \lambda_{ZLB,t}\left(u_{c,t}\xi_t - \beta E_t \frac{u_{c,t+1}\xi_{t+1}}{\pi_{t+1}}\right)$$
$$-\lambda_{p,t}\left(u_{c,t}\xi_t\pi_t(\pi_t - 1) - \frac{\eta}{\theta}h_t u_{c,t}\xi_t\left(\frac{1+\eta}{\eta} - w_t\right) - \beta E_t u_{c,t+1}\xi_{t+1}\pi_{t+1}(\pi_{t+1} - 1)\right)\Big\}$$
$$-\lambda_s\left[E_0\sum_{t=0}^{\infty}\beta^t u_{c,t}\xi_t\left[-g_t + (1 + \frac{v_{h,t}}{u_{c,t}w_t})w_t h_t\right] - u_{c,0}\xi_0\overline{B}_{-1}\right]. \tag{9}$$

2.3.1 Optimality

Denoting by $s_t \equiv -g_t + (1 + \frac{v_{h,t}}{u_{c,t}w_t})w_t h_t$ the per-period government's surplus the first-order conditions for the optimum can be written as

$$u_{c,t}\xi_t - \lambda_{f,t} + \lambda_{p,t}\frac{\eta}{\theta}h_t u_{cc,t}\xi_t\left(\frac{1+\eta}{\eta} - w_t\right) - u_{cc,t}\xi_t\left(\lambda_{ZLB,t} - \lambda_{ZLB,t-1}\frac{1}{\pi_t}\right) - \lambda_s\xi_t\left(u_{cc,t}s_t + u_{c,t}s_{c,t}\right)$$
$$+\lambda_s u_{c,0}\xi_0\overline{B}_{-1}\mathcal{I}_{t=0} = 0, \tag{10}$$

$$v_{h,t}\xi_t + \lambda_{f,t} - \lambda_s u_{c,t}\xi_t s_{h,t} + \lambda_{p,t}\frac{\eta}{\theta}u_{c,t}\xi_t\left(\frac{1+\eta}{\eta} - w_t\right) = 0, \tag{11}$$

$$-\theta\lambda_{f,t}(\pi_t - 1) - \lambda_{ZLB,t-1}\frac{u_{c,t}\xi_t}{\pi_t^2} = 0, \tag{12}$$

$$-\lambda_s s_{w,t} - \lambda_{p,t}\frac{\eta}{\theta}h_t = 0, \tag{13}$$

where $s_{c,t} \equiv -\frac{v_{h,t}}{u_{c,t}^2}u_{cc,t}h_t$ and $s_{h,t} = \frac{v_{hh,t}}{u_{c,t}}h_t + \left(1 + \frac{v_{h,t}}{u_{c,t}w_t}\right)w_t$, and $s_{w,t} = h_t$.

(10) is the first order condition for consumption in t. $\mathcal{I}$ denotes the indicator function, and therefore the last term in this equation applies solely in period 0 and if the outstanding liability of the government $\overline{B}_{-1}$ is non-zero.[8] (11) and (12) are the first order conditions with respect to hours and inflation. From (13), the first order condition with respect to wages, we get that $\lambda_{p,t} = -\lambda_s\frac{\theta}{\eta}$ (constant).

Finally, from complementary slackness, we have

$$\lambda_{ZLB,t}\left(u_{c,t}\xi_t - \beta E_t\frac{u_{c,t+1}\xi_{t+1}}{\pi_{t+1}}\right) = 0.$$

Together with the constraints from the planner's program, these equations yield the system that needs to be solved to obtain the optimal allocation.

2.4 Optimal Debt Management

Given the solution to the planner's program we follow the approach of ABN and FMS (2010) to recover the optimal portfolio of the government. Since in our model there are two states of the world in every period the optimal portfolio can be found as a combination of two different maturities provided that

[8] In the case of positive initial debt levels this term captures the incentive of the government to manipulate interest rates through changes in taxes (see FMOS (2016)).

these assets give us a matrix of returns that is of full rank. As is standard in the literature, we consider a short bond, of one period maturity, and a long bond of maturity equal to N periods.

Let $x^t = ((\xi_0, g_0), ..., (\xi_t, g_t))$ denote the history of shocks until t and let (b_t^1, b_t^N) denote the optimal portfolio of the *real* value of short and long bonds issued in t. (b_t^1, b_t^N) can be found as a solution to the following system of equations:

$$
(14) \qquad \begin{bmatrix} \frac{1}{\pi\left(x^t,(\overline{\xi},\overline{g})\right)} & \frac{1}{\pi\left(x^t,(\overline{\xi},\overline{g})\right)} q_{t+1}^{N-1}\left(x^t,(\overline{\xi},\overline{g})\right) \\ \frac{1}{\pi\left(x^t,(\overline{\xi},\overline{g})\right)} & \frac{1}{\pi(\left(x^t,(\underline{\xi},\underline{g})\right)} q_{t+1}^{N-1}\left(x^t,(\underline{\xi},\underline{g})\right) \end{bmatrix} \cdot \begin{bmatrix} b_t^1 \\ b_t^N \end{bmatrix} = \begin{bmatrix} S_{t+1}\left(x^t,(\overline{\xi},\overline{g})\right) \\ S_{t+1}\left(x^t,(\underline{\xi},\underline{g})\right) \end{bmatrix}.
$$

where $S_{t+1}\left(x^t,(\underline{\xi},\underline{g})\right)$ is the present discounted value of the government surplus when the economy remains in the LT, meaning that following history x^t we have $(\xi_{t+1}, g_{t+1}) = (\underline{\xi}, \underline{g})$.[9] Analogously, $S_{t+1}\left(x^t,(\overline{\xi},\overline{g})\right)$ is the surplus when the economy experiences in $t+1$ an increase in the value of ξ_t and permanently escapes the LT.

According to (14) the government issues a portfolio in period t that ensures that the intertemporal constraint is satisfied with equality in both states in $t+1$ given the future policies for taxes and inflation. System (14) has a unique solution if and only if $q_{t+1}^{N-1}\left(x^t,(\overline{\xi},\overline{g})\right) \neq q_{t+1}^{N-1}\left(x^t,(\underline{\xi},\underline{g})\right)$. Note further that it is meaningful to use (14) to obtain (b_t^1, b_t^N) only in cases where the risk of falling (remaining) in the LT in the next period is present. When $\xi_{t+1} = \overline{\xi}$ and given how uncertainty enters in the model, there are no further gains from debt management, and the government will be able to accomplish the same outcomes when it issues one type of bond as when it issues any combination of the two bonds. In our subsequent analysis we will therefore characterize the government's portfolio at $t = 0$ and in during LT episodes.

3 Solving the Model: Analytics and Algorithms

This section characterizes the solution of the optimal policy in the model. In Section 3.1 we focus on the long run properties of policy and allocations; we show that following any history of shocks, taxes, inflation, and all other model variables return to constant long run values. In Section 3.2, we consider the short run properties of the solution to the planner's program and propose numerical algorithms to globally approximate the equilibrium.

3.1 Long Run Distortion smoothing

Given the structure of shocks described in Section 2.2, if we simulate the model many times, the duration of LT episodes will vary across the samples. We consider one realization of these simulations whereby the economy enters in the LT in period 1 and period $T > 1$ is the last period where $\xi_t = \underline{\xi}$. Thereafter, $\xi_t = \overline{\xi}$. Notice that T is a random variable; in our analysis we use a generic value T.

Proposition 1. *Consider a history of preference and spending shocks x^T for $T > 1$ and let $x_{T+1} = (\overline{\xi}, \overline{g})$.*

[9] Recall that it is possible to get $(\xi_{t+1}, g_{t+1}) = (\underline{\xi}, \underline{g})$ as a non-zero probability event if and only if $\xi_t = \underline{\xi}$ for $t > 0$.

After period $T+1$ i) the optimal inflation rate is 0. ii) the optimal tax rate equals $\overline{\tau}$ and the debt level equals $\overline{B}$ (constants). iii) $\overline{\tau}$ and $\overline{B}$ are independent of x^T and T.

Proof. i) follows from equation (12) and the fact that $\lambda_{ZLB,T+1} = \lambda_{ZLB,T+2} = \lambda_{ZLB,T+3}... = 0$ (from complementary slackness). Then, we have that

$$-\theta\lambda_{f,t}(\pi_t - 1) = 0 \to \pi_t = 1.$$

To show ii) first note that since inflation equals zero for $t > T+1$, $w_t = \frac{1+\eta}{\eta}$. From (10) and (11), and replacing $g_t = \overline{g}$, we have

$$u_{c,t} + v_{h,t} - \lambda_s u_{c,t}\left[\frac{v_{hh,t}}{u_{c,t}}h_t + \left(1 + \frac{v_{h,t}}{u_{c,t}}\frac{\eta}{1+\eta}\right)\frac{1+\eta}{\eta}\right] - \lambda_s u_{cc}(c_t)\left[-\overline{g} + \left(1 + \frac{v_{h,t}\eta}{u_{c,t}(1+\eta)}\right)\frac{1+\eta}{\eta}h_t\right]$$

$$+\lambda_s u_c(c_t)\left(-\frac{v_{h,t}}{u^2_{c,t}}u_{cc,t}h_t\right) = 0, \tag{15}$$

where $c_t + \overline{g} = h_t$.

Notice that (15) defines an nonlinear equation in consumption that applies in all periods after $T+1$. The solution defines a constant consumption level $\overline{c}$ and since hours are also constant, taxes are constant. The per-period surplus, s_t, is therefore also constant (equal to $\overline{s}$).

Turning to the debt levels and recalling that once uncertainty is removed from the model, a short bond is sufficient to complete the markets, we have

$$B^1_{t-1} = \sum_{j=1}^{\infty}\beta^{j-1}\frac{u_{c,t+j}}{u_{c,t}}\left[-\overline{g} + \left(1 + \frac{v_{h,t+j}\eta}{u_{c,t+j}(1+\eta)}\right)\frac{1+\eta}{\eta}h_{t+j}\right] = \frac{\overline{s}}{1-\beta} = \overline{B},$$

for $t = T+2, T+3, ...$

iii) follows easily from the above derivations. ■

Under complete markets, all the distortions stemming from price rigidity and taxation are eliminated in the long run when the ZLB ceases to bind. The objective of debt management therefore is to construct a portfolio of government bonds that achieves price stability and tax smoothing once the economy exits the LT. This objective is common with the models of ABN.

3.2 Short-run properties of the planner's solution and algorithm

The previous subsection demonstrated that in our model the long run allocation is history independent; following a sequence of preference and spending shocks $\{x_t\}_1^T = \{(\underline{\xi}, \underline{g}), ..., \underline{\xi}, \underline{g})\}$, the long run values of consumption, hours, inflation and taxes are independent of the random variable T. This property is convenient; in this section we will construct numerical algorithms to approximate the solution of the planner's program, and the algorithms will be based on model simulations; long run history independence relieves us from the task of finding terminal conditions separately, for each sample, based on the realization of T.

Notice further that the derivations in Section 3.1 can be used to shed light on the properties of the complete market allocation in the absence of the ZLB constraint. If we ignored the constraint (8), then inflation would equal zero in every period. Moreover, from equation (15) (and replacing $\overline{g}$ with g_t) it follows easily that consumption is a function of the spending level in t solely. These properties are standard and well known to the literature (see for example Lucas and Stockey (1983) and Marcet and Scott (2009)). When markets are complete, the government does not have to use inflation to stabilize public debt (as it does for example in the incomplete market models of SGU (2004), Lustig et al (2008) and FMOS (2013)). Instead, it can exploit variations in bond prices to smooth tax distortions in the short run, as in ABN. Since inflation bestows a welfare loss to society, using it to reduce the real payout of debt is completely wasteful. Moreover, since period t spending levels are the only force behind short run variations in the level of consumption (and hence also hours and taxes) the optimal allocation is history independent.

History independence is, however, only a long run property in our model with the ZLB constraint. In the short run, the presence of the constraint makes the history of shocks matter for the allocation; this can be easily seen by noticing the presence of the lagged value of the multiplier $\lambda_{ZLB,t-1}$ in the first order conditions. This property differentiates our model from the existing literature on optimal debt management under complete markets, and introduces an important element of complexity in the global approximation of the equilibrium, as we will later argue.

In what follows, we assume that preferences are of the form:

$$u(c_t) + v(h_t) = \frac{c_t^{1-\gamma_c} - 1}{1-\gamma_c} - \psi \frac{h_t^{1+\gamma_h}}{1+\gamma_h}, \tag{16}$$

where γ_c represents the relative risk aversion coefficient and γ_h is the inverse of the Frisch elasticity of labor supply. In Section 3.2.1 we will consider the special case $\gamma_c = 0$ (quasi-linear preferences). This case is convenient for two reasons: First, as we show below, the multiplier $\lambda_{ZLB,t-1}$ is constant, which greatly attenuates the influence of history dependence on the optimal allocation. Second, under quasi-linear preferences, the model solution is tractable enough to allow for useful analytical insights on optimal debt management. In Section 3.2.2, we study the (more plausible) case where $\gamma_c > 0$.

3.2.1 A special case: quasi-linear preferences

Under quasi-linear preferences ($\gamma_c = 0$) , the system of first order conditions becomes:

$$\xi_t = \lambda_{f,t}, \tag{17}$$

$$v_h(h_t)\xi_t + \xi_t - \lambda_s \xi_t \left[v_{hh}(h_t)h_t + \left(1 + v_h(h_t)\frac{1}{w_t}\right) w_t \right] + \lambda_p \frac{\eta}{\theta} \xi_t \left(\frac{1+\eta}{\eta} - w_t \right) = 0, \tag{18}$$

$$-\theta \xi_t (\pi_t - 1) - \lambda_{ZLB,t-1} \frac{\xi_t}{\pi_t^2} = 0, \tag{19}$$

and the slackness condition becomes

$$\lambda_{ZLB,t}\left(\xi_t - \beta E_t \frac{\xi_{t+1}}{\pi_{t+1}}\right) = 0.$$

Analytics The following Proposition explains the properties of inflation, hours and taxes in the model.
Proposition 2: Inflation, Hours and Taxes in Quasi-linear Utility *Assume that preferences are of the form (16) with $\gamma_c = 0$. Assume $\xi_t = \underline{\xi}$ for $t = 1, 2, .., T$ and $\xi_t = \overline{\xi}$ for $t \geq T+1$.*
i) The optimal path of inflation is given by:

$$\pi_t = \begin{cases} 1 & t = 0, 1 \text{ and } t = T+2, T+3, ... \\ \beta \frac{\phi \underline{\xi} + (1-\phi)\overline{\xi}}{\underline{\xi}} \equiv \underline{\pi} & t = 2, 3, ..., T+1 \end{cases} \tag{20}$$

ii) Hours are constant over time so that $h_t = \overline{h}$. The optimal tax rate satisfies:

$$\tau_t = \begin{cases} \omega_{\lambda_s}\left(1 - \lambda_s \gamma_h - \widetilde{\eta}\right) & t = 0, t \geq T+2 \\ \omega_{\lambda_s}\left(1 - \lambda_s \gamma_h - \frac{1}{\frac{1+\eta}{\eta} + \kappa \underline{\pi}^2(\underline{\pi}-1)}\left[1 + \lambda_s \kappa \underline{\pi}^2(\underline{\pi}-1)\right]\right) & t = 1 \\ \omega_{\lambda_s}\left(1 - \lambda_s \gamma_h - \frac{1}{\frac{1+\eta}{\eta} + \kappa \underline{\pi}(\underline{\pi}-1)^2}\left[1 + \lambda_s \kappa \underline{\pi}(\underline{\pi}-1)^2\right]\right) & t = 2, 3, ...T \\ \omega_{\lambda_s}\left(1 - \lambda_s \gamma_h - \frac{1}{\frac{1+\eta}{\eta} - \kappa \underline{\pi}(\underline{\pi}-1)}\left[1 - \lambda_s \kappa \underline{\pi}(\underline{\pi}-1)\right]\right) & t = T+1 \end{cases} \tag{21}$$

where $\omega_{\lambda_s} \equiv \frac{1}{1-\lambda_s(1+\gamma_h)}$ and $\widetilde{\eta} = \frac{\eta}{1+\eta}$. Moreover, $\kappa = \frac{\theta}{\eta \overline{h}}$.

Proof: Towards i): From equation (19), the optimal inflation satisfies $-\theta(\pi_t - 1)\pi_t^2 = \lambda_{ZLB,t-1}$. Inflation differs from zero whenever the ZLB has binded in the previous period. Moreover, $\pi_{t+1}\left(x^t, (\overline{\xi}, \overline{g})\right) = \pi_{t+1}\left(x^t, (\underline{\xi}, \underline{g})\right)$. If the economy has just escaped the LT, the inflation rate remains equal to the level it would be if the economy remained in the trap for a one more period.

From the ZLB constraint we have:

$$\beta E_t \frac{\xi_{t+1}}{\xi_t} \frac{1}{\pi_{t+1}} = 1,$$

for $t = 1, 2, ...T$. Since π_{t+1} is not random (conditional on date t information) we have

$$\pi_{t+1} = \beta E_t \frac{\xi_{t+1}}{\xi_t} = \beta \frac{\phi \underline{\xi} + (1-\phi)\overline{\xi}}{\underline{\xi}},$$

for $t = 1, 2, ...T$. For $t = 0$ we have $-\theta(\pi_1 - 1)\pi_1^2 = \lambda_{ZLB,0} = 0$ and therefore $\pi_1 = 1$.

To show ii) first note that combining (17) and (18) together with $\lambda_s = -\frac{\eta}{\theta}\lambda_p$, we get that

$$v_{h,t} + 1 - \lambda_s \left(v_{hh,t}h_t + v_{h,t}\right) + \lambda_p \frac{1+\eta}{\theta} = 0, \tag{22}$$

which gives $h_t = \overline{h}$. Moreover, noting that under preferences (16) we have $v_{hh}(h_t)h_t = \gamma_h v_h(h_t)$, it follows that:

$$\tau_t(1 - \lambda_s(1+\gamma_h)) = \left(1 - \lambda_s\gamma_h - \frac{1}{w_t}\right) + \frac{\lambda_s}{w_t}\left(\frac{1+\eta}{\eta} - w_t\right). \tag{23}$$

When inflation equals zero (and therefore $\left(\frac{1+\eta}{\eta} - w_t\right) = 0$), we have $\frac{\tau_t}{\omega_{\lambda_s}} = (1 - \lambda_s\gamma_h - \widetilde{\eta})$. For periods $t = 1, 2, ..T+1$ we can use the Phillips curve (3) and substitute the process of inflation, to complete the derivation of (21). For the sake of brevity the algebra is relegated to the appendix.■

Several comments are in order: Notice first that under quasi-linear utility it suffices to pin down hours inflation and taxes in order to compute the equilibrium. Since $u(c_t) = c_t$ consumption levels have no influence on the nominal interest rate and therefore the value of c_t does not need to be considered when finding the equilibrium in this model; c_t is a residual that is set to satisfy the resource constraint.[10] Second, notice that under preferences (16), the optimal tax rate would be constant if we assumed perfect competition in the goods sector (i.e. $\eta = -\infty$). Under perfect competition, the real wage remains constant thereby implying that taxes are also constant.[11] In contrast, under finite η taxes move to compensate for the movements in wages, which may rise or fall depending on whether the government raises inflation today, or plans to do so in the next period.

Solving the model with quasi-linear preferences Under $\gamma_c = 0$ inflation is the only policy margin that can adjust to satisfy constraint (8). Since inflation is exogenously given and $u_{cc,t} = 0$, the value of $\lambda_{ZLB,t}$ is not at all important for the computation of the equilibrium in the model. From previous derivations we have that $\lambda_{ZLB,t-1} = -\theta\underline{\pi}^2(\underline{\pi} - 1)$ remains constant when the ZLB is binding. This is an example where the dependence of the system of first order conditions on the term $\lambda_{ZLB,t-1}$ does not significantly impact the optimal allocation. Hours are constant over time as is the case in models without the ZLB constraint and complete markets.

Though our previous derivations may have given to the reader the impression that this model admits a closed form solution, in practice it does not. We still have to recover the value of λ_s that satisfies the intertemporal budget constraint with equality. λ_s enters in a nonlinear fashion in the expressions we derived previously. The model needs to be solved with numerical methods.

[10] Practically, the problem is not defined if we assume that $\theta \to \infty$, since in this case utility diverges to minus infinity. Large values of θ could give us that $c_t < 0$ for some t. Here, we have not imposed non-negativity constraints on consumption; we verify that the value of θ we pick when we calibrate the model always gives us a consumption level exceeding zero.

[11] See for example Scott (2008). Under complete markets, taxes are affected by the shocks when the value of the elasticity of labor supply changes over the cycle. But under preferences (16) the elasticity is constant and so the planner commits to a constant tax schedule. To see this, notice that because hours are constant and $u_c(c_t) = 1$, the labor-supply condition is given by $\psi\overline{h}^{\gamma_h} = w_t(1 - \tau_t)$. Under perfect competition, the real wage remains constant thereby implying that taxes are also constant.

To formally state the task, let us define by $\overline{s}$, $\underline{\underline{s}}$, $\underline{s}$ and $\overline{\overline{s}}$ the 4 values of the government surplus (s_t for $t = 0, 1$ $1 < t < T+1$ and $t = T+1$ respectively) that derive from (21). We can show that

$$s_t \equiv \tau_t w_t \overline{h} = \overline{h}\left(w_t - \frac{1+\eta}{\eta}\right) + \overline{h}\frac{1+\eta}{\eta}\left(1 - \psi\overline{h}^{\gamma_h}\frac{\eta}{1+\eta}\right) \equiv \overline{h}\left(w_t - \frac{1+\eta}{\eta}\right) + \overline{s}.$$

In other words, the surplus in t is equal to the long run value $\overline{s}$ plus the term $\overline{h}\Big(w_t - \frac{1+\eta}{\eta}\Big)$, which measures the deviation of wages in t from their long run value $\frac{1+\eta}{\eta}$. Given λ_s we have

$$s_t = \begin{cases} \overline{s}(\lambda_s), & t = 0, t \geq T+2 \\ \underline{\underline{s}} \equiv \overline{s}(\lambda_s) + \frac{\theta}{\eta}\underline{\pi}^2(\underline{\pi} - 1) - (\underline{g} - \overline{g}), & t = 1 \\ \underline{s} \equiv \overline{s}(\lambda_s) + \frac{\theta}{\eta}\underline{\pi}(\underline{\pi} - 1)^2 - (\underline{g} - \overline{g}), & t = 2, 3, ...T \\ \overline{\overline{s}} \equiv \overline{s}(\lambda_s) - \frac{\theta}{\eta}\underline{\pi}(\underline{\pi} - 1), & t = T+1 \end{cases} \tag{24}$$

The following numerical procedure can be utilized to approximate the equilibrium in the model:

Algorithm 1

- **Step 1** Guess a value for $\lambda_s = \lambda_s^0$. Solve equation (24) and formulate the expected present value of the surplus in period 0 as follows:

$$S_0 = (1-\omega)\frac{\overline{s}}{1-\beta} + \omega\left[\overline{s} + \beta\frac{\underline{\xi}}{\overline{\xi}}\left(\underline{\underline{s}} + \frac{\beta\phi}{1-\beta\phi}\underline{s}\right) + \frac{\beta^2(1-\phi)}{1-\beta\phi}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right)\right].$$

- **Step 2** If $|S_0 - \overline{B}_{-1}| > \epsilon$ (where ϵ is a pre-specified tolerance level) update the value λ_s^0 and return to Step 1. Iterate to convergence.

3.2.2 CRRA Preferences

In this section we show that assuming $\gamma_c > 0$ complicates considerably the task of finding equilibria under complete markets. We demonstrate this property using a simple example that establishes that the multiplier $\lambda_{ZLB,t}$ cannot be constant in equilibrium. With time varying multipliers, history dependence exerts a more significant influence, and the model cannot be solved using standard numerical algorithms of complete market models. We propose a numerical procedure based on the incomplete market model algorithms of FMOS (2014, 2015).

To demonstrate the above analytically, we assume $\theta = \infty$. In this case the government will never want to use inflation when the ZLB constraint binds and therefore, policy adjustments are made solely through taxes. To satisfy (8) the government will engineer a recession through a rise in the tax rate in periods where $\xi_t = \underline{\xi}$ and commit to a boom through a subsequent fall in the tax rate when the economy escapes from the LT.

The system of first order conditions becomes:

$$u_{c,t}\xi_t - \lambda_{f,t} - u_{cc,t}\xi_t\Delta\lambda_{ZLB,t} - \lambda_s\xi_t\left(u_{cc,t}s(t) + u_{c,t}s_{c,t}\right) + \lambda_s u_{c,0}\xi_0\widetilde{B}_{-1}\mathcal{I}_{t=0} = 0, \tag{25}$$

$$v_{h,t}\xi_t + \lambda_{f,t} - \lambda_s u_{c,t}\xi_t s_{h,t} = 0, \tag{26}$$

where $s_{c,t} \equiv -\frac{v_{h,t}}{u_{c,t}^2}u_{cc,t}h_t$ and $s_{h,t} = \frac{v_{hh,t}}{u_{c,t}}h_t + \left(1 + \frac{v_{h,t}}{u_{c,t}}\frac{\eta}{1+\eta}\right)\frac{1+\eta}{\eta}$, and $s_{w,t} = h_t$.

Analytics The following proposition shows the properties of the tax schedule in the model

Proposition 3 *Assume that $\gamma_c > 0$, $\widetilde{B}_{-1} = 0$ and $\theta = \infty$. The optimal tax rate is given by:*

(27)

$$\tau_t = \frac{1}{\frac{1+\eta}{\eta}(1 - \lambda_s(1+\gamma_h))}\left[\left(\frac{1+\eta}{\eta} - 1\right) - (\lambda_s\gamma_h + 1)\frac{1+\eta}{\eta} + \frac{\lambda_s u_{cc,t}g_t}{u_{c,t}}\left(\frac{1+\eta}{\eta} - 1\right) + \frac{u_{cc,t}}{u_{c,t}}\Delta\lambda_{ZLB,t}\right].$$

Proof: See Appendix.[12]

Impossibility of having a constant $\lambda_{ZLB,t}$ under $\gamma_c > 0$. Since under the assumptions of Proposition 3 it holds that $\tau_t = 1 - \psi(c_t + g_t)^{\gamma_h}c_t$, equation (27), together with the ZLB constraint, define a system of 2 equations in c_t and $\lambda_{ZLB,t}$. Let us assume that in equilibrium we have $\lambda_{ZLB,t} = \overline{\lambda}_{ZLB} < 0$ for $t = 1, 2, .., T$ (constant). From (27) it is obvious that the model can be resolved through finding 4 values for c_t: $\bar{c}$, $\underline{\underline{c}}$, $\underline{c}$ and $\bar{\bar{c}}$ for $t = 0$, $t = 1$, $T \geq t > 1$ and $t = T + 1$ respectively, independent of the duration T. These values must be such that the ZLB is satisfied in $t \geq 1$ and $t \leq T$.

Let us conjecture that such a path is consistent with optimization. Then from (27) we have $\underline{\underline{c}} < \underline{c}$. Consumption is lower in period 1 than in $t = 2, 3, ..., T$ because $\lambda_{ZLB,0} = 0$ by definition. The zero lower bound constraint is satisfied with equality in $t = 2, 3, ..., T$ if

$$\beta\frac{u'(\bar{\bar{c}})\bar{\xi}(1-\phi) + u'(\underline{c})\underline{\xi}\phi}{u'(\underline{c})\underline{\xi}} = 1.$$

Moreover, in $t = 1$ we have

$$\beta\frac{u'(\bar{\bar{c}})\bar{\xi}(1-\phi) + u'(\underline{c})\underline{\xi}\phi}{u'(\underline{\underline{c}})\underline{\xi}} < \beta\frac{u'(\bar{\bar{c}})\bar{\xi}(1-\phi) + u'(\underline{c})\underline{\xi}\phi}{u'(\underline{c})\underline{\xi}} = 1,$$

which contradicts that $\lambda_{ZLB,1} < 0$.

[12]Notice that the assumption $\overline{B}_{-1} = 0$ made in Proposition 3 is simply for convenience. Assuming $\overline{B}_{-1} > 0$ would require to characterize a separate tax schedule for $t = 0$ following the argument of FMOS (2016). Since our focus is on the properties of the optimal allocation when the ZLB binds (after period 0) we simplify the analysis assuming zero initial debt.

The above result shows that when solving the model with $\gamma_c > 0$ we need to confront the fact that it is not possible to find solutions where the influence of the history is limited, so that the duration T is irrelevant for the allocation (as in the previous model with quasi-linear utility).[13] We next propose a numerical procedure based on the Parameterized Expectations Algorithm of den Haan and Marcet (1990) to solve the model taking into account explicitly that the profiles of the endogenous variables vary with T.

Solving the model with CRRA preferences Algorithm 2

- Step 1. Guess an initial value for $\lambda_s = \lambda_s^0$.
- Step 2. Formulate an approximation of the conditional expectations $E_t\left(\frac{u_{c,t+1}\xi_{t+1}}{\pi_{t+1}}\right) \approx \Phi(\lambda_{ZLB,t})$ and $E_t\left(u_{c,t+1}\xi_{t+1}\pi_{t+1}\left(\pi_{t+1}-1\right)\right) \approx \Psi(\lambda_{ZLB,t})$ for the periods that the economy is in the LT.[14]

  Φ and Ψ are polynomials of the state variable $\lambda_{ZLB,t}$ when the ZLB binds.
- Step 3. Simulate a large number of samples for the preference and spending shocks under the Markov process defined previously. As T varies across samples the realized values of the multipliers and consumption should also vary. To obtain $(c_t, \pi_t, \lambda_{ZLB,t})$
  - if the ZLB binds in t use $u_{c,t} = \frac{\beta}{\underline{\xi}}\Phi(\lambda_{ZLB,t})$ for c_t, (12) to get π_t and (10) to get $\lambda_{ZLB,t}$.
  - in $t = T+1$: Set $\lambda_{ZLB,t} = 0$ and use (12) and (10) to obtain π_t and c_t.
  - in $t = 0$ or $t > T+1$ solve the static (history independent) problem setting $\lambda_{ZLB,t} = 0$ and $\pi_t = 1$.

  Use the simulation data to regress $\frac{u_{c,t+1}\xi_{t+1}}{\pi_{t+1}}$ and $u_{c,t+1}\xi_{t+1}\pi_{t+1}\left(\pi_{t+1}-1\right)$ on the state variable $\lambda_{ZLB,t}$ (on the level, square, cube and so on depending on the order of the approximation). Update the coefficients of the polynomials Φ and Ψ and iterate to convergence.
- Step 4. Formulate the expected present discounted value of the government's surplus S_0. Update the value of λ_s^0 and iterate to convergence.

3.3 Summary

A key condition to solve dynamic models using the Bellman equation is that only past variables influence the set of feasible current actions. This condition is obviously violated in all the models studied in this

[13]Since we assumed $\theta = \infty$, we may have given to the reader the impression that the above result is not general enough to allow for positive inflation. It is; under a constant $\lambda_{ZLB,t}$, inflation is also constant and the above contradiction still applies.

[14]Notice that the state vector does not include the spending shocks, this property derives from the simple structure for uncertainty we have assumed. In an alternative setup where shocks to spending are not perfectly correlated with shocks to preferences, g_t needs to be included in the list of state variables. It is straightforward to extend the algorithms proposed to this case.

paper; it is also violated in the complete markets optimal policy problem without the ZLB constraint. In all of these cases the presence of future expected variables (in either (6) or (8)) implies that the whole history of past shocks may matter for the allocation (see for example Marcet and Marimon (2009) and the references therein).

In the standard model of complete markets (without the ZLB constraint) this problem is practically not felt. The history of shocks turns out not to matter for the allocation and, in this case, one basically needs to solve a sequence of 'static' first order conditions to recover the endogenous variables. In contrast, when markets are incomplete (for example AMSS, SGU, FMOS (2015, 2016) among many others), then (6) is no longer sufficient for the equilibrium. Rather, the entire (infinite) sequence of intertemporal budget constraints needs to be included in the program and the associated multipliers are not constant through time.[15] As is well known, the history of shocks then influences the optimal policies chosen by the government.

The previous subsections draw this analogy for constraint (8). We have seen that under quasi-linear preferences, $\lambda_{ZLB,t}$ is constant through time and the optimal policy program admits a simplistic solution that can be summarized in a few values for taxes and s_t, independent of the duration of the shocks T. Algorithm 1 is very similar to the numerical procedures used by FMOS (2014, 2015) to solve models of optimal policy under complete markets, without the ZLB constraint. In contrast, in the case where $\gamma_c > 0$, $\lambda_{ZLB,t}$ is not constant over time and as we have seen, the optimal allocation is no longer independent of the realization of T. Algorithm 2 adds two further steps to the computation of the equilibrium (to approximate the conditional expectation terms with the polynomials Φ and Ψ). It is very similar to the numerical procedures used by FMOS (2014, 2015, 2016) to solve models of optimal fiscal policy under incomplete financial markets.

Finally, a key finding that emerges from the analysis of this section is that the presence of the future expected variables in equation (8) bestows an important influence of the history of shocks to the allocation only when consumption enters significantly in the ZLB equation. In other words, the presence of future inflation alone is not sufficient for the history of shocks to exert a significant effect. This finding appears new to the literature.

4 Optimal Fiscal Policy and Debt Management

4.1 Some analytical results

In this section we describe our findings for optimal debt management. Before proceeding to the numerical experiments we provide insights on the optimal debt management strategy, assuming that $\gamma_c = 0$ and $\phi = 0$. Though this parameterization of the model is far from the calibrated values we will later adopt, it enables us to derive analytical expressions that show the forces that determine the optimal portfolios. Recall also that when $\phi = 0$ debt management only needs to be determined in period 0. Therefore, in

[15] For example, under incomplete markets and short term debt λ_s follows a risk adjusted random walk process in AMSS. In models with long term debt this multiplier could display more complex dynamics depending on the characteristics of long debt (see for example FMOS (2015, 2016)).

this section we study the optimal policy at $t = 0$.

Under the assumptions employed here, we can easily show that $\pi_2 = \underline{\pi} = \beta\frac{\overline{\xi}}{\underline{\xi}}$. Moreover, the long bond prices in period 1 are given by: $q_1^{N-1}\left(x^0, (\underline{\xi}, \underline{g})\right) = \frac{\beta^{N-1}}{\underline{\pi}}\frac{\overline{\xi}}{\underline{\xi}} = \beta^{N-2}$, if the economy falls in the LT, and $q_1^{N-1}\left(x^0, (\overline{\xi}, \overline{g})\right) = \beta^{N-1}$, if the preference shock does not occur in $t = 1$.

We can derive the present discounted value of the government's surplus in each of the two possible states in $t = 1$ as:

$$(28) \qquad S_1\left(x^0, x_1\right) = \begin{cases} \frac{\overline{s}}{1-\beta} & x_1 = (\overline{\xi}, \overline{g}) \\ \\ \underline{\underline{s}} + \beta\frac{\overline{\xi}}{\underline{\xi}}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right) & x_1 = (\underline{\xi}, \underline{g}) \end{cases}$$

From (14) the optimal portfolio is given by:

$$(29) \qquad b_0^1 = \frac{\beta^{N-2}S_1\left(x^0, (\overline{\xi}, \overline{g})\right) - \beta^{N-1}S_1\left(x^0, (\underline{\xi}, \underline{g})\right)}{\beta^{N-2}(1-\beta)} \quad \text{and} \quad b_0^N = \frac{-S_1\left(x^0, (\overline{\xi}, \overline{g})\right) + S_1\left(x^0, (\underline{\xi}, \underline{g})\right)}{\beta^{N-2}(1-\beta)},$$

showing that the signs of the bond positions (b_0^1, b_0^N) hinge crucially on the relative magnitudes of $S_1\left(x^0, (\overline{\xi}, \overline{g})\right)$ and $S_1\left(x^0, (\underline{\xi}, \underline{g})\right)$. As can be easily seen from (29), whenever the present value of the surplus increases during LT episodes, the government will issue long bonds and the standard results of the debt management literature will apply in our model. However, the opposite holds if during LTs fiscal revenues decrease and the intertemporal budget goes into deficit.

To gain insights on how S_1 is impacted by the preference shock note that combining (24) and (28) yields

$$(30) \qquad S_1\left(x^0, (\underline{\xi}, \underline{g})\right) = \overline{s} + \beta\frac{\overline{\xi}}{\underline{\xi}}\left(\frac{\overline{s}}{1-\beta}\right) - (\underline{g} - \overline{g}).$$

Let us use (30) to study the following scenarios:

1. Assume that $\overline{B}_{-1} = 0$ and $\underline{g} = \overline{g}$. It then holds that $\overline{s} = 0$.[16] (29) gives us $b_0^1 = b_0^N = 0$.

   Why is it optimal to not issue any long or short debt at all? Suppose that the government had chosen $b_0^N > 0$ and $b_1^0 < 0$. Since $q_1^{N-1}\left(x^0, (\underline{\xi}, \underline{g})\right) = \beta^{N-2} > q_1^{N-1}x\left(^0, (\overline{\xi}, \overline{g})\right) = \beta^{N-1}$, the real payout of government debt in $t = 1$ would increase in state $(\overline{\xi}, \overline{g})$. Then the government would have to commit to increase taxes (permanently) to finance the debt. The opposite holds $b_0^N < 0$ and $b_0^1 > 0$. Portfolios different from $(0, 0)$ give rise to unnecessary fluctuations in the tax schedule.

2. Assume that $\overline{B}_{-1} = 0$ and $\underline{g} > \overline{g}$. Assume further that the shock to spending **is** sufficiently small

[16] The present value of the surplus at date 0 is $\overline{s} + \beta\left[\omega S_1\left(x^0, (\underline{\xi}, \underline{g})\right) + (1-\omega)S_1\left(x^0, (\overline{\xi}, \overline{g})\right)\right]$. It must be that $\overline{s} = 0$ to satisfy the intertemporal budget at date zero. The value of λ_s will be pinned down accordingly.

so that $\overline{s} \approx 0$. (29) gives:[17]

$$b_0^1 \approx \frac{\beta(\underline{g} - \overline{g})}{1-\beta} > 0 \quad \text{and} \quad b_0^N \approx -\frac{(\underline{g} - \overline{g})}{\beta^{N-2}(1-\beta)} < 0. \tag{31}$$

Therefore, the government optimally issues short term debt (financed through long term savings) in period 0.

Since bond prices rise during LTs, taxes will be smoother if the value of debt depreciates when the government's budget goes into deficit. By holding long private bonds and issuing short term debt, the government can benefit from the capital gain.

3. Finally, consider the case where $\overline{B}_{-1} > 0$ and $\underline{g} = \overline{g}$. We now have $\overline{s} > 0$ to finance the initial debt level. Moreover, $S_1\left(x^0, (\underline{\xi}, \underline{g})\right) > S_1\left(x^0, (\overline{\xi}, \overline{g})\right)$. By (29), we obtain $b_0^N > 0$.

   When the initial debt level is positive, the government commits to a sequence of positive surpluses in future periods to repay the debt. Since $\frac{\overline{\xi}}{\underline{\xi}} > 1$, $\overline{s} + \beta\frac{\overline{\xi}}{\underline{\xi}}\left(\frac{\overline{s}}{1-\beta}\right)$ exceeds $\frac{\overline{s}}{1-\beta}$ and therefore, during the LT, the present value of the future surpluses increases. Unless $b_0^N > 0$ the government would have to reduce taxes following the shock in preferences, and increase taxes if the shock does not occur.

To summarize, the optimal portfolio is determined through the balance of the following forces. When the preference shock hits, long bond prices increase. Governments that issue long-term debt in period 0 suffer a capital loss. Whether or not it is optimal to incur this loss depends on the behavior of the present value of the surplus. When $S_1\left(x^0, (\underline{\xi}, \underline{g})\right) < S_1\left(x^0, (\overline{\xi}, \overline{g})\right)$ (as in the case where spending increases and $\overline{B}_{-1} = 0$) the optimal debt management strategy is to hold long term savings. However, when debt is positive there is a second force that makes issuing long debt optimal. Due to the 'discounting impact' of the fall in interest rates, we may have $S_1\left(x^0, (\underline{\xi}, \underline{g})\right) > S_1\left(x^0, (\overline{\xi}, \overline{g})\right)$; to stabilize taxes the government must issue long debt.

4.2 Calibration

In order to solve the model numerically we need to give values to the structural parameters. Each period represents a quarter, and therefore we set $\beta = 0.99$. We set $\gamma_h = 1$ so that the Frisch elasticity of labor supply is equal to 1, and choose ψ so that in the deterministic steady state, the household spends 20 percent of its unitary time endowment in market work. Notice that the value of ψ depends on the initial debt level assumed. We will study cases where $\overline{B}_{-1} = 0$ as in ABN, but also cases where the initial debt is positive. In each case we adjust the value of ψ to hit the hours target. Finally, we treat $\gamma_c = 1$ as our benchmark. In Section 5 we will experiment with higher values of the risk aversion coefficient.

To calibrate η and θ, we follow SGU (2004) and choose values of -6 and 17.5, respectively. In the deterministic steady state, the level of public expenditure is set to 20% of output and therefore we have

[17] Given the previous derivations a shock in spending in period 1 will give us $\overline{s} > 0$. From section 3.2.1 we know that the equilibrium value λ_s will adjust to have $S_0 = \overline{B}_{-1} = 0$. The deficit during the liquidity needs to be compensated with a (small) positive surplus in other periods.

$\overline{g} = 0.04$. In the numerical experiments we will consider values for $\underline{g} \in \{1, 1.04, 1.08\}\overline{g}$.[18]

To calibrate the preference shock process, we proceed as follows: First, we set, $\phi = 0.8$. This gives us an average duration of LT episodes equal to 5 quarters, well within the range of values considered in the literature. Second, we normalize $\overline{\xi} = 1$. Third, to calibrate $\underline{\xi}$, we take the following steps: For each version of the model that we solve, we compute the optimal allocation in an economy without preference shocks. Denote the short bond price that derives from this (history independent) allocation by $q^{HI} = \beta \frac{(1-\phi)u_c^L + \phi u_c^H}{u_c^H}$ where u_c^L (u_c^H) denotes the marginal utility of consumption when spending is at $\overline{g}$ ($\underline{g}$). We then find the value of $\underline{\xi}$ such that $\widetilde{q} \equiv \beta \frac{(1-\phi)u_c^L + \phi u_c^H \underline{\xi}}{\underline{\xi} u_c^H} = 1 + \epsilon$, where $\epsilon = 0.0011$.[19]

Finally, we set $\omega = 0.5$ in our benchmark calibration. In Section 6 we will show that our results are robust to alternative calibrations for ω. Table 1 summarizes the parameter values discussed above.

4.3 Behavior of Endogenous Variables

Figure 1 traces the behavior of consumption, taxes, inflation and λ_{ZLB} after the preference shock in period 1. The top left panel plots consumption during the LT. The middle top panel plots consumption in $T + 1$. The top right and bottom left panels plot taxes (during the LT and in $T + 1$) and the middle bottom panel traces the behavior of inflation. Each graph shows simulations over 15 model periods. Moreover, each plot shows three curves. The solid curve is the case where $\underline{g} = \overline{g}$, the dashed line, $\underline{g} = 1.04\overline{g}$ and the crossed line, $\underline{g} = 1.08\overline{g}$.

Notice that consumption levels, both during the LT episode but also in period $T + 1$, converge (monotonically) to constant values. The planner decreases consumption below the steady state when the shocks hit in $t = 1$ and commits to gradually increase consumption over time if the LT persists.[20] The ZLB constraint is satisfied along the optimal path because the planner also increases the consumption level at the 'exit' (period $T + 1$, middle panel). The responses are similar to the analogous objects in EW (2003, 2006).

Turning to the top right and bottom left panels we see that the adjustment of consumption to the shocks, is explained by the adjustment in taxes. During the LT, taxes increase initially and then drop gradually as the episode persists. In the absence of any shocks to spending (solid line) taxes return to their long run value after roughly 10 periods, whereas when spending levels rise during the LT, the tax rates, following their initial rise, will drop below their long run levels. This model property is consistent with equation (27). Finally, the optimal policy at the 'exit', is to reduce the tax rate, and thus engineer

[18]To pick these values, we followed FMOS (2013) and SGU. In these papers, the log of government spending has first order autocorrelation equal to 0.9 and the standard deviation of the shocks is 0.03. We assume that our spending process derives from the same long run distribution, however, the serial correlation now is 0.8. The conditional standard deviation is therefore 0.04. This gives the 1 and 2 standard deviations values above the mean close to 4 and 8 percent above $\overline{g}$.

[19]To clarify the above, note that for different parameterizations of the spending process, we need to assume different values for $\underline{\xi}$ to make the ZLB bind when $\gamma_c > 0$. Since spending shocks will lower bond prices, $\underline{\xi}$ needs to drop with the difference $\underline{g} - \overline{g}$. Moreover, as we have seen, the value of the preference shock is irrelevant if the ZLB constraint does not bind. This makes finding u_c^L and u_c^H sufficient to pin down $\widetilde{q}$.

Clearly, under $\gamma_c = 0$ the short bond price is always $\beta \frac{(1-\phi)+\phi\underline{\xi}}{\underline{\xi}}$ and therefore, the parameterization of spending is irrelevant for $\underline{\xi}$. We find $\underline{\xi} = 0.9469$ in this case. When $\gamma_c = 1$ we have $\underline{\xi} = 0.9430$ (0.9391) when $\underline{g} = 1.04\overline{g}$ ($\underline{g} = 1.08\overline{g}$).

[20]To construct Figure 1, we assumed that the initial debt level is equal to zero. The long run steady state of the model (i.e., after uncertainty is removed) gives the same consumption level as in period 0.

the consumption boom.

The middle-bottom panel shows the response of the (annualized) inflation rate in percentage points. Notice that inflation displays very little volatility under the benchmark calibration of the model. This property can be traced to the (high) value of θ we have assumed in the calibration. As is well known, models of optimal policy that feature a Phillips curve consistent with the US macro data, require large price adjustment costs and as a result, predict larger welfare losses from inflation than from tax variability (see for example SGU and FMOS (2013)). This also holds in our context when $\gamma_c > 0$.[21] Our model predicts that in response to the binding ZLB constraint, most of the policy adjustment comes from taxes and consumption, inflation exerts a relatively minor role. In Section 5 we will introduce changes to the calibration of the model, to increase the importance of inflation.

Furthermore, notice that the bottom-middle graph shows that the response of inflation to the preference shock is strikingly similar between the version of the model without spending shocks and the versions with positive spending shocks. Indeed, as we have seen, government spending shocks will not result in changes in the level of inflation under complete markets. Inflation only responds to $\lambda_{ZLB,t}$, and the behavior of this variable is very similar across models (e.g. bottom right panel). As described previously, we adjust the value $\underline{\xi}$ downwards when we introduce spending shocks to the model. In all cases the ZLB constraint is 'just binding', this explains the 'similarity' in the values of $\lambda_{ZLB,t}$ across the three model versions.

The findings of this section demonstrate the properties of key endogenous variables that will be useful to understand the optimal debt management policy of the government. The qualitative features of the responses of consumption, taxes and inflation to the preference and spending shocks we studied in Figure 1, will be preserved in all the parameterizations of the model that we will subsequently assume.

4.4 Optimal Portfolios

We now show our main results for debt management. In Table 2, we consider the case of zero initial debt. The table is divided in two panels. The top panel shows the key moments of debt management under $\gamma_c = 0$, and the bottom panel under $\gamma_c = 1$. Moreover, each panel reports separately portfolios when $\underline{g} = \overline{g}$, $\underline{g} = 1.04\overline{g}$ and $\underline{g} = 1.08\overline{g}$. Columns 2-3 show the optimal mixture between short and long bonds expressed as percentages of steady state GDP; Columns 4-5 show the price of long term debt under $x_{t+1} = (\underline{\xi}, \underline{g})$ (Column 4) and $x_{t+1} = (\overline{\xi}, \overline{g})$ (Column 5). Finally, Columns 6-7 report the present discounted value of the surplus.

Notice that in the case $\gamma_c = 1$, we report the above quantities over 4 different periods. At $t = 0$ we report the optimal portfolio that is determined prior to the realization of the preference shock. We also report the prices and the surpluses that will prevail in $t = 1$, depending on the state of the economy in that period. The remaining rows report the portfolios in $t = 1, 4, 9$ and the prices and surpluses in $t = 2, 5, 10$ respectively. Our goal is to identify changes in the optimal debt management policy during LT episodes. Under $\gamma_c = 0$ the portfolios do not change after $t = 1$, since the bond prices and the

[21] Recall that when we assume quasi-linear preferences, the adjustment of inflation to the preference shocks is independent of θ. In this case we obtain an inflation rate 10 times as high as under $\gamma_c = 1$.

surpluses remain constant. Therefore, we only report model quantities at $t = 0, 1$.

Model without spending shocks. Consider first the case $g_t = \overline{g}$. The values of short and long bond positions are close to zero. This holds for all periods reported, both under quasi-linear utility and in the case where $\gamma_c = 1$.

These predictions are consistent with our previous theoretical findings. As reported in the last two columns of the table, it holds that $S\left(x^t, (\underline{\xi}, \underline{g})\right) \approx S\left(x^t, (\overline{\xi}, \overline{g})\right) \approx 0$ in all periods. It is trivial to show that the solution to (14) will give us long and short term debt close to zero.

A well known prediction of the debt management literature is that governments take positions in the bond market that are several multiples of GDP. This is shown in several papers (e.g. ABN, FMS (2010), Nosbusch (2008)) that study optimal portfolios assuming that government debt is initially zero. The intuition is that government spending shocks change significantly the present value of the surplus, however, they do not impart a large effect on bond prices. In the case of preference shocks, however, the opposite holds. Bond prices show considerable variability; the intertemporal value of the surplus, does not.

Model with spending shocks. We now turn to the case where $\underline{g} > \overline{g}$. Table 2 shows that (b_t^1, b_t^N) equals (76.7%, -82.4%) in $t = 0$ when $\gamma_c = 1$ and spending increases by 4 percent during the LT. In $t = 1, 4, 9$ we have (75.8%, -81.5%), (74.5%, -80.3%) and (74.0%, -79.8%) respectively. As explained above, governments want to issue short term debt when $corr(q_{t+1}^{N-1}, S_{t+1}) < 0$ (where *corr* denotes the correlation coefficient). The numbers reported in Columns 4-7 show that this property holds in the model when we introduce spending shocks. Bond prices increase in periods when $x_t = (\underline{\xi}, \underline{g})$ (to roughly 0.956) and drop (to around 0.914) when the economy escapes from the LT. Due to the rise in public spending the government runs a deficit during the LT.

Notice further that when we increase the variance of the spending shocks, we find even larger positions of short debt and long savings. When $\underline{g}$ is 8% higher than in the steady state the optimal portfolios are (153.4%, -164.8%), (151.6%, -163.0%), (149.2%, -160.5%) and (148.3%, -159.6%) for $t = 0, 1, 4, 9$ respectively. This may seem surprising if we expect that the larger variability of spending shocks, should bring us closer to the ABN benchmark, where long debt is optimal. The reason this does not occur in our simulations is that while preference shocks generate large swings in asset prices, spending shocks do not.[22] Instead, spending shocks impact considerably the intertemporal budget of the government, and the larger they are the larger the deficit during the LT. Under zero initial debt, in the presence of both preference and spending shocks, governments will always want to issue short debt.

Finally, the patterns we documented above are preserved in the quasi-linear utility case. In this case we observe that the size of the positions that the government takes in the bond market is smaller than under $\gamma_c = 1$ and across all specifications of positive spending shocks. As we will later demonstrate quasi-linear preferences is indeed a 'special case' in terms of the quantitative impact of the shocks; assuming $\gamma_c > 0$ always gives us very similar results to the $\gamma_c = 1$ model. Recall that under quasi-linear

[22]This property would obtain even if we kept $\underline{\xi}$ constant across simulations. The property that preference shocks generate large variability in interest rates is discussed (for example) in Ravenna and Seppälä (2006).

preferences, taxes are not used to influence the behavior of consumption, and inflation is 10 times more volatile than in the benchmark. These differences explain the findings documented in Table 2.

Portfolio rebalancing (under zero initial debt). A key property that emerges clearly from Table 2 is that the government does not change considerably its debt management strategy over time. This is not surprising under quasi-linear utility where the value of the surplus and the bond prices are constant after $t = 1$, but as the table shows, the property holds also when $\gamma_c = 1$. The results reported in Table 2 suggest that substantial revisions of the debt management strategy are not needed to deal with further shocks during LTs when we assume $\overline{B}_{-1} = 0$.

Recall that a well known prediction of macroeconomic models of debt management is that the optimal portfolio is constant over time. This is the case in standard models in which allocations are history independent. However, as we have seen, history independence does not hold under the ZLB, this leads to considerable changes in consumption, and taxes over time under positive risk aversion. As Table 2 shows, these changes do not impart a large effect on bond prices and on the present value of the surplus. This property explains the *limited portfolio rebalancing* we find in this section.

5 Sensitivity and Extensions

5.1 Positive Government Debt

Our benchmark model (following most papers in the optimal debt management literature) assumed that initial government debt is equal to zero.[23] We now consider the more empirically plausible scenario of positive government debt. In Table 3, we assume that $\overline{B}_{-1}$ is 60% of GDP at the annual horizon (240% at the quarterly horizon).

Notice first that with positive initial government debt, the optimal debt management strategy changes dramatically relative to previous findings. We now find that the government prefers to issue long-term debt. For example, when $\gamma_c = 1$ and $\underline{g} = \overline{g}$, the optimal portfolio is (-29.3%, 304.6%) at $t = 0$. Moreover, when $\underline{g}$ is 4% above the steady-state value, we have (b_0^1, b_0^N)=(46.4%, 223.5%).

Our previous remarks can explain the changes in the optimal debt management strategy. When initial debt is positive, the government must commit to run positive surpluses on average in order to redeem its initial liability. When the preference shock hits the economy in period 1, the value of S increases when government spending does not rise considerably, due to the discounting effect of interest rates on the future primary surpluses. As the table shows, in all cases we have $S_{t+1}\left(x^t, (\underline{\xi}, \underline{g})\right) > S_{t+1}\left(x^t, (\overline{\xi}, \overline{g})\right)$, and hence $corr(q_{t+1}, S_{t+1}) > 0$. As in ABN, the government prefers to 'finance long'.

The relative positions of short and long bonds hinge on the magnitude of the spending shocks. Larger shocks lead to a more balanced portfolio since they lower the impact of the preference shocks on $S_{t+1}\left(x^t, (\underline{\xi}, \underline{g})\right)$ and therefore the covariance of q_{t+1} and S_{t+1}. When spending shocks are 8 percent above the steady state, for example, the optimal portfolio becomes $(122.0\%, 142.4\%)$ in $t = 0$. The share of short-term debt in the total market value of government debt is now nearly 50 percent.

[23] The positive debt case is analyzed in Nosbusch (2008). He finds that the optimal portfolio continues to feature only long term debt.

Notice that, hypothetically, we can obtain portfolios under positive initial debt, whereby the government holds long private bonds; however, this in practice requires to assume incredible increases in government spending (so as to reverse the sign of $corr(q_{t+1}, S_{t+1})$) and as explained previously, assuming an 8 percent increase in g, is already two standard deviations above the initial value. Moreover, were we to increase spending by 10 or 12 percent above the steady state, we would continue to find that the government optimally issues some long term debt. We thus conclude that reasonably calibrated values of $\underline{g}$ and $\overline{B}_{-1}$ yield the prediction that issuing long term debt is optimal.

Portfolio rebalancing (under positive initial debt). Consider the evolution of the optimal portfolios with t. As Columns 2 and 3 of the table show, the government revises the optimal debt management strategy over time. For example, when $\gamma_c = 1$ and in the case of no spending shocks, the portfolio changes from (-29.3%, 304.6%) in $t = 0$, to (-20.3%, 295.1%) in $t = 1$ and (-3.2%, 277.1%) in $t = 9$. The government reduces its short term savings by roughly 26 percent of GDP within 10 quarters and also reduces its long term debt issuance by roughly 27 percent of GDP. These numbers are considerable, however, the qualitative features of the optimal policy are stable. We do not (for example) observe 'reversals' in the portfolio, whereby the government initially issues only long bonds and after a few periods issues mostly short term debt. The active management of the maturity structure during the LT when $\overline{B}_{-1} > 0$ can be explained from the behavior of consumption. As we have seen, consumption in $T+1$ gradually increases during the LT, above the long run value. As a result, the present discounted value $S_{t+1}\left(x^t, (\overline{\xi}, \overline{g})\right)$ increases over time, since the government runs positive surpluses in the long run to balance the budget. These changes explain the findings of this subsection. [24]

5.2 Risk Aversion

How sensitive are the above findings to higher values of the risk aversion coefficient? We have seen that moving from $\gamma_c = 0$ to $\gamma_c = 1$ changes quantitatively the impact of the shocks on the optimal debt management strategy. However, quasi-linear preferences were used here mainly to derive analytical results for the optimal portfolios, and therefore the quantitative impact of varying γ_c from 0 to 1 offers little guidance to policy. A more instructive exercise is to vary the risk aversion coefficient between 1 and 5 (since this is widely viewed as a reasonable range for the parameter) in order to gauge the effect that preferences exert on the optimal portfolio. We consider here the case where $\gamma_c = 5$.

The results are reported in Table 4. The top panel corresponds to zero initial debt and the bottom panel to positive debt. For brevity, we only show the portfolios in the table. Notice when $\overline{B}_{-1} = 0$, risk aversion does not influence our results. We continue to obtain that short term debt is optimal and the size of the bond positions are very similar to the case $\gamma_c = 1$. However, under the case of positive initial debt, the optimal portfolio varies somewhat with the risk aversion coefficient; at higher values for γ_c we observe that long term debt is lowered. For example, consider the case $\overline{g} = \underline{g}$. The value of long bonds

[24] Let $\overline{u}_c$ denote the marginal utility in $t > T + 2$. We have $S_{T+1}\left(x^T, (\overline{\xi}, \overline{g})\right) = s_{T+1} + \beta \frac{\overline{u}_c}{u_{c,T+1}} \frac{\overline{s}}{1-\beta}$, which clearly rises with the level of c_{T+1} when $\overline{s} > 0$. From Table 3 we see that the changes in $S_{t+1}\left(x^t, (\overline{\xi}, \overline{g})\right)$ are not large, yet the optimal portfolio displays considerable sensitivity to the changes. This finding is similar to FMS (2010) who document that debt management under complete markets features excess sensitivity to small changes in the underlying economic environment. Here, the changes in the properties of consumption derive from the history dependence property of the optimal allocation.

equaled 304.6% in the $\gamma_c = 1$ economy, it now equals 261.8%. Long term debt is reduced by roughly 40 percentage points of GDP.

What explains this property? We find that the higher degree of risk aversion has almost no effect on long bond prices. This in turn is explained by the fact that long term interest rates in the model are approximately equal to the product of expected future short term rates. Again, since short rates are expected to equal 0 for many periods, the value of the risk aversion coefficient does not exert an important influence. Instead, the most noticeable impact of γ_c is on the term $S_{t+1}\left(x^t, (\overline{\xi}, \overline{g})\right)$. Since the government must to commit to a positive surplus value in the long run, the optimal consumption level at the 'exit' exerts an influence on the present value S. When we increase risk aversion the government commits to a smaller increase in c_{T+1} relative to the long run value and (we find that) $u_{c,T+1}$ increases by less than when $\gamma_c = 1$. The discounting impact on $S_{T+1}\left(x^T, (\overline{\xi}, \overline{g})\right) = s_{T+1} + \beta \frac{\overline{u}_c}{u_{c,T+1}} \frac{\overline{s}}{1-\beta}$ is more moderate.

5.3 Price Stickiness

We now show that our results are robust towards lowering the value of θ. To build intuition we first provide an analytical result using the examples of Section 4.1.

Result 1: *Assume $\gamma_c = 0$ and $\phi = 0$, $\overline{B}_{-1} = 0$ and $\underline{g} > \overline{g}$. i) For any $\theta > 0$ the optimal portfolio (b_0^1, b_0^N) is given by (31). ii) Under $\theta = 0$ the optimal portfolio is indeterminate and one of the solutions is (31).*

Proof. See appendix.[25]

The above result should not be surprising since from previous derivations we know that under $\gamma_c = 0$ inflation and the present value of the surplus, are independent of θ, when $\theta > 0$. Under $\theta = 0$ the ZLB constraint does not 'really bind' in the sense that the optimal allocation is not impacted by the preference shocks and $\lambda_{ZLB,t} = 0$. In this case the government can set inflation in $t = 1$ at an even higher levels than $\beta \frac{1}{\underline{\xi}}$ and choose, for each level of inflation, the portfolio that is consistent with constant taxes for all t. Since inflation is indeterminate, so is the optimal portfolio.

The above properties hold approximately in the case where $\gamma_c > 0$; To discern the impact of lower price adjustments costs, we assume in Table 5, that $\theta = 5.8333$. This value is chosen as follows: SGU show that with $\theta = 17.5$ the linearized version of equation (3) gives a new-Keynesian Phillips curve fits the US macro data and is consistent with on average 3 quarters of sticky prices (i.e. when the price adjustment friction is as in Calvo (1983)). When we divide θ by three, we target one quarter of price stickiness, this number is consistent with recent micro studies on the frequency of price adjustments (e.g. Bills and Klenow (2004)). Notice that the optimal portfolios in the zero initial debt case, shown in Table 5 do not change considerably relative to the benchmark. Lowering price adjustment costs has only minor effects on bond prices (since prices are pinned down by future rates, which are zero anyway) and do not exert a significant influence on the values of S. As we saw previously, government spending shocks and the initial debt level are the most important factors that influence S, the tax adjustments we

[25] As discussed above, when the planner sets same inflation rate so that the ZLB constraint is satisfied, for any $\theta > 0$. This makes bond prices not depend on the price adjustment costs. Moreover, when $\phi = 0$ inflation costs cancel out in the calculation of the surplus, and hence, the optimal portfolio is essentially independent of θ.

documented in Figure 1 have only minor effects on the intertemporal surplus. This explains the findings of this section.

In the online appendix we illustrate that these properties hold also for the positive debt case. We conclude that θ is not a crucial parameter for our results.

5.4 Persistent LTs

In the baseline calibration of the model, we chose $\phi = 0.8$ so that LT episodes last on average for 5 quarters. We now set $\phi = 0.9$, targeting an average duration of 10 quarters. The results are shown in Table 6. For brevity, we consider the case of zero initial government debt and report the values of the portfolios in the table. The case of positive initial debt is shown in the online appendix.

The key property that emerges from the table is that when we increase ϕ, the government fans out its positions to hedge against spending and preference shocks. For example, when $\underline{g} = 1.04\overline{g}$ we find that optimal short term debt in $t = 0$ increases by roughly 25 percentage points of GDP when $\gamma_c = 1$. The rise is close to 50 percentage points when $\underline{g} = 1.08\overline{g}$. We obtain similar changes when we increase the risk aversion coefficient (see the bottom panel in Table 6).

These findings can be explained as follows. First, long bond prices increase more following the preference shock, owing to the fact that the persistence of the shock increases. Since the variability of interest rates across states increases, this impact tends to reduce the size of the bond positions that the government must take in the market to insure. Second, under positive spending shocks, the term $S_{t+1}\left(x_t, (\underline{\xi}, \underline{g})\right)$ during the LT episode becomes more negative, since in our parameterization, spending shocks have the same persistence as preference shocks. This effect tends to increase the magnitude of the difference between $S_{t+1}\left(x_t, (\underline{\xi}, \underline{g})\right)$ and $S_{t+1}\left(x_t, (\overline{\xi}, \overline{g})\right)$ and hence, increase the absolute values of the positions. The numbers reported in Table 6 show that the second effect overpowers the first. As a result, the optimal portfolio features a higher issuance of short-term debt and a more negative position in the long bonds.

5.5 LTs as rare events

In the baseline calibration of the model, we had set $\omega = 0.5$. Though we are not aware of any empirical evidence that supports this choice, the value was chosen so that the exercise focuses on cases where the LT episode is very likely. The reader is reminded that the debt management literature has assumed $\omega = 0$, and therefore completely abstracted from shocks to preferences as a source of variability in the economy. At the opposite end, recent papers that study LTs assume $\omega = 1$ (e.g. CER, EW (2006)) to focus on the effect of the shocks to preferences.

We now show that the value of ω is not at all important for our results. We set $\omega = 0.005$ so that at the beginning of the planning horizon, the probability that the economy falls into a LT is 0.5% as opposed to 50%. Again, for the sake of brevity, we report results in Table 7 only for the case $\overline{B}_{-1} = 0$. Notice that the optimal portfolios are almost identical across the two calibrations of ω. For example, consider the case $\gamma_c = 1$ in $t = 0$. We have $(b_t^1, b_N) = (79.8\%, -85.8\%)$ under $\omega = 0.5$ and $(b_t^1, b_N) = (79.9\%, -87.5\%)$ under $\omega = 0.005$. In $t = 9$ we have $(79.6\%, -85.87\%)$ and $(79.7\%, -87.3\%)$

respectively. The findings are similar when $\gamma_c = 0$ or 5. The results for the positive initial debt case were also found to be very similar. For the sake of brevity we leave them outside the tables.

5.6 Summary and Discussion

Our analysis has two important and related goals. First we ask whether the complete market approach to debt management is suitable to offer offers a robust/useful benchmark to debt managers during LT episodes. Second, we seek to assess whether the recent shortening of the maturity structure in the hands of the private sector witnessed in developed countries, brings us closer to or takes us further from the optimal strategy identified in the model.

We have shown that the optimal policy is robust across a number of different specifications of the model. In particular, the degree of risk aversion, the degree of price stickiness, and the likelihood of falling in a LT are of little relevance for the portfolios. The optimal portfolios are more sensitive to the duration of LTs in absolute levels, but the qualitative features of the solution are preserved. The reader should note that the above elements (perhaps with the exception of price stickiness) are also the ones that are more difficult to be accurately pinned down from the data.[26] LTs have been quite rare historically, though many economists believe that they will be recurrent events in the future. Moreover, evidence on the duration of the episodes is hard to come by. Finally, any value for the risk aversion coefficient within the interval [1,5] is considered plausible in the literature. Had our results varied considerably across these parameters, there would be little we could say about optimal debt management. A positive note therefore is that our model's predictions are robust across a number of parameters that are not easy to measure.

In contrast, we found that government portfolios vary considerably with the initial debt level, and the level of spending in the LT. These elements are determined as initial conditions and government policies. Debt managers surely know the initial value of debt, and may have a good view of what spending policies are likely to be pursued by the government at the ZLB.[27]

We emphasize that our goal here is not to find an exact value for the optimal portfolio and suggest it to debt managers, or (even) come up with a formula that translates model numbers into portfolios.[28] We are obviously aware of the numerous simplifications our modelling approach brings to the very complex interactions between monetary and fiscal policies and financial markets. The Ramsey outcome derives from a frictionless market, where the costs of debt issuance and portfolio rebalancing are zero. Still, were we to posit that these costs are positive, the complete market approach to debt management remains an instructive tool. Consider the previous cases where the government issues only long term debt (e.g., when $\overline{B}_{-1} > 0$). As in Lustig et al (2008), assume that to lend the short bond to the private sector is very costly so that $b_t^1 \geq 0$ is a further constraint in the Ramsey program.[29] Then, the optimal policy

[26]If price adjustment costs are a proxy for the government's ability to generate positive inflation during the LT, it is clear that also this parameter is debatable.

[27]Had we modelled 'optimal spending' explicitly, we would obtain values of $\underline{g}$ very close to $\overline{g}$ (see for example Nakata (2015)).

[28]In practice it even formula (14) maybe of limited use since the discounted value S is difficult to estimate. See the considerable literature on the fiscal theory of the price level.

[29]This for example, can be the case if holding negative amounts of some maturity exposes the government to the risk

will be to issue only long debt and set $b_t^1 = 0$ for all t. In this case, markets are incomplete, and yet taxes are much smoother than if $b_t^1 > 0$. In other words, if in the frictionless market, it is optimal to issue only long debt, it remains optimal when we account for financial frictions as previous papers, in the macro debt management literature, have.

Whether these conclusions also generalize to the more recent literature on 'Quantitative Easing' remains to be explored. As discussed in the 'Introduction' this literature builds models with segmented markets, in which groups of agents have preferences over different maturities. In these models governments may not want to issue only long or only short debt, however, if the 'tax smoothing' objective of debt management, studied here, is properly accounted for, then governments will face a tradeoff between satisfying the demand for a particular maturity and using the maturity to smooth tax distortions over time. We believe that this tradeoff is interesting and merits to be studied separately in future work.

6 Welfare Analysis

How large is the welfare gain from completing the market when the economy is in the LT? In this section we compare welfare outcomes between our benchmark model and a model where the government has to balance the budget in every period (autarky). A well known result from the previous literature, is that the welfare gains from debt management are moderate, even when comparing with the autarky outcome (see for example Nosbusch (2008)).[30] We investigate whether this property also holds in our model where debt is optimal during the LT.

As discussed previously, when governments can issue debt in different maturities, distortions are smoothed over time. Moreover, during the LT tax adjustments accompany changes in inflation to satisfy the ZLB constraint. The government increases taxes during the trap and lowers them when the economy exits from the trap, to engineer a recession and an expected recovery that increases the real interest rate. Under financial autarky such adjustments may not be feasible, and therefore a larger burden may fall on inflation to satisfy the ZLB constraint in equilibrium.

Solving the model under financial autarky To solve the optimal allocation under financial autarky we basically use the same program as previously. However, we now need to have the government balance its budget and therefore, we replace the intertemporal constraint (6) with the following equation

$$g_t = w_t h_t \left(1 + \frac{v_{h,t}}{u_{c,t} w_t}\right),$$

of private sector default (See Lustig et al (2008) and FMOS (2015)).

[30] AMSS compare the welfare outcomes under complete and incomplete markets (when the government can issue short debt). Nosbusch (2008) makes a comparison between the complete market and the financial autarky allocations. We follow the latter paper for two reasons: First, to characterize the upper bound on welfare gains that the economy may experience from debt issuance. Second, because recent research has shown that fiscal policy outcomes under incomplete markets, hinge crucially on the types of short or long bonds that governments issue. Recall that under complete markets, it is innocuous whether the government buys back its debt in every period and then reissues (as we have assumed). When markets are incomplete this is not the case (see FMOS (2015)). We leave to future work the task of characterizing welfare outcomes under incomplete markets.

which needs to hold for every t. In the appendix we state the program formally. We show that the first order conditions are the essentially the same as in equations (10) to (13), the only difference is that now the multiplier λ_s is not constant over time. This property should follow immediately from the previous discussion; through removing optimal debt management, we are in a world of incomplete financial markets; it is no longer sufficient to maximize the household's preferences subject to a single implementability constraint (6), so that the excess burden of taxation, λ_s, is constant over time.[31] A numerical procedure similar to Algorithm 2 (Steps 2-3) can be used to solve the model under autarky (see appendix).

Welfare criterion To compute the welfare effects we adopt the following criterion:

$$u(c_0^A(1+\mu)) + v(h_0^A) + \sum_{T=1}^{\infty}(1-\phi)\phi^{T-1}\left[\sum_{j=1}^{T}\left(\beta^j \underline{\xi} u(c_j^A(1+\mu)) + v(h_j^A)\right) + \beta^{T+1}\left(u(c_{T+1}^A(1+\mu)) + v(h_{T+1}^A)\right)\right] =$$

$$u(c_0^R) + v(h_0^R) + \sum_{T=1}^{\infty}(1-\phi)\phi^{T-1}\left[\sum_{j=1}^{T}\left(\beta^j \underline{\xi} u(c_j^R) + v(h_j^R)\right) + \beta^{T+1}\left(u(c_{T+1}^R) + v(h_{T+1}^R)\right)\right], \tag{32}$$

where A denotes the autarky allocation and R is the Ramsey outcome under complete markets.

Note that in (32) we compute the increment in consumption μ needed to make the agent as well off in autarky as under the complete market, from period 0 to the random period $T+1$. Since our model does not possess a full blown stochastic environment, we do not want to underestimate the welfare effects from debt management by including periods in which there are no further shocks to the economy. In these periods the structure of the financial market will not matter for allocations.[32] This justifies using (32) as an appropriate welfare criterion in our model.

Welfare comparisons The results from the welfare analysis are reported in Table 8. The second and third columns in the table present the welfare levels under complete markets and autarky and Column 4 reports the value of μ. The second row shows the results under the assumption that $\underline{g} = \overline{g}$. Notice that in this case, the welfare increment from completing the market is essentially equal to zero. This prediction of the model should not be surprising. As we have seen, though the government uses taxes to impact the interest rate, the deficit does not increase substantially during the LT. In Section 4.3 we documented that as a result, the optimal portfolio features both long and short debt very close to zero. The autarky outcome is very close to the complete market.

The third and fourth rows of the table document the consumption increments when we add shocks to government spending. We now find that 'complete markets' generates a consumption increment equal to 0.1937% when $\underline{g} = 1.04\overline{g}$ and 0.3966% when $\underline{g} = 1.08\overline{g}$. The first number is similar to the analogous finding in Nosbusch (2008).

[31] This argument is explained in AMSS. What is different here (relative to AMSS) is that since the government does not issue debt, the first order conditions from the planner's program will not give us a risk adjusted random walk for $\lambda_{s,t}$.

[32] To see this assume that $\overline{B}_{-1} = 0$; note that this is the only 'sensible' case we can consider since under autarky debt by definition equals zero. It is then easy to show that $c_t^A = c_t^R$, $t = 0$ and $t > T+1$. These equalities carry over to the remaining endogenous variables. If we extend the welfare calculation beyond the LT, the welfare increment will be lower relative to a full blown model where LTs are recurrent events over the infinite horizon.

In Figure 2 we show the behavior of consumption taxes and hours worked in 4 different versions of the models. The solid (blue) line corresponds to the complete markets model and the dashed (green) line to the autarky model when $\underline{g} = \overline{g}$. The crossed (red) line and the dashed-dotted (black) line correspond to complete markets and autarky under $\underline{g} = 1.08\overline{g}$. The welfare effects documented in Table 8 can be explained through using the behavior of the variables shown in Figure 2. Notice that without spending shocks the response of the endogenous variables is indeed very similar between autarky and complete markets. However, when spending levels increase during the LT, consumption drops considerably in autarky, and continues to be below the steady state even when the economy escapes from the LT (e.g. middle top panel). Taxes too display considerable variability. These differences explain the welfare patterns discussed above.

7 Conclusion

A model of optimal government portfolios, under complete financial markets, and with shocks that can drive the economy to a liquidity trap, was presented in this paper. The theoretical model predicts that it is optimal to finance short when the initial debt level of the government is low. Under this condition, government portfolios do not have to be rebalanced throughout the LT, and are not affected by the specification of preferences or the degree of price stickiness in the economy. However, when initial government debt is high, the optimal strategy is to issue long debt, the conclusion we reach is similar to that of preexisting models of government debt management. On the methodological side this paper discusses how to solve models of optimal portfolios with liquidity traps using a global approximation method.

Our findings should be viewed as a useful benchmark for the optimal maturity structure of debt during liquidity traps. Recent interventions by central banks in the bond market, have decreased the amounts of long term government in the hands of the private sector when interest rates hit the zero lower bound. This paper identifies the conditions under which such a policy is optimal in a frictionless financial market and under the assumption that fiscal, monetary and debt management authorities coordinate in the Ramsey policy. The main takeaways and insights from this paper, however, apply broadly to models with frictions, including the recent literature on Quantitative Easing. Future research should embed the 'fiscal hedging approach' to debt management in models with segmented bond markets, transaction costs, and separate monetary and fiscal policies.

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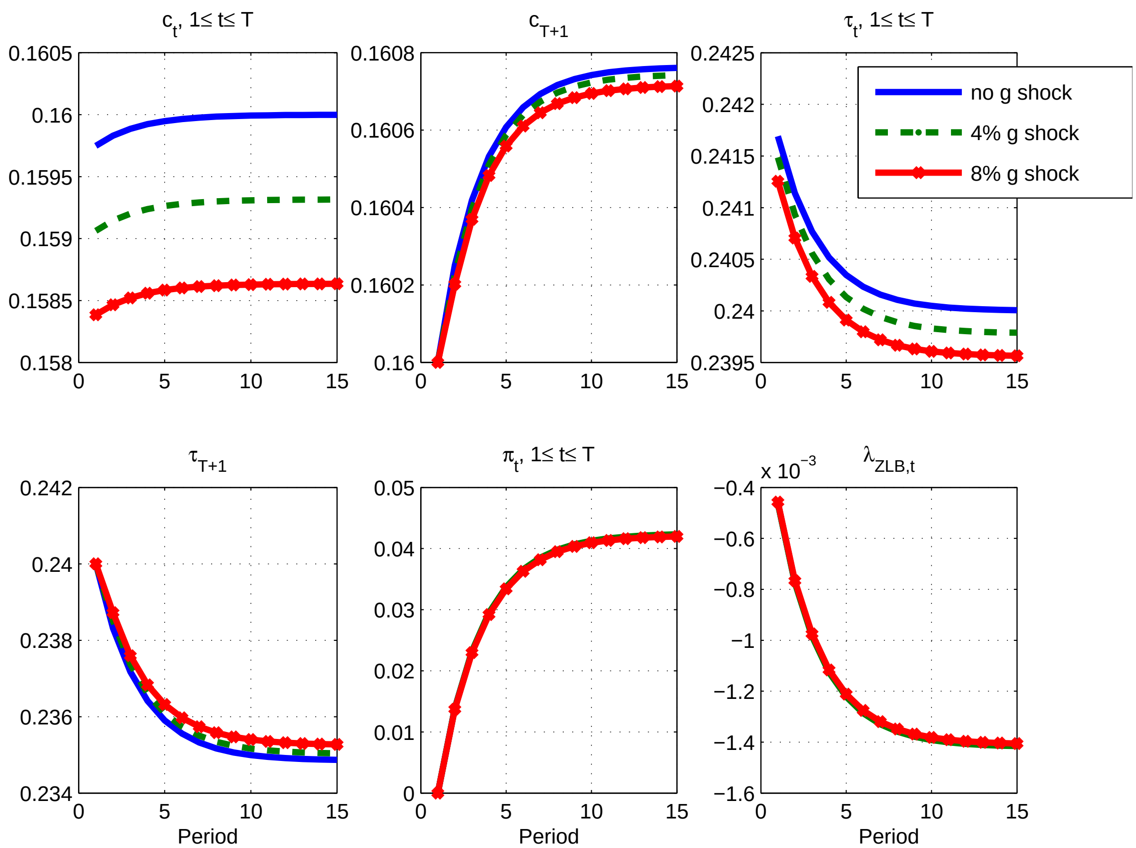
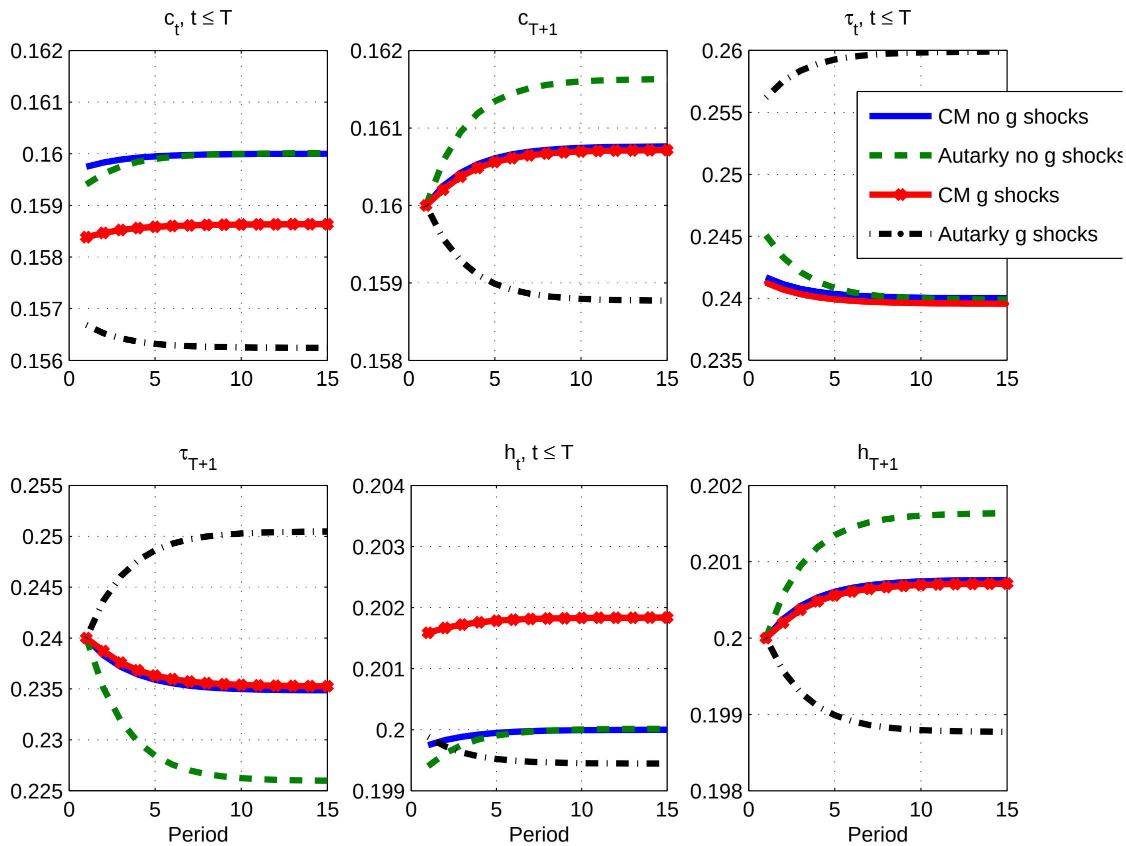
Figure 1: **Responses of endogenous variables to the preference shocks: various values of $\underline{g}$.**



Notes: The figure shows the adjustment of consumption, taxes and inflation to the preference shock under the ZLB constraint. The solid line corresponds to the case $\underline{g} = \overline{g}$, the dashed line to $\underline{g} = 1.04\overline{g}$ and the crossed (red) line to $\underline{g} = 1.08\overline{g}$. The top left panel shows the response of consumption during the LT episode. The middle top, shows the response at the 'exit' from the LT (period $T + 1$). The top right and bottom left show responses of taxes (during and right after the LT respectively). The middle bottom panel shows inflation and the bottom right traces the behavior of $\lambda_{ZLB,t}$.

Figure 2: **Responses of endogenous variables to the preference shocks: complete markets vs. autarky**



Notes: The figure shows the adjustment of consumption, taxes and hours to the preference and spending shocks under complete markets and financial autarky. The solid (crossed) lines show the complete market allocation without (with) spending shocks. The dashed (dashed-dotted) lines represent the autarky model without (with) g shocks. The top left panel shows the response of consumption during the LT and the middle-top the response at the 'exit' (period $T+1$). The top right (bottom left) panel shows taxes in the LT ('exit') and finally, the middle-bottom and bottom right panels show hours in the LT and in period $T+1$ respectively.

Table 1: Calibrated Parameters

Symbol	Value	Description
β	0.99	Discount factor
γ_c	1	Relative risk aversion
$\frac{\eta}{1+\eta}$	1.2	Gross value added markup
θ	17.5	Degree of price stickiness
γ_h	1	Inverse elasticity of labor supply
ω	0.5	Probability $\xi_1 = \underline{\xi}$
ϕ	0.8	Persistence of preference shock
$\bar{g}$	0.04	Steady state government spending
$\bar{h}$	0.2	Steady state hours worked
$\bar{c}$	0.16	Steady state consumption

Notes: The table reports parameter values under the benchmark calibration of the model. See text for further details.

Table 2: Optimal portfolios for $\gamma_c = 0$ and $\gamma_c = 1$ when $\widetilde{B}_{-1} = 0$

$[\gamma_c = 0]$

Period	b_t^1	b_t^N	$q_{t+1}^{N-1}\left(x^t\left(\underline{\xi},\underline{g}\right)\right)$	$q_{t+1}^{N-1}\left(x^t\left(\overline{\xi},\overline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\underline{\xi},\underline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\overline{\xi},\overline{g}\right)\right)$
			$g_1 = \overline{g}$			
0	0.000	0.000	0.953	0.914	-0.003	0.000
>0	0.000	0.000	0.953	0.914	0.000	0.003
			$g_1 = \overline{g} \times 1.04$			
0	0.176	-0.188	0.953	0.914	-0.005	0.004
>0	0.176	-0.188	0.953	0.914	-0.006	0.007
			$g_1 = \overline{g} \times 1.08$			
0	0.351	-0.377	0.953	0.914	-0.006	0.007
>0	0.351	-0.377	0.953	0.914	-0.011	0.010

$[\gamma_c = 1]$

Period	b_t^1	b_t^N	$q_{t+1}^{N-1}\left(x^t\left(\underline{\xi},\underline{g}\right)\right)$	$q_{t+1}^{N-1}\left(x^t\left(\overline{\xi},\overline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\underline{\xi},\underline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\overline{\xi},\overline{g}\right)\right)$
			$g_1 = \overline{g}$			
0	0.000	0.000	0.956	0.914	0.000	0.000
1	-0.001	0.000	0.956	0.914	0.000	0.000
4	-0.003	0.000	0.957	0.913	-0.001	-0.001
9	-0.003	0.000	0.957	0.913	-0.001	-0.001
			$g_1 = \overline{g} \times 1.04$			
0	0.767	-0.824	0.956	0.914	-0.004	0.003
1	0.758	-0.815	0.956	0.914	-0.004	0.003
4	0.745	-0.803	0.957	0.913	-0.005	0.002
9	0.740	-0.798	0.957	0.913	-0.005	0.002
			$g_1 = \overline{g} \times 1.08$			
0	1.534	-1.648	0.956	0.914	-0.008	0.006
1	1.516	-1.630	0.956	0.914	-0.009	0.005
4	1.492	-1.605	0.957	0.913	-0.009	0.005
9	1.483	-1.596	0.957	0.913	-0.009	0.005

Notes: The top panel shows the optimal portfolios for $\gamma_c = 0$, whilst the bottom panel considers the case of $\gamma_c = 1$. Columns 2-3 show the optimal mixture between short and long bonds; Columns 4-5 show the price of long term debt under $x_{t+1} = \left(\underline{\xi}, \underline{g}\right)$ (Column 4) and $x_{t+1} = \left(\overline{\xi}, \overline{g}\right)$ (Column 5); Columns 6-7 report the present discounted value of the surplus. In the first set of rows of each panel uncertainty is only driven by preference shocks, whilst in the remaining set of rows a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$ (second set) and $\underline{g}$ 8% higher than $\overline{g}$ (third set). Initial debt is zero.

Table 3: Optimal portfolios for $\gamma_c = 0$ and $\gamma_c = 1$ when $\widetilde{B}_{-1} = 60\%$

$[\gamma_c = 0]$

Period	b_t^1	b_t^N	$q_{t+1}^{N-1}\left(x^t\left(\underline{\xi}, \underline{g}\right)\right)$	$q_{t+1}^{N-1}\left(x^t\left(\overline{\xi}, \overline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\underline{\xi}, \underline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\overline{\xi}, \overline{g}\right)\right)$
				$g_1 = \overline{g}$		
0	-0.108	0.644	0.953	0.914	0.002	0.481
> 0	-0.108	0.644	0.953	0.914	0.506	0.484
				$g_1 = \overline{g} \times 1.04$		
0	0.068	0.456	0.953	0.914	0.000	0.484
> 0	0.068	0.456	0.953	0.914	0.501	0.487
				$g_1 = \overline{g} \times 1.08$		
0	0.244	0.267	0.953	0.914	-0.002	0.488
> 0	0.244	0.267	0.953	0.914	0.495	0.491

$[\gamma_c = 1]$

Period	b_t^1	b_t^N	$q_{t+1}^{N-1}\left(x^t\left(\underline{\xi}, \underline{g}\right)\right)$	$q_{t+1}^{N-1}\left(x^t\left(\overline{\xi}, \overline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\underline{\xi}, \underline{g}\right)\right)$	$S_{t+1}\left(x^t\left(\overline{\xi}, \overline{g}\right)\right)$
				$g_1 = \overline{g}$		
0	-0.293	3.046	0.956	0.914	0.524	0.498
1	-0.203	2.951	0.956	0.914	0.524	0.499
4	-0.078	2.819	0.957	0.913	0.524	0.499
9	-0.032	2.771	0.957	0.913	0.524	0.500
				$g_1 = \overline{g} \times 1.04$		
0	0.464	2.235	0.956	0.914	0.520	0.501
1	0.544	2.150	0.956	0.914	0.520	0.502
4	0.656	2.032	0.957	0.913	0.520	0.502
9	0.697	1.989	0.957	0.913	0.520	0.503
				$g_1 = \overline{g} \times 1.08$		
0	1.220	1.424	0.956	0.914	0.516	0.504
1	1.294	1.347	0.956	0.914	0.516	0.505
4	1.396	1.240	0.957	0.913	0.516	0.506
9	1.434	1.200	0.957	0.913	0.517	0.506

Notes: The top panel shows the optimal portfolios for $\gamma_c = 0$, whilst the bottom panel considers the case of $\gamma_c = 1$. Columns 2-3 show the optimal mixture between short and long bonds; Columns 4-5 show the price of long term debt under $x_{t+1} = \left(\underline{\xi}, \underline{g}\right)$ (Column 4) and $x_{t+1} = \left(\overline{\xi}, \overline{g}\right)$ (Column 5); Columns 6-7 report the present discounted value of the surplus. In the first set of rows of each panel uncertainty is only driven by preference shocks, whilst in the remaining set of rows a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$ (second set) and $\underline{g}$ 8% higher than $\overline{g}$ (third set). Initial debt is 60% at annual horizon.s

Table 4: Optimal portfolios for $\gamma_c = 5$

$[\widetilde{B}_{-1} = 0]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.001	0.742	-0.798	1.484	-1.596
1	-0.001	0.001	0.733	-0.789	1.468	-1.579
4	-0.003	0.001	0.720	-0.777	1.444	-1.555
9	-0.003	0.001	0.715	-0.772	1.435	-1.545

$[\widetilde{B}_{-1} = 60\%]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	-0.234	2.618	0.496	1.839	1.225	1.059
1	-0.154	2.533	0.568	1.762	1.291	0.990
4	-0.038	2.411	0.673	1.652	1.385	0.892
9	0.007	2.362	0.714	1.608	1.422	0.853

Notes: The table reports optimal portfolios for the case of $\gamma_c = 5$. The top panel shows the optimal portfolios for $\widetilde{B}_{-1} = 0$, whilst the bottom panel considers the case of $\widetilde{B}_{-1} = 60\%$ of GDP at annual horizon. Columns 2-3 show the optimal mixture between short and long bonds under preference shocks. In Columns 4-5 a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$, whilst in Columns 6-7 a spending shock is added which brings $\underline{g}$ 8% higher than $\overline{g}$.

Table 5: Optimal portfolios for $\theta = 5.833$ when $\widetilde{B}_{-1} = 0$

$[\gamma_c = 0]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.000	0.176	-0.188	0.351	-0.377
> 0	0.000	0.000	0.176	-0.188	0.351	-0.377

$[\gamma_c = 1]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.000	0.776	-0.835	1.552	-1.668
1	-0.001	0.000	0.766	-0.825	1.533	-1.649
4	-0.002	0.000	0.753	-0.812	1.509	-1.623
9	-0.003	0.000	0.749	-0.807	1.501	-1.615

$[\gamma_c = 5]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.001	0.743	-0.799	1.487	-1.599
1	-0.001	0.001	0.734	-0.790	1.470	-1.582
4	-0.003	0.001	0.721	-0.778	1.446	-1.557
9	-0.003	0.001	0.717	-0.774	1.437	-1.548

Notes: The top panel shows the optimal portfolios for $\gamma_c = 0$, the middle for $\gamma_c = 1$ and the bottom panel considers the case of $\gamma_c = 5$. Columns 2-3 show the optimal mixture between short and long bonds under preference shocks. In Columns 4-5 a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$, whilst in Columns 6-7 a spending shock is added which brings $\underline{g}$ 8% higher than $\overline{g}$. Price stickiness is lower ($\theta = 5.833$). Initial debt is zero.

Table 6: Optimal portfolios for $\varphi = 0.9$ when $\widetilde{B}_{-1} = 0$

$[\gamma_c = 0]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.000	0.234	-0.249	0.468	-0.498
> 0	0.000	0.000	0.234	-0.249	0.468	-0.498

$[\gamma_c = 1]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.001	1.031	-1.097	2.060	-2.193
1	-0.001	0.001	1.021	-1.089	2.042	-2.176
4	-0.004	0.001	1.005	-1.074	2.014	-2.147
9	-0.006	0.001	0.997	-1.066	1.999	-2.132

$[\gamma_c = 5]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.001	0.993	-1.057	1.987	-2.117
1	-0.002	0.001	0.983	-1.049	1.970	-2.099
4	-0.005	0.001	0.967	-1.034	1.940	-2.070
9	-0.006	0.001	0.959	-1.026	1.924	-2.054

Notes: The top panel shows the optimal portfolios for $\gamma_c = 0$, the middle for $\gamma_c = 1$ and the bottom panel considers the case of $\gamma_c = 5$. Columns 2-3 show the optimal mixture between short and long bonds under preference shocks. In Columns 4-5 a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$, whilst in Columns 6-7 a spending shock is added which brings $\underline{g}$ 8% higher than $\overline{g}$. The persistence of preference shocks is higher ($\varphi = 0.9$). Initial debt is zero.

Table 7: Optimal portfolios for $\omega = 0.005$ when $\widetilde{B}_{-1} = 0$

$[\gamma_c = 0]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.000	0.176	-0.193	0.353	-0.386
> 0	0.000	0.000	0.176	-0.193	0.353	-0.386

$[\gamma_c = 1]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.000	0.769	-0.841	1.537	-1.683
1	-0.001	0.000	0.759	-0.832	1.519	-1.664
4	-0.003	0.000	0.745	-0.818	1.493	-1.637
9	-0.003	0.000	0.740	-0.813	1.483	-1.627

$[\gamma_c = 5]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	0.000	0.000	0.743	-0.814	1.487	-1.628
1	-0.001	0.000	0.734	-0.805	1.470	-1.610
4	-0.003	0.000	0.721	-0.792	1.445	-1.584
9	-0.003	0.000	0.715	-0.787	1.435	-1.574

Notes: The top panel shows the optimal portfolios for $\gamma_c = 0$, the middle for $\gamma_c = 1$ and the bottom panel considers the case of $\gamma_c = 5$. Columns 2-3 show the optimal mixture between short and long bonds under preference shocks. In Columns 4-5 a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$, whilst in Columns 6-7 a spending shock is added which brings $\underline{g}$ 8% higher than $\overline{g}$. The probability of the economy falling into the LT is lower ($\omega = 0.005$). Initial debt is zero.

Table 8: Welfare. Zero initial debt

	Complete	Autarky	μ
$\underline{g} = \overline{g}$	-14.9350	-14.9350	0.00%
$\underline{g} = 1.04\overline{g}$	-14.9728	-14.9857	0.1937%
$\underline{g} = 1.08\overline{g}$	-15.0107	-15.0372	0.3966%

Notes: The table reports welfare outcomes, measured as the increment in consumption μ which is needed to make the agent as well off in autarky as under the complete market. Initial debt is zero.

A Appendix

A.1 Derivations

Proposition 2 The optimal tax schedule can be derived as follows: From (17) and (18) we have

$$v_{h,t} + 1 - \lambda_s \left[\gamma_h v_{h,t} + \left(1 + \frac{v_{h,t}}{w_t}\right) w_t\right] - \lambda_s \left(\frac{1+\eta}{\eta} - w_t\right) = 0.$$

Dividing by w_t and rearranging, we obtain

$$\frac{v_{h,t}(h_t)}{w_t} + 1 + \frac{1}{w_t} - 1 - \lambda_s \gamma \left[_h \left(\frac{v_{h,t}(h_t)}{w_t} + 1\right) - \gamma_h + \left(1 + \frac{v_{h,t}}{w_t}\right)\right] - \frac{\lambda_s}{w_t}\left(\frac{1+\eta}{\eta} - w_t\right) = 0,$$

or

$$\tau_t(1 - \lambda_s(\gamma_h + 1)) = 1 - \frac{1}{w_t} - \lambda_s \gamma_h + \frac{\lambda_s}{w_t}\left(\frac{1+\eta}{\eta} - w_t\right).$$

From (3) and making use of the fact that $h_t = \overline{h}$ we have

(33) $$0 = \frac{\eta}{\theta}\left(\frac{1+\eta}{\eta} - w_t\right)\overline{h} + \beta(\underline{\pi} - 1)\underline{\pi}^2,$$

at $t = 1$,

(34) $$(\underline{\pi} - 1)\underline{\pi} = \frac{\eta}{\theta}\left(\frac{1+\eta}{\eta} - w_t\right)\overline{h} + \beta(\underline{\pi} - 1)\underline{\pi}^2,$$

for $1 < t \leq T$ and

(35) $$(\underline{\pi} - 1)\underline{\pi} = \frac{\eta}{\theta}\left(\frac{1+\eta}{\eta} - w_t\right)\overline{h},$$

for $t = T + 1$. Using the above expressions we can get (21). ■

Proof of Proposition 3 Combining (25) and (26) and assuming $\overline{B}_{-1} = 0$ we get

$$u_{c,t} + v_{h,t} - \lambda_s u_{c,t} s_{h,t} - u_{cc,t}\Delta\lambda_{ZLB,t} - \lambda_s\left(u_{cc,t}s(t) + u_{c,t}s_{c,t}\right) = 0$$

where $s_{c,t} \equiv -\frac{v_{h,t}}{u_{c,t}^2}u_{cc,t}h_t$ and $s_{h,t} = \frac{v_{hh,t}}{u_{c,t}}h_t + \left(1 + \frac{v_{h,t}\eta}{u_{c,t}(1+\eta)}\right)\frac{1+\eta}{\eta}$. (25) can be written as:

$$1 + \frac{v_{h,t}}{u_{c,t}} - \lambda_s\left(\gamma_h \frac{v_{h,t}}{u_{c,t}} + \frac{1+\eta}{\eta} + \frac{v_{h,t}}{u_{c,t}}\right) - \lambda_s\left[\frac{u_{cc,t}}{u_{c,t}}\left(-g_t + \frac{1+\eta}{\eta}h_t + \frac{v_{h,t}}{u_{c,t}}h_t\right) - \frac{v_{h,t}}{u_{c,t}}\frac{u_{cc,t}}{u_{c,t}}h_t\right] = \frac{u_{cc,t}}{u_{c,t}}\Delta\lambda_{ZLB,t}.$$

Rearraging and dropping the terms that cancel out, we get:

$$\frac{v_{h,t}}{u_{c,t}}\left[1 - \lambda_s(1 + \gamma_h)\right] + 1 - \lambda_s\frac{1+\eta}{\eta} - \lambda_s\frac{u_{cc,t}}{u_{c,t}}\left(-g_t + \frac{1+\eta}{\eta}h_t\right) = \frac{u_{cc,t}}{u_{c,t}}\Delta\lambda_{ZLB,t}.$$

Using $c_t + g_t = h_t$ and $\frac{u_{cc,t}}{u_{c,t}} c_t = -1$ we get

$$\left[1 + \frac{v_{h,t}\eta}{u_{c,t}(1+\eta)}\right][1 - \lambda_s(1+\gamma_h)]\frac{1+\eta}{\eta} =$$
$$[1 - \lambda_s(1+\gamma_h)]\frac{1+\eta}{\eta} - 1 + \lambda_s\frac{1+\eta}{\eta} + \lambda_s\frac{u_{cc,t}}{u_{c,t}}g_t\left(\frac{1+\eta}{\eta} - 1\right) - \lambda_s\frac{1+\eta}{\eta} + \frac{u_{cc,t}}{u_{c,t}}\Delta\lambda_{ZLB,t},$$

or

$$\tau_t[1 - \lambda_s(1+\gamma_h)]\frac{1+\eta}{\eta} =$$
$$\left(\frac{1+\eta}{\eta} - 1\right)\left(1 + \lambda_s\frac{u_{cc,t}}{u_{c,t}}g_t\right) - \lambda_s(1+\gamma_h)\frac{1+\eta}{\eta} + \frac{u_{cc,t}}{u_{c,t}}\Delta\lambda_{ZLB,t},$$

which is equation (27) in Proposition 3. ■

Proof of Result 1. To prove i) holds, first note that with $\theta > 0$ the planner will set the inflation rate $\underline{\pi} = \beta\frac{\overline{\xi}}{\underline{\xi}}$. If $\pi_1 > \underline{\pi}$ then (19) does not hold and welfare decreases due to 'wasteful' inflation. Then from (30) $S_1\left(x^0, (\underline{\xi}, \underline{g})\right)$ is independent of θ since $\overline{s}$ does not depend on θ. Moreover, $q_1\left((\underline{\xi}, \underline{g})\right) = \beta^{N-2}$ also does not depend on θ. Under $\overline{s} \approx 0$ (31) continues to hold.

ii) When $\theta = 0$ (19) does not constrain π_1 to equal $\beta\frac{\overline{\xi}}{\underline{\xi}}$. Therefore, we may have $\pi_1 > \underline{\pi}$ in which case we will get $\lambda_{ZLB,1} = 0$. The bond price is given by $q_1\left((\underline{\xi}, \underline{g})\right) = \beta^{N-1}\frac{\overline{\xi}}{\underline{\xi}\pi_1} \leq \beta^{N-2}$. The optimal long and short positions are given by

$$b_0^1 \approx \frac{(\underline{g} - \overline{g})}{(\frac{\overline{\xi}}{\underline{\xi}\pi_1} - 1)} > 0 \quad \text{and} \quad b_0^N \approx -\frac{(\underline{g} - \overline{g})}{\beta^{N-1}(\frac{\overline{\xi}}{\underline{\xi}\pi_1} - 1)} < 0, \tag{36}$$

and they vary as π_1 varies. ■

The General Case of Quasi-linear Utility. We derive the present value of the government's surplus under quasi-linear preferences. Recall that we have 4 distinct values for s_t: $\overline{s}$, $\underline{\underline{s}}$, $\underline{s}$ and $\overline{\overline{s}}$. Suppose first that we are in period $t > T+1$. Then, trivially, the present value of the surplus is $\frac{\overline{s}}{1-\beta}$; from Proposition 1, the government sets the tax rate to a constant each period and spending equals $\overline{g}$. Moreover, suppose the economy has just escaped the LT (hence we are in period $T+1$. Then the present value of the surplus is $\overline{\overline{s}} + \beta\frac{\overline{s}}{1-\beta}$. The government sets $\overline{\overline{s}}$ for one period and thereafter it commits to a surplus equal to $\overline{s}$.

Consider now period $t = 2, 3, ...$ Recall that T is not known ex ante. The value of the surplus of the government is given by the following recursive form:

$$\underline{\xi}\underline{S} = \underline{\xi}\underline{s} + \beta\phi\underline{\xi}\underline{S} + \beta(1-\phi)\overline{\xi}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right),$$

or

$$\underline{S} = \frac{1}{1-\beta\phi}\left[\underline{s} + \beta(1-\phi)\frac{\overline{\xi}}{\underline{\xi}}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right)\right].$$

For period 1 if $\xi_1 = \underline{\xi}$, we get:

$$\underline{\xi} S_1 = \underline{\xi}\underline{\underline{s}} + \beta\phi\underline{\xi}\underline{S} + \beta(1-\phi)\overline{\xi}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right),$$

so that:

$$S_1 = \underline{\underline{s}} + \frac{\beta\phi}{1-\beta\phi}\underline{\underline{s}} + \frac{\beta(1-\phi)}{1-\beta\phi}\frac{\overline{\xi}}{\underline{\xi}}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right).$$

Finally, in period zero we have:

$$S_0 = (1-\omega)\frac{\overline{s}}{1-\beta} + \omega\left[\overline{s} + \beta\frac{\underline{\xi}}{\overline{\xi}}\left(\underline{\underline{s}} + \frac{\beta\phi}{1-\beta\phi}\underline{\underline{s}}\right) + \frac{\beta^2(1-\phi)}{1-\beta\phi}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right)\right].$$

To derive the bond prices consider first period 0. If the LT shock does not occur we have that $q^{N-1}\left(x^0(\overline{\xi},\overline{g})\right) = \beta^{N-1}$. However, if the preference shock occurs we need to determine the price $q^{N-1}\left(x^0(\underline{\xi},\underline{g})\right)$ as the weighted sum of all future possible realizations of the quantity $\beta^{N-1}\frac{\xi_N}{\xi_1}\frac{P_1}{P_N}$. We therefore have: With probability $1-\phi$ the shock lasts for one period and we have: $\beta^{N-1}\frac{\overline{\xi}}{\underline{\xi}}\frac{1}{\underline{\pi}}$. With probability $(1-\phi)\phi$ the shock ends in period 3 and therefore $\beta^{N-1}\frac{\overline{\xi}}{\underline{\xi}}\frac{1}{\underline{\pi}^2}$. With probability $(1-\phi)\phi^{N-2}$ we have $\beta^{N-1}\frac{\overline{\xi}}{\underline{\xi}}\frac{1}{\underline{\pi}^{N-1}}$ and finally with probability ϕ^{N-1} the shock does not end before period $t+N$ and therefore we have $\beta^{N-1}\frac{1}{\underline{\pi}^{N-1}}$.

Put together the price equals

$$q^{N-1}\left(x^0,(\underline{\xi},\underline{g})\right) = \beta^{N-1}\left[\frac{\overline{\xi}}{\underline{\xi}}\sum_{j=1}^{N-1}(1-\phi)\phi^{j-1}\frac{1}{\underline{\pi}^j} + \frac{1}{\underline{\pi}^{N-1}}\right],$$

which is the formula in text. We also have that $q^{N-1}\left(x^t,(\underline{\xi},\underline{g})\right) = q^{N-1}\left(x^0,(\underline{\xi},\underline{g})\right)$ in every period $t+1$ that the LT shock persists.

We now determine the price $q^{N-1}\left(x^t,(\overline{\xi},\overline{g})\right)$ when $x_t = (\underline{\xi},\underline{g})$ (when the economy is in the LT is t but the in $t+1$ the LT ends. Since inflation is positive in $t+1$ but this does not impact the price q_{t+1}^{N-1} we have that $q^{N-1}\left(x^t,(\overline{\xi},\overline{g})\right) = \beta^{N-1}$.

Consider now system (14). In the case of quasi-linear preferences we can derive the optimal portfolios as follows:

$$\begin{bmatrix} 1 & \beta^{N-1} \\ 1 & \beta^{N-1}\left(\frac{\overline{\xi}}{\underline{\xi}}\sum_{j=1}^{N-1}(1-\phi)\phi^{j-1}\frac{1}{\underline{\pi}^j} + \frac{1}{\underline{\pi}^{N-1}}\right)\end{bmatrix}\cdot\begin{bmatrix} b_0^1 \\ b_0^N\end{bmatrix} = \begin{bmatrix} \frac{1}{1-\beta}\overline{s} \\ \underline{\underline{s}} + \frac{\beta(1-\phi)}{1-\beta\phi}\frac{\overline{\xi}}{\underline{\xi}}\left(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}\right) + \frac{\beta\phi}{1-\beta\phi}\underline{\underline{s}}\end{bmatrix},$$

for the ex ante portfolio in $t=0$ and

$$\begin{bmatrix} 1 & \frac{\beta^{N-1}}{\overline{\pi}} \\ 1 & \frac{\beta^{N-1}}{\underline{\pi}}\left(\frac{\overline{\xi}}{\underline{\xi}}\sum_{j=1}^{N-1}(1-\phi)\phi^{j-1}\frac{1}{\underline{\pi}^j} + \frac{1}{\underline{\pi}^{N-1}}\right)\end{bmatrix}\cdot\begin{bmatrix} b_t^1 \\ b_t^N\end{bmatrix} = \begin{bmatrix} \overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s} \\ \frac{\beta(1-\phi)}{1-\beta\phi}\frac{\overline{\xi}}{\underline{\xi}}(\overline{\overline{s}} + \frac{\beta}{1-\beta}\overline{s}) + \frac{\underline{s}}{1-\beta\phi}\end{bmatrix},$$

for $t > 0$ and when $\xi_t = \underline{\xi}$ so that the economy remains in the liquidity trap.

A.2 Autarky Allocation

To solve the model under autarky we formulate the Lagrangian as:

$$\mathcal{L} = E_0 \sum_t \beta^t \Bigg\{ u(c_t)\xi_t + v(h_t)\xi_t + \lambda_{f,t}\left(h_t - c_t - g_t - \frac{\theta}{2}(\pi_t - 1)^2\right) - \lambda_{ZLB,t}\left(u_{c,t}\xi_t - \beta E_t \frac{u_{c,t+1}\xi_{t+1}}{\pi_{t+1}}\right)$$
$$-\lambda_{p,t}\left(u_{c,t}\xi_t\pi_t(\pi_t - 1) - \frac{\eta}{\theta}h_t u_{c,t}\xi_t(\frac{1+\eta}{\eta} - w_t) - \beta E_t u_{c,t+1}\xi_{t+1}\pi_{t+1}(\pi_{t+1} - 1)\right)$$
$$-\lambda_{s,t}u_{c,t}\xi_t\left(s_t - 0\right)\Bigg\}. \tag{37}$$

As discussed in text, in the Lagrangian (37) we have replaced the intertemporal constraint of the government, with the sequence of constraints $s_t = 0$ (budget balance). The multiplier $\lambda_{s,t}$ is now time varying as in the incomplete market models of AMSS and FMOS (2013, 2015). The first-order conditions for the optimum can be written as

$$u_{c,t}\xi_t - \lambda_{f,t} + \lambda_{p,t}\frac{\eta}{\theta}h_t u_{cc,t}\xi_t\left(\frac{1+\eta}{\eta} - w_t\right) - u_{cc,t}\xi_t\left(\lambda_{ZLB,t} - \lambda_{ZLB,t-1}\frac{1}{\pi_t}\right) - \lambda_{s,t}\xi_t\left(u_{cc,t}s_t + u_{c,t}s_{c,t}\right)$$
$$-u_{cc,t}\xi_t\pi_t(\pi_t - 1)(\lambda_{p,t} - \lambda_{p,t-1}) = 0 \tag{38}$$

$$v_{h,t}\xi_t + \lambda_{f,t} - \lambda_{s,t}u_{c,t}\xi_t s_{h,t} + \lambda_{p,t}\frac{\eta}{\theta}u_{c,t}\xi_t\left(\frac{1+\eta}{\eta} - w_t\right) = 0, \tag{39}$$

$$-\theta\lambda_{f,t}(\pi_t - 1) - \lambda_{ZLB,t-1}\frac{u_{c,t}\xi_t}{\pi_t^2} = 0, \tag{40}$$

$$-\lambda_{s,t}s_{w,t} - \lambda_{p,t}\frac{\eta}{\theta}h_t = 0. \tag{41}$$

Notice that these equations show that the optimal allocation depends on the value of $\lambda_{ZLB,t-1}$ but also on the value of $\lambda_{p,t-1}$ (equivalently on $\lambda_{s,t-1}$). Therefore, in theory, to solve the model under $\gamma_c > 0$ we need to formulate the approximating polynomials for the terms $E_t\frac{\xi_{t+1}u_{c,t+1}}{\pi_{t+1}}$ and $E_t\xi_{t+1}u_{c,t+1}\left(\pi_{t+1} - 1\right)\pi_{t+1}$ as $\Phi\left(\lambda_{ZLB,t}, \lambda_{s,t}\right)$ and $\Psi\left(\lambda_{ZLB,t}, \lambda_{s,t}\right)$. However, given our assumptions over how uncertainty enters in the model, $\lambda_{s,t}$ will not exert any influence on the allocation in $t+1$ beyond what is summarized by $\lambda_{ZLB,t}$. The reader can be sure of this by noting first that in models under autarky and without the Phillips curve, $\lambda_{p,t-1}$ (and hence $\lambda_{s,t-1}$) will not appear in the FOCs. Second, we know that in models without the ZLB, inflation does not respond to government and preference spending shocks. This continues to hold in the autarky allocation, because debt is zero and the government cannot use inflation to reduce the real payout of debt (as for example in SGU and FMOS (2013)). For these reasons $\lambda_{ZLB,t}$ is the only Lagrange multiplier that needs to be stored in the state vector under autarky. Steps 2 and 3 of Algorithm 2 can be applied to solve the model.

B Online Appendix- Additional Tables

Table 9: Optimal portfolios for $\varphi = 0.9$ when $\widetilde{B}_{-1} = 60\%$

$[\gamma_c = 0]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	-0.277	0.831	-0.043	0.583	0.191	0.334
> 0	-0.277	0.831	-0.043	0.583	0.191	0.334

$[\gamma_c = 1]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	-1.325	4.186	-0.312	3.111	0.703	2.035
1	-1.225	4.081	-0.219	3.014	0.787	1.947
4	-1.064	3.913	-0.070	2.859	0.924	1.806
9	-0.985	3.830	0.004	2.783	0.991	1.736

$[\gamma_c = 5]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	-1.109	3.587	-0.137	2.562	0.836	1.537
1	-1.016	3.489	-0.051	2.471	0.914	1.455
4	-0.857	3.323	0.095	2.319	1.048	1.316
9	-0.771	3.232	0.175	2.236	1.121	1.240

Notes: The top panel shows the optimal portfolios for $\gamma_c = 0$, the middle for $\gamma_c = 1$ and the bottom panel considers the case of $\gamma_c = 5$. Columns 2-3 show the optimal mixture between short and long bonds under preference shocks. In Columns 4-5 a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$, whilst in Columns 6-7 a spending shock is added which brings $\underline{g}$ 8% higher than $\overline{g}$. The persistence of preference shocks is higher ($\varphi = 0.9$). Initial debt is 60% og GDP at annual horizon.

Table 10: Optimal portfolios for $\theta = 5.833$ when $\widetilde{B}_{-1} = 60\%$

$[\gamma_c = 0]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	-0.108	0.644	0.068	0.456	0.244	0.267
> 0	-0.108	0.644	0.068	0.456	0.244	0.267

$[\gamma_c = 1]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	-0.336	3.093	0.430	2.271	1.196	1.451
1	-0.249	3.002	0.507	2.191	1.264	1.379
4	-0.139	2.885	0.605	2.088	1.353	1.287
9	-0.104	2.848	0.635	2.056	1.380	1.258

$[\gamma_c = 5]$

Period	$g_1 = \overline{g}$		$g_1 = \overline{g} \times 1.04$		$g_1 = \overline{g} \times 1.08$	
	b_t^1	b_t^N	b_t^1	b_t^N	b_t^1	b_t^N
0	-0.239	2.623	0.492	1.843	1.223	1.062
1	-0.159	2.539	0.564	1.767	1.287	0.994
4	-0.045	2.418	0.667	1.658	1.380	0.897
9	-0.001	2.372	0.707	1.616	1.415	0.860

Notes: The top panel shows the optimal portfolios for $\gamma_c = 0$, the middle for $\gamma_c = 1$ and the bottom panel considers the case of $\gamma_c = 5$. Columns 2-3 show the optimal mixture between short and long bonds under preference shocks. In Columns 4-5 a spending shock is added which drives $\underline{g}$ 4% higher than $\overline{g}$, whilst in Columns 6-7 a spending shock is added which brings $\underline{g}$ 8% higher than $\overline{g}$. Price stickiness is lower ($\theta = 5.833$). Initial debt is 60% og GDP at annual horizon.