DISTRIBUTIONAL COMPARATIVE STATICS WITH HETEROGENEOUS AGENTS

Andreas Hefti*

December 29, 2016

Abstract

We propose a general approach to study the differential effects of exogenous shocks in economic models with heterogeneous agents. Our setting applies to models that can be stated as "competition for market shares" in a broad sense. Examples that fit our type of structure are ubiquitous in competition theory, monopolistic competition, political economics and applied game theory. We show that even in presence of any number of arbitrarily heterogeneous agents, a single recursive relation characterizes the distributional pattern of equilibrium market shares, and related measures. We identify conceptual conditions under which the market share function rotates, thereby either causing more or less equality among the agents, and study the distributional comparative statics in examples from various economic fields.

JEL Classification: D30, D41, D43, D50, C62, C65, C72

Keywords: Heterogeneity; Inequality; Monopolistic Competition; General Equilibrium; Contest Theory

First Version: Nov 2016

This Version: Dez 2016

*Corresponding author: Andreas Hefti, Bluemlisalpstrasse 10, CH-8006 Zurich, andreas.hefti@econ.uzh.ch, +41787354964. Author affiliations: Department of Economics, University of Zurich and School of Management and Law, Zurich University of Applied Sciences. I wish to thank Armin Schmutzler, David Wettstein, Shuo Liu and Reto Foellmi as well as the participants of the Swiss-Basel Theory Meeting 2014, EEA-ESEM 2014 and UECE Lisbon Game Theory Meeting 2015 for many valuable comments.

1 Introduction

In reality, heterogeneous agents are omnipresent. Firms differ in their production possibilities, workers have different abilities or skills, and consumers have different preferences or income. Heterogeneity in the distribution of such fundamentals typically yields differences in equilibrium outcomes, which can be a major concern for several reasons. For instance, a regulating authority would like to know which conditions affect market concentration or the sales distribution, and which interventions (taxes, subsidies, emission permits,...) have what distributional consequences. A sports tournament designer may want to know which prize structure makes competition most unpredictable by equalizing winning chances between stronger and weaker teams. It is not uncommon for economic models to neglect agent heterogeneity, either by assuming perfect symmetry, by resorting to a single representative agent,[1] or by focusing more on some aggregate and less on the distributional impact of an exogenous shock. The cost of such a simplification is that the distributional comparative statics of the model cannot be studied.

In this paper we seek to resolve the problem of analyzing the comparative-statics with heterogeneous agents at the general and the applied level. To this end, we develop a systematic approach to study the comparative-statics in models with payoff functions that can be formulated as "competition for market shares" in a broad sense. An arbitrary number of agents may differ in their type (production costs, preferences, disposable income, ability,...), and each agent chooses some economic variable, such as a product price, the quantity to supply, or an advertising intensity that, jointly with the actions of all other agents, determines the equilibrium market share of every agent.[2] The respective equilibrium market share function, a density on the agent type space, is the core object of the analysis. This conceptualization of the market share function fits to many different examples, with a contextual interpretation of what a "market share" means. For example, in monopolistic competition with CES-demand, heterogeneous firms (by means of technology or product quality) compete in prices, and their respective market share is their share of overall consumption expenditure. Contests constitute a different class of examples. In a single-prize contest, the market share of a contestant is her probability of winning the prize. Alternatively, in the contest model of advertising, the market share of a firm amounts to the

[1]There are various definitions of what a representative agent could and should mean, and in many cases there is no respective representative agent (see Jackson and Yariv (2016) for a recent discussion).

[2]The equilibrium concept is defined in section 2.1, and encompasses the standard cases of monopolistic competition equilibria, Walrasian equilibria, aggregate-taking behavior equilibria and, with an appropriate modification, Nash equilibria.

share of consumers that acquire the product, in response to its relative advertising intensity.

Our main question is how an exogenous shock, such as a common demand or innovation shock in competition models, introduction or change of a sale tax, or a more or less effort-sensitive prize function in a contest, may distort the respective market share function and related measures, such as the distribution of payoffs, prices, efforts, quantities or similar.[3] Put generally, we seek to study under which circumstances a change in the economic environment makes already strong agents even stronger, or when we could expect to see a reduction of market concentration. The answer to this question boils down to understanding when and how our equilibrium market share function rotates, and we provide general and particular answers. We first uncover the main principles that guide the distortion of the market share function, without having a specific application in mind. In a second step, we use these results to derive novel and testable predictions about the distribution of market shares and related measures, as a function of idiosyncratic and aggregate conditions in a subset of applications from various economic fields.

Main results We show that the possible rotational patterns of the equilibrium market share function are described by a recursive relation between agent types, captured by a single optimality equation, for an arbitrary degree of agent heterogeneity and without the need to analyze the entire system of equilibrium equations. This recursion equation decomposes the relation between the market shares of any two different types into a direct-aggregative and an indirect effect of the shock, where the former is decisive for the type of rotation, and the latter influences quantitative aspects of the rotation. A rule of thumb is that if marginal benefits increase more (less) than proportional for the stronger agent, this tends to increase (decrease) the inequality of the equilibrium market shares. Thus, if the shock is such that marginal benefits increase uniformly for all agents, equality across agents increases, while shocks that affect the incentives of all agents proportional to their current market shares have no distributional consequences.

Our general analysis identifies a remarkable role of power functions for the distributional comparative statics. If costs and benefits associated with attaining a certain market share are type-invariant power functions of the market share, then the rotations induced by a shock have the property that the *relative* change in market share is always strictly increasing in agent type. A consequence is that if a shock induces a rotation, the strongest (weakest) agents al-

[3]While we are mostly interested in the distributional effects of common shocks, our methods can also be applied to idiosyncratic shocks (section 3.2.4).

ways gain (lose) comparably most market share. Such power function occur naturally in several cases, ranging from monopolistic competition with CES-demand, competitive production with homogeneous production technology to imperfectly discriminatory contests with Tullock success functions.

At the applied level, our approach allows us to address previously underexplored questions concerning market concentration and distribution in a variety of models at a general level. An important example is monopolistic competition with CES-demand (Dixit and Stiglitz, 1977). This model has been a central ingredient in many later equilibrium models, such as the Melitz-model of international trade (Melitz, 2003). The Melitz-model introduces firm-side heterogeneity in terms of a linear production function that depends on an idiosyncratic efficiency parameter. While heterogeneous firms, in terms of technology or product quality, play an indispensable role for understanding the empirical regularities observed in trade data (Redding, 2011; Gervais, 2015), the general determinants of firm-side market shares of consumption expenditures have received little attention,[4] which is our core focus. We consider monopolistic competition model with CES demand, possibly with idiosyncratic product quality levels, and study how the equilibrium market shares of the firms depend on the central aggregate parameters of this model for different classes of production technologies. If production belongs to the family of homogeneous technologies, an increase in the substitutability of final products causes a rich-gets-richer effect, where already large firms expand their market size while small firms become marginalized. We extend the analysis to the more challenging case where firms have heterogeneous production elasticities. A major insight is that the neutrality of income for firm market shares is driven by the assumption of identical elasticities. With heterogeneous elasticities growing income yields quantity growth jointly with a growing inequality of firm market shares. An industry-wide increase in efficiency has a similar effect, while it necessarily generates winners and losers in terms of absolute profits. This occurs because large firms face less downward pressure on prices as efficiency increases while they can expand their quantities by relatively more. It follows that with spillover process innovations, the most dominant firms also have the strongest incentive to

[4] An exception is Mrazova et al. (2016), who relate the observed distribution of firm sales or markups to the underlying distribution of technology in a monopolistic competition setting. They show that if productivity and sales parameters follow the same generalized power function, then they must be related by means of a simple power function. A class of demand functions, including CES demand as a special case, is derived that links sales and productivity distribution in such a way.

innovate, while laggards are hurt by such innovations.

A very different class of examples, where distributional concerns may arise for a variety of reasons, are contests (Konrad, 2009). Indeed, many real-world competitions, such as obtaining a research grant, lobbying, elections, sports tournaments or advertising for market shares can best be described by means of a *contest* for a scarce resource. We consider a generalized contest model with a possibly endogenous prize function, and study the distributional comparative statics of important aggregate parameters, such as prize value, the degree of influence on the success chances of the individual contestants, or the distribution of prize money across multiple prizes. Among other results, we show that a prize increase favors strong contestants in terms of success chances, provided that the cost elasticities of efforts increases across agents. In case of a Tullock contest, we show that an increase in contest noise, i.e., a reduction in the ability to influence own success chances, always levels the playing field by equalizing winning odds. Introducing (or increasing) a second prize in a contest tends to make the chances of winning the first prize (typically also of winning any prize) more equal. Particularly, an even split of an available prize budget across two prizes equalizes success chances most, at the cost of minimizing total effort. We also study how the sensitivity of the prize function to efforts affects the equilibrium market share function. Our leading example is advertising for market shares, where ads influence consumer attention as well as the willingness to pay of attentive consumers. According to our model, we should observe increasing market concentration if advertising has persuasive or attaching effects, while market shares become more uniformly distributed if ads are more of a disturbing nature. Finally, the results from this section have some implications for tournament design, essentially because of an efficiency-equity trade-off. Letting the prize function become more sensitive to the winner's effort tends to make success chances more unequal, while aggregate effort increases. If the goal is to make the competition less predictable, which is a central design aspect of sports tournaments, then a "handicapping" of winning efforts can achieve this goal, but comes at the costs of a reduction in total efforts.

This paper further considers the following applications. In the Logit model of product competition choice we show that if the deterministic part of utility becomes more decisive (choice is less noisy) this is associated with an increasing inequality of the supplier market shares. One suggestive consequence is that the weakest firms (in terms of low product quality or high supply costs) have the strongest incentive to increase the noisiness in the choice procedure e.g., by resorting to obfuscation tactics. We further consider the distributional implications of an import tax on domestic and foreign firms market shares in the domestic market.

We also include simple Walrasian general equilibrium models, where we jointly discuss the distributional patterns on the firm and the consumer side. Specifically, we consider a model where consumers with quasilinear preferences sell their resources to firms and acquire the produced consumption good on a competitive market. We show that an introduction (or increase) of a quantity tax has differential implications for firm-side market shares only if costs are not a common-elastic power function. If firm heterogeneity is driven by different elasticities of marginal costs, then the tax increases equality of market shares on the supply side (a subsidy would have opposite effects). A similar result applies for the distribution of consumption over consumers. The tax will affect consumer-side market share only if the utility functions for the good are not common-elastic power functions. With Log-utility, the tax reduces consumption inequality of the good among consumers. A common efficiency shock on the firm side has an effect similar to a subsidy, and therefore tends to increase consumption inequality.

We further study the effects of a uniform contraction of the available production resources for the distribution of firm market shares and consumer consumption in a competitive single input-output private ownership economy, where income is composed of labor and capital earnings. We find that if the source of income inequality is the resource endowment, then consumption inequality always increases as the resource is depleted. If however income inequality originates from the share distribution, then more or less consumption inequality can result depending on aggregate production possibilities.

Finally, we ask how the distribution of consumption and leisure across consumers depends on the state of technology in the economy and on the importance of leisure relative to consumption. In case of the former, a common positive technology shock, in terms of more efficient labor, boosts consumption, real wages and profits. As profits and wages increase proportionally, the incentives to work more and benefit from the higher wage or to rather enjoy more leisure and finance consumption from the higher capital income counterbalance each other, making labor supply and consumption shares invariant to technology. However, absolute consumption differences between the wealthy and the poor increases. If consumption becomes more important relative to leisure, consumers tend to supply more labor, increasing firm profits. This benefits capital owners more, and therefore consumption-side market shares become less equally distributed. It may even happen that in equilibrium the poorest end up with a lower consumption level, despite a higher propensity to consume.

Article structure We outline the general model in section 2, where we state the definition of equilibrium and establish existence of a unique equilibrium (Theorem 1). In section 3 we define the concept of a rotation of the market share function, develop the essential analytical tools to study such rotations and derive some general insights about the possible causes of rotations. The main results on existence and type of rotation are presented in sections 3.2.1 and 3.2.2 (Theorems 2 - 4). The methods and results developed are applied in section 4 to a number of examples. All proofs are in the appendix.

2 The model

Let I be an index set and $i \in I$ an agent. In many economic models, the payoff of each agent i can be decomposed as

$$\Pi(i) = \text{Market share}(i) * \text{Market value}(i) - \text{Costs}(i) \tag{1}$$

In a given model, an agent typically needs to choose a variable (or a set of variables) with the goal of maximizing (1), possibly subject to a number of constraints. Specifically, let

$$\Pi(i) = \max_{t(i)\geq 0} p(i, t(i), T; x)V(i, t(i), T; x) - \Phi(i, t(i); x), \tag{1'}$$

where $t(i) \in \mathbb{R}$ is agent i's choice variable (this could be a price, a quantity, an effort,...), and $T = \int_i t(i)$ is an aggregate. Further, $p(i) = p(i, t(i), T; x)$ is i's market share, $V(i) = V(i, t(i), T; x)$ the market value and $\Phi(i, t(i))$ the costs of agent i given her choice of $t(i)$ and the aggregate T. Finally, $x \in X$ is an exogenous parameter drawn from some parameter set, which influences at least one of the component functions of $\Pi(i)$.

This paper seeks to analyze how the equilibrium distribution of market shares $p(i), i \in I$, and related quantities such as $\Pi(i)$ or $t(i)$, depend on x in presence of heterogeneous agents. We first need to be precise, of course, what we mean by equilibrium, and how heterogeneity is introduced, which are the topics of the next section.

2.1 Heterogeneity and Equilibrium: Definition, Existence and Uniqueness

To solve our model we invoke two conventions. Both are without loss of generality with respect to the purpose of this paper. First, we reformulate the model as a direct competition for market shares. Second, we introduce continuum agents.

Market shares It is convenient to reformulate the optimization problem (1') as

$$\Pi(i) = \max_{p(i)\geq 0} p(i)V(i, p(i), T; x) - \Phi(i, p(i), T; x). \tag{2}$$

In (2) the agent directly chooses her market share $p(i)$, instead of indirectly over $t(i)$.[5] Such a transformation is possible if the functions $p(\cdot)$, $V(\cdot)$ and $\Phi(\cdot)$ are bijective in $t(i)$, which we always assume.[6] Thus, the way we solve for the equilibrium is different from how the agents truly act in the model. Working with the transformed model is useful for our comparative-static purposes, but it is more natural to think that agents directly choose certain variables (efforts, prices, quantities) that determine their respective market shares when obtaining an intuition.

Continuum agents We set $I = [0, 1]$ for the agent population. The formal advantage of working with continuum agents is that they will allow for a simpler representation of equilibrium objects. Most importantly, the market share $p(i)$ will be a (density) function $p : [0, 1] \to \mathbb{R}_+$, rather than a discrete mapping, that changes its support as the number of agents change. Importantly, continuum agents are without loss of generality in our setting. It is possible to identify the equilibrium market share $p^d(i)$ for any given number of atomistic agents $n \in \mathbb{N}$ by a corresponding equilibrium density $p(i)$ with support $[0, 1]$ by means of rescaling with the factor $1/n$ (see appendix A.1 for details). For example, if $n = 3$ and $p^d(1) = 1/2$, $p^d(2) = 1/3$, $p^d(3) = 1/6$, then $p(i) = 3/2$, $i \in [0, 1/3)$, $p(i) = 1$, $i \in [1/3, 2/3)$ and $p(i) = 1/2$, $i \in [2/3, 1]$, and $\int p(i)di = 1/3(3/2+1+1/2) = 1$. Moreover, our formulation encompasses the case of "true" continuum agents, where the equilibrium density is a strictly monotonic, continuous function $p(i)$. While there is no direct atomistic-agent analogue to this case, we argue in appendix A.1 that this can be seen as the limiting case of a large number of distinct atomistic agents.

[5]The solutions of problem (2) for the various agents could be such that the sum of all $p(i)$ is greater or less than one, but it must be equal to one in any equilibrium (see Definition 1).

[6]We are slightly abusing notion here. If $p(i) = p(i, t(i), T; x)$ then, assuming invertibility, $t(i) = h(i, p(i), T; x)$ and plugging this, e.g., into the market share function $V(i, t(i), T; x)$ would yield a new function $\hat{V}(i, p(i), T; x)$ which we again label with the function symbol V.

2.1.1 Equilibrium Definition

We are now ready to formally state the definition of an equilibrium. Consider the slightly more general payoff function[7]

$$\Pi(i) = B(i, p(i), T) - \Phi(i, p(i), T). \tag{3}$$

Definition 1 (Equilibrium) *An equilibrium is a bounded function* $p : [0, 1] \to \mathbb{R}_+$ *and a number* $T \in (0, \infty)$ *such that*

i) For each $i \in [0, 1]$*,* $p(i)$ *solves* $\max_{p(i) \geq 0} \Pi(i)$*, where* $\Pi(i)$ *is given by* (3)*.*

ii) $\int_0^1 p(i)di = 1$

An intuitive interpretation of the equilibrium conditions is that in any equilibrium each agent is choosing her action variable $t(i)$ to maximize her payoff while holding a correct belief about the aggregate T. Section 4 shows that this equilibrium definition identifies, e.g., Walrasian (price) equilibria, the monopolistic equilibrium with CES-utility (or logit) consumers and, with an appropriate modification, the Nash equilibrium in games with a sum-aggregative representation of payoffs (such as contests or the Cournot model). We next show that under a natural set of technical assumptions on the benefit $B(\cdot)$ and cost $C(\cdot)$ functions, the above notion of equilibrium is well-defined.

2.1.2 Heterogeneity and main assumptions

Let $g(i, p(i), T) \equiv \frac{\partial B(\cdot)}{\partial p(i)}$ and $\varphi(i, p(i), T) \equiv \frac{\partial \Phi(\cdot)}{\partial p(i)}$ denote marginal revenue and marginal costs, respectively. Note that, for given $T > 0$, the FOC pertaining to maximizing (3) at an interior point $p(i) > 0$ is

$$g(i, p(i), T) = \varphi(i, p(i), T). \tag{4}$$

Our formal analysis will be squarely centered around this innocent-looking expression. If not mentioned otherwise we will take the following assumption as satisfied.

Assumption 1 *Let* $\Pi(i)$ *be given by* (3)*.*

[7] We ignore x here, having in mind that x can take on any fixed value in a parameter interval X.

(A1) *For any* $T > 0$, $i \in [0,1]$ *and* $p(i) \geq 0$*:* $\Pi(i)$ *is a* C^2*-function of* $p(i)$ *and strongly quasiconcave in* $p(i)$*,* $g(i,0,T) > 0$ *and* $g(i,\cdot,T)$ *is bounded from above,* $\varphi(i,0,T) = 0$*,* $\frac{\partial\varphi(i,p,T)}{\partial p} > 0$ *and* $\lim_{p\to\infty} \varphi(i,p,T) = \infty$*.*

(A2) *For any* $i \in [0,1]$*:* $g(i,p,\cdot)$ *and* $\varphi(i,p,\cdot)$ *are* C^1*-functions,* $g(i,1,0) > 0$*,* $g(i,1,\cdot)$ *is bounded from above,* $\varphi(i,p,0) = 0$*,* $\varphi(i,p,\cdot)$ *is strictly increasing and* $\lim_{T\to\infty} \varphi(i,p,T) = \infty$ *whenever* $p > 0$*, and* $g_T(i,p,T) < \varphi_T(i,p,T)$ *whenever* $g(i,p,T) = \varphi(i,p,T)$*.*

(A3) *For any* $p, T > 0$*:* $B(\cdot,p,T), g(\cdot,p,T)$ *are (weakly) decreasing and* $\Phi(\cdot,p,T), \varphi(\cdot,p,T)$ *(weakly) increasing on* $[0,1]$*.*

We also maintain, for simplicity, that inaction is possible by setting $B(i,0,T) = \Phi(i,0,T) = 0$.[8] Assumptions (A1) and (A2) amount to natural differentiability, boundary and slope assumptions. In particular, these assumptions assert the existence of a unique equilibrium, and their precise role will be clarified below. Note that we impose no assumption on how the (marginal) revenue depends on $p(i)$ or T, but we maintain that marginal costs are increasing in T. Intuitively, this means that maintaining a certain market share $p(i)$ at a higher aggregate effort level T requires to bear higher efforts and expenses.[9]

The order assumption (A3) is how we introduce heterogeneity to the model. (A3) implies that agents are sorted left-to-right in thus that (marginal) benefits are (weakly) decreasing and (marginal) costs (weakly) increasing in agent index i. A simple example is that agents are heterogeneous according to their (production) efficiency, such that

$$\Pi(i) = p(i)V(p(i),T) - c(i)\Phi(p(i),T), \tag{5}$$

where $c(\cdot)$ is increasing.

2.2 Existence and uniqueness

Assumptions (A1) and (A2) ascertain the existence of a unique equilibrium.

Theorem 1 (Existence and uniqueness) *Any model with payoffs* (3) *that satisfy assumption 1 has a unique equilibrium* $(p(i), T)$*. All equilibrium payoffs* $\Pi(i)$ *are positive, and* $p(\cdot)$ *is a bounded, decreasing and strictly positive density.*

[8] This is not problematic because we shall not consider entry or exit decisions.

[9] This will become self-evident in the context of specific applications.

The proof evolves in two steps, corresponding to the two requirements in the equilibrium definition. Its baseline reasoning is illustrated in Figure 1. (A1) implies that a unique optimizer

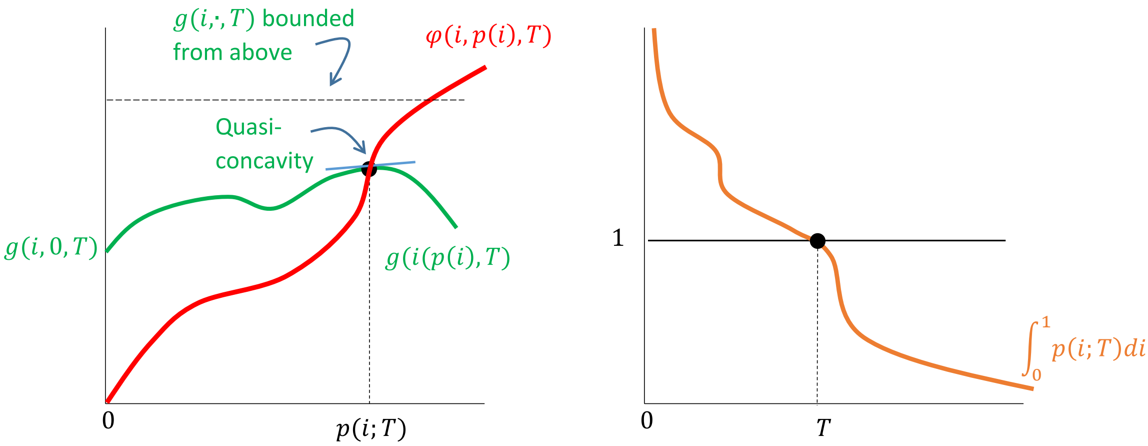

Figure 1: Equilibrium existence and uniqueness

$p(i;T) > 0$ exists for any given $T > 0$ and any fixed $i \in [0,1]$. The existence of such an optimizer follows because a zero market share is not optimal $(g(i,0,T) > 0 = \varphi(i,0,T))$, the gains from increasing one's market share are limited ($g(\cdot,T)$ bounded from above) and marginal costs are strictly increasing in $p(i)$ and unbounded. Uniqueness of this optimizer is implied by strong quasiconcavity. Assumption (A2) then assures that there is a unique $T > 0$ such that $\int p(i;T)di = 1$. To see why, suppose that $g(i,p,T)$ is bounded above and away from 0 for any $p \geq 0$ and any $T > 0$, consistent with (but stronger than) assumption (A2). In such a case even the best agent ($i = 0$) seeks to set her market share $p(i;T)$ arbitrarily small as marginal costs become arbitrarily large ($T \to \infty$). Similarly, even the worst agent aims at an arbitrarily large $p(i;T)$ if $T \to 0$ and marginal costs become arbitrarily small. These two facts imply that $\lim_{T\to\infty} \int p(i;T)di = 0$ and $\lim_{T\to 0} \int p(i;T)di = \infty$, and existence of a $T > 0$ with $\int p(i,T)di = 1$ follows because $\int p(i;\cdot)di$ is continuous. Uniqueness then follows from the last assumption in (A2), which assures that $\int p(i;\cdot)$ is strictly decreasing at $\int p(i,T)di = 1$.[10]

Among other, assumption (A3) precludes leap-frogging in equilibrium:

Corollary 1 (Basic equilibrium properties) *Consider two agents $i, j \in [0,1]$ with $i < j$. In equilibrium $p(i) \geq p(j)$ and $\Pi(i) \geq \Pi(j)$ (no leap-frogging). Moreover, $p(i) > p(j)$ if $g(i,p,T) \geq g(j,p,T)$ and $\varphi(i,p,T) \leq \varphi(j,p,T)$ $\forall p, T > 0$, where at least one inequality is strict, and $\Pi(i) >$*

[10] This assumption is also essential for the comparative-static results, because it assures that T increases if marginal benefits increase exogenously for all agents (Lemma 1).

$\Pi(j)$ *if* $B(i,p,T) \geq B(j,p,T)$ *and* $\Phi(i,p,T) \leq \Phi(j,p,T)$ $\forall p, T > 0$, *where at least one inequality is strict. Further,* $p(i) = p(j)$ *if both* $g(i,p,T) = g(j,p,T)$ *and* $\varphi(i,p,T) = \varphi(j,p,T)$ $\forall p, T > 0$.

The last statement of Corollary 1 implies that if there is a non-trivial interval $\hat{I} \subset [0,1]$ of homogeneous agents, meaning that $B(i,p,T) = B(j,p,T)$ and $\Phi(i,p,T) = \Phi(j,p,T)$ $\forall i,j \in \hat{I}$ and any $p, T > 0$, then $p(i) = p(j)$ $\forall i,j \in \hat{I}$. This means that if $[0,1]$ is partitioned by a finite number of non-trivial *distinct* homogeneous agent intervals I_n, such that $g(i,p,T) \geq g(j,p,T)$ and $\varphi(i,p,T) \leq \varphi(j,p,T)$, one inequality strict, whenever $j > i$, $i \in I_n$ but $j \notin I_n$, then $p(\cdot)$ must be a step-wise decreasing density with finitely many steps, where $p(i) = p(j)$ iff $i,j \in I_n$. In such a case we can assume, wlog, that $p(\cdot)$ is right-continuous. In the trivial case, where $n = 1$ and thus $I_1 = [0,1]$, meaning that all agents are homogeneous, the equilibrium is symmetric, i.e. $p(i) = 1$ $\forall i \in [0,1]$. Finally, if $g(i,p,T) \geq g(j,p,T)$ and $\varphi(i,p,T) \leq \varphi(j,p,T)$, one inequality strict, for *any* two $i,j \in [0,1]$ with $i < j$, such that no two agents are identical, then $p(\cdot)$ is a strictly decreasing density.

We summarize this discussion by illustrating the possible structures of $p(\cdot)$ in an example. Suppose that for any $i \in [0,1]$ (2) is given by

$$\Pi(i) = p(i)V(p(i),T) - c(i)C(p(i),T), \tag{6}$$

where the cost coefficient function $c(\cdot) > 0$ either is a finite step-wise increasing function, or a strictly increasing C^1-function, according to the following definition.

- *Class I* consists of all increasing, right-continuous step functions for which $\exists i_0 \in (0,1)$: $i < i_0 \leq j \Rightarrow c(i) < c(j)$, $i,j \in [0,1]$.
- *Class II* consists of all strictly increasing functions $c \in C^1([0,1],[1,\bar{c}])$.

We refer to the final qualification for class I functions as the *somewhere strictly increasing (SI)* property, which means that $c(\cdot)$ has a step "somewhere in the middle", and is equivalent to the requirement that $c(i)$ is not constant on $(0,1)$. Class II functions trivially satisfy SI. Class I functions capture the case of finitely many different *cost types*. That is, the steps partition the population into equivalent cost types, and all members of an equivalence class (agents "sitting on the same step") are homogeneous to each other and, by Corollary 1, display an indistinguishable equilibrium behavior. As an interpretation, one could think of every cost type

k being "represented" by an agent i_k, who solves problem (6) for the entire group. The more important interpretation however is to view these steps as the continuum analogue to the case of finitely many atomistic agents (see appendix A.1).

The discussion following Corollary 1 shows that if $c(i)$ is class I, a step function capturing agent cost groups $k = 1, ..., K$ with measures $\gamma_1, ..., \gamma_K > 0$, $\sum_k \gamma_k = 1$, then $p(i)$ will be a corresponding step-wise decreasing (density) function, meaning that $p([0,1]) = \{p(i_1), ..., p(i_K)\}$ and $\int_0^1 p(i)di = \sum_{k=1}^K \gamma_k p(i_k)$.[11] If $c(\cdot)$ is class II, then it follows from Theorem 1, (4) and the Implicit Function Theorem that $p(i)$ must be differentiable density with $p'(i) < 0$ on $(0,1)$. Summarizing, this means that the respective equilibrium share function $p(\cdot)$ inherits the class membership of $c(\cdot)$.

3 Heterogeneity: Toolbox and general results

Throughout section 3.1 we develop the machinery that will help us analyze how the equilibrium distribution of market shares depends on the parameters of a model. The most important concepts for the later analysis are the definition of rotations (Definition 2), Proposition 2, Corollary 2 and the notion of a monotone ratio.

3.1 Rotations

Let $X \subset \mathbb{R}$ be an open parameter interval, $A \equiv [0,1] \times X$, and consider a function $p : A \to \mathbb{R}_+$ with the properties that $p(0,x) \geq p(1,x) > 0$, and $p(\cdot,x)$ is weakly decreasing $\forall x \in X$. In the following, we develop our theory of distributional comparative-statics for the two classes of density functions that naturally emerge with heterogeneous agents.

- Density p belongs to **class I** if $p(\cdot,x)$ is a step-wise decreasing, right-continuous density with finitely many steps that has the **somewhere strictly decreasing (SSD) property**, i.e. $\forall x \in X$ $\exists i_0 \in (0,1)$: $p(i,x) > p(j,x)$ for $i < i_0 \leq j$.

- Density p belongs to **class II** if $p(i,x)$ is C^1 and $\frac{\partial p(i,x)}{\partial i} < 0$ for $i \in (0,1)$, $x \in X$

[11]Hence in case of class I finding the equilibrium generally requires solving a $(K+1)$-system of equations in the unknowns $p(i_1), ..., p(i_K)$ and T.

The simplest case of a class I density is the two-types case. If the fraction of "good" (e.g., low-cost) types is $\gamma \in (0,1)$, and we let $i = 0$ represent good types and $i = 1$ bad types, $p(\cdot)$ has the form

$$p(\cdot) = \begin{cases} p_0 & i \in [0,\gamma) \\ p_1 & i \in [\gamma, 1] \end{cases} \qquad , p_1 = \frac{1-\gamma p_0}{1-\gamma} \qquad , p_0 \geq p_1 \tag{7}$$

How do $p(\cdot, x')$ and $p(\cdot, x)$ differ in general if $x' \neq x$? Among the simplest and most interesting movements of $p(\cdot, x)$ as x varies is the idea of rotation.

Definition 2 (Rotations) *Let $x \neq x' \in X$, and consider the two functions $p(\cdot, x')$ and $p(\cdot, x)$. We say that $p(\cdot, x')$ is an **outward-rotation (OR)** of $p(\cdot, x)$, or $p(\cdot, x)$ is an **inward-rotation (IR)** of $p(\cdot, x')$, if $\exists\ 0 < i_0 \leq i_1 < 1$ such that*

$$\begin{aligned} p(i,x') > p(i,x) \quad & i \in (0, i_0) \\ p(i,x') < p(i,x) \quad & i \in (i_1, 1) \\ p(i,x') = p(i,x) \quad & i \in (i_0, i_1) \end{aligned} \tag{8}$$

where the last condition only is required if $i_0 < i_1$. We say that a parameter change $dx > 0$ induces a (local) OR (IR) of $p(\cdot, x)$ if $\exists \delta > 0$ such that $p(\cdot, x')$ is OR (IR) of $p(\cdot, x)$ for any $x' \in (x, x+\delta)$.

Figure 2 presents some examples of rotations. If $p(\cdot, x')$ is OR (IR) of $p(\cdot, x)$, this means that the inequality of market shares (or market concentration) has increased (decreased). Given that the decreasing density $p(\cdot, x)$ has the SSD property $\forall x \in X$, the corresponding distribution function $F(\cdot, x)$ is a strictly increasing, continuous and concave function, that is strictly above the 45°-line for $i \in (0,1)$. Moreover, if $p(i, x')$ is OR of $p(i, x)$, then $F(\cdot, x)$ stochastically dominates $F(\cdot, x')$.[12]

3.1.1 Detecting rotations

We now derive useful conditions asserting that $p(i, x')$ is an OR (or IR) of $p(i, x)$. For a given density on $[0,1]$, let $[i_0]_x = \{i \in I : p(i,x) = p(i_0, x)\}$ be the equivalence class of agents with representative i_0. If $p(\cdot, x)$ is of class I, then there is a finite number of such equivalence classes, and if $p(\cdot, x)$ is of class II, each agent forms her own equivalence class. The union of all

[12]See Proposition B.1, Online Appendix.

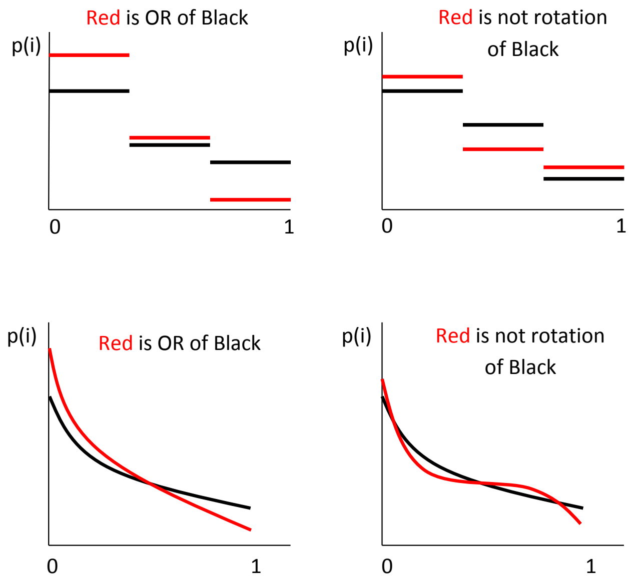

Figure 2: Class I and II rotations

equivalence classes form a partition of I. Given a density $p(\cdot, x)$, we define the binary relation $\rhd$ by $j \rhd i$ if $j > i$ and $j \notin [i]_x$. If $p(\cdot, x)$ is of class II, the relations $\rhd$ and $>$ coincide, and if $p(\cdot, x)$ is of class I, $j \rhd i$ means that j is of a "worse" type than i.[13]

If $p(\cdot, x') - p(\cdot, x)$ is strictly decreasing over its equivalence classes, i.e., if $p(\cdot, x') - p(\cdot, x)$ is strictly decreasing for any two different agent types, then $p(\cdot, x')$ is OR of $p(\cdot, x)$:

Proposition 1 (Difference test) *Let $x, x' \in X$ and suppose that $\infty > p(\cdot, x'), p(\cdot, x) > 0$ are right-continuous, decreasing SSD densities with identical equivalence classes. If*

$$p(i, x') - p(i, x) > (<) p(j, x') - p(j, x) \quad \textit{whenever } j \rhd i \in (0, 1) \tag{9}$$

is satisfied, then $p(\cdot, x')$ is an OR (IR) of $p(\cdot, x)$.

If $p(\cdot, \cdot)$ is strictly submodular, then (9) is satisfied, but note that if $p(\cdot, x)$ is a step-function, $p(\cdot, \cdot)$ cannot be strictly submodular.[14] There exists an alternative sufficient condition for the OR-property, which is particularly useful in applications.

[13]Given that in our comparative-static exercises we maintain the initial heterogeneity as, e.g., specified by the cost coefficient function $c(i)$, we do not require to make $\rhd$ x-specific as the equivalence classes do not change (but the heights of the steps do).

[14]In particular, if $p(\cdot, \cdot)$ is strictly submodular, then $p(\cdot, x)$ must be strictly decreasing $\forall x \in X$. See Online Appendix B.1 for a comment on how the difference and ratio test differ from (log-)supermodularity.

Proposition 2 (Ratio test) *Suppose that the premise of Proposition 1 is satisfied. If*

$$\frac{p(i,x')}{p(j,x')} > (<) \frac{p(i,x)}{p(j,x)} \quad \textit{whenever } j \rhd i \in (0,1) \tag{10}$$

is satisfied, then $p(\cdot,x')$ is OR (IR) of $p(\cdot,x)$.

Note that Propositions 1 and 2 encompass both class I and II densities. Comparing conditions (9) and (10), the following observation is useful.[15] If $u' > u > 0$, $v' > v > 0$ and $\frac{u'}{v'} \geq \frac{u}{v} > 1$ then we have increasing differences, i.e., $u' - v' > u - v$. Hence, with increasing arguments a (weakly) increasing ratio implies strictly increasing differences. Further, both conditions are equivalent if $p(i,x), p(i,x')$ are linear in i (which they cannot be if $p(\cdot)$ is of class I). Finally, in the two-types case (7) properties (9), (10), the OR-property and stochastic dominance of the respective distribution functions are all equivalent:

Proposition 3 (Two-types case) *Let $x, x' \in X$ and suppose that the densities $p(\cdot,x), p(\cdot,x')$ are specified by* (7) *with distribution functions $F(\cdot,x), F(\cdot,x')$. Then properties* (8), (9), (10) *and strict stochastic dominance $F(i,x') > F(i,x)$ are equivalent.*

The practical significance of conditions (9) and (10) is that we can derive corresponding calculus tests to detect a rotation. This will enable us to use a local criterion to decide about a global property of $p(\cdot)$. We only present the differential version of condition (10) here, because this turns out to be most relevant for our applications.[16]

Corollary 2 (Ratio test) *Suppose that $p(\cdot)$ is a class I or II density. Further, if $p(\cdot)$ is class I, let $p(i,x)$ be differentiable in x, except at step points. Let $x_0 \in Int(X)$. If*

$$\frac{\partial}{\partial x}\left(\frac{p(i,x)}{p(j,x)}\right) > 0 \qquad \forall x \geq x_0 \textit{ and any } j \rhd i \in (0,1) \tag{11}$$

whenever the derivative exists, then for any $x > x_0$ (10) *holds with "$>$" and hence $p(i,x)$ is OR of $p(i,x_0)$. If the first inequality in* (11) *is reversed, then, for any $x > x_0$,* (10) *holds with "$<$", and $p(i,x)$ is IR of $p(i,x_0)$.*

[15] See Online Appendix B.1 for a more general comparison.
[16] Further results are in the Online Appendix (section B.1).

By defining

$$\Delta_i(x) \equiv \frac{\frac{\partial p(i,x)}{\partial x}}{p(i,x)} = \frac{dp(i)}{p(i,x)}$$

condition (11) can be succinctly reformulated as

$$\Delta_i(x) > \Delta_j(x) \qquad \forall x \geq x_0 \text{ and any } j \rhd i \in (0,1) \tag{11'}$$

Monotone ratios We say that the function $p(i,x)$ has **monotone ratios** if for any $x, x' \in X$ with $x' > x$, condition (10) holds with a fixed inequality for any $j \rhd i$.[17] Monotone ratios yield a particularly pronounced type of rotation, where the effect of dx on $p(i)$ increases monotonically towards the tails of the distribution. Specifically, if $p(i,x)$ has monotone ratios and $p(i,x')$ is OR of $p(i,x)$, then any agent's relative change in her market share compared to any weaker agent type $j > i$ increases if $x' \neq x$. An equivalent interpretation is that the relative change in market shares is strictly increasing in agent type. Hence with an OR (IR) the strongest agents ($i \in i[0]$) gain (lose) most while the weakest agents ($i \in i[1]$) lose (gain) most.

3.2 Distributional comparative-statics: General Results

Let (marginal) benefits depend on an exogenous parameter with $B_x(i,p,T,x), g_x(i,p,T,x) > 0$.[18] Hence any change $dx \neq 0$ shifts $g(\cdot)$, but potentially by different magnitudes for different agents, and our rotational tools allow to extract the equilibrium effects of dx for the distribution of market shares $p(i)$. The main challenge to overcome is that $g(\cdot)$ and $\varphi(\cdot)$ in any equilibrium equation (4) generally are allowed to depend on p and T, which both are endogenous to the model. A first important stepping stone is to note that, with the assumptions made, the equilibrium aggregate increases in x:

Lemma 1 *The equilibrium aggregate $T = T(x)$ verifies $T'(x) > 0$.*

It should be emphasized that $T'(x) > 0$ holds independent of the equilibrium behavior of $p(i)$. Intuitively, an exogenous increase in marginal benefits increases "efforts" (activities to improve

[17] If $p(\cdot,x)$ is of class II, then the monotone ratio condition is equivalent to strict log-super(sub)modularity of $p(i,x)$.

[18] For expositional ease we formally consider only benefit shifters. It will become self-evident how to adopt the analysis if x were a cost shifter instead.

the own market share) of all agents and thus also aggregate effort.[19]

3.2.1 Existence of distributional effects

We now use (4) and our rotation tools to determine how $p(\cdot;x)$ depends on x, always assuming $g(\cdot), \varphi(\cdot)$ (heterogeneity) to be such that the equilibrium density $p(i)$ is either of class I or II. Let $x_0 \in X$. Because in equilibrium $p(i;x)$ is implicit in (4) for any given $i \in [0,1]$, we have that

$$g(i,p(i),T,x_0)\varphi(j,p(j),T) = g(j,p(j),T,x_0)\varphi(i,p(i),T) \qquad i,j \in [0,1] \tag{12}$$

or, in short-hand notation, $g(i)\varphi(j) = g(j)\varphi(i)$. In the following we always let $i \in [0,1)$ and take any $j \rhd i$. Total differentiation of (12) with respect to x implies

$$dp(i)\left(g_p(i)\varphi(j) - g(j)\varphi_p(i)\right) = dp(j)\left(g_p(j)\varphi(i) - g(i)\varphi_p(j)\right) + r, \tag{13}$$

where $dp(i) = \frac{\partial p(i;x_0)}{\partial x}$ and

$$r = \big(g_T(j)T'(x_0) + g_x(j)\big)\varphi(i) + \varphi_T(i)T'(x_0)g(j) - \big(g_T(i)T'(x_0) + g_x(i)\big)\varphi(j) - \varphi_T(j)T'(x_0)g(i)$$

Define

$$\Delta_i \equiv \frac{dp(i)}{p(i;x_0)}, \qquad \varepsilon_i \equiv \frac{g_p(i)p(i)}{g(i)}, \qquad \eta_i \equiv \frac{\varphi_p(i)p(i)}{\varphi(i)},$$

where $\eta_i > 0$ and $\varepsilon_i < \eta_i$ by strong quasiconcavity (assumption A1). These definitions together with (12) allow to restate (13) as a surprisingly simple type-recursive relation:

$$\Delta_i = \Delta_j k_{ij} + R_{ij}, \qquad k_{ij} = \frac{\eta_j - \varepsilon_j}{\eta_i - \varepsilon_i} > 0, \tag{14}$$

where

$$R_{ij} = \frac{1}{\eta_i - \varepsilon_i}\left(A(i) - A(j)\right), \qquad A(s) \equiv \left(\frac{g_T(s)}{g(s)} - \frac{\varphi_T(s)}{\varphi(s)}\right)T'(x_0) + \frac{g_x(s)}{g(s)}, \quad s = i,j. \tag{15}$$

[19] Lemma 1 also holds if x is an idiosyncratic, rather than a common, shock. In particular, if $M = \{i \in [0,1] : g_x(i,p,T,x) > 0\}$ is a set of positive (Lebesgue) measure and $g_x(i,p,T,x) = 0 \ \forall i \in M^C$, then $T'(x) > 0$.

Equation (14) decomposes the relation between Δ_i and Δ_j into a direct-aggregative effect (R_{ij}),[20] and an indirect effect (k_{ij}). We write R and k instead of R_{ij}, k_{ij} whenever there is no confusion. Decomposition (14) is key to understanding if and how dx affects the distribution of market shares. Our first main theorem establishes that there are distributional effects of $dx \neq 0$ if and only if x has a non-zero direct-aggregative effect on two different agents i, j.

Theorem 2 (Distributional Effects) *If $R = 0$ $\forall i, j \in [0,1]$ and any $x \in X$, then $\Delta_i = 0$ $\forall i \in [0,1]$ and hence $p(\cdot, x') = p(\cdot, x)$ for any $x, x' \in X$. If for a given $x \in X$ $\exists i, j \in [0,1]$ such that $R \neq 0$, then $\exists \delta > 0$ such that $p(\cdot, x') \neq p(\cdot, x)$ for any $x' \in (x - \delta, x + \delta) \backslash \{x\}$.*

Note that $R = 0$ $\forall i, j \in [0,1]$ iff $A(i) = A(j)$ $\forall i, j \in [0,1]$, i.e., iff $A(\cdot)$ is independent of i. As an example, if $g(i) = g(T, x)$ and $\varphi(i) = \varphi(i, p)h(T)$, then $A(s)$ is independent of s, and hence $p(i, x)$ is invariant to x. More generally, if both $g(i)$ and $\varphi(i)$ are multiplicatively separable[21] in (i, p) and T, the bracket in $A(s)$ is independent of s. In such a case, $dx \neq 0$ will induce distributional effects if and only if $g(i)$ is not multiplicatively separable in (i, p) and x.

3.2.2 Rotational effects

We know from Theorem 2 that for dx to induce a distributional effect on $p(\cdot)$, we require that $R \neq 0$ for at least two agents $i \neq j$, but can we say more about how $p(\cdot)$ responds to a (possibly small) change in x?

Definition 3 *For a given $x_0 \in X$, we say that R is uniformly positive (negative), if $R(x_0) > (<) 0$ $\forall i, j \in [0,1]$. If $R(x) > (<) 0$ $\forall i, j \in [0,1]$ holds for any $x \in X$, then R is globally uniformly positive (negative).*

The first main result of this section establishes that the direct-aggregative effect R, if uniform, is decisive for the resulting distributional pattern.

Theorem 3 (Rotational Effects) *If R is globally uniformly positive (negative), then for any $x' > x_0$ we have that $p(i, x') > (<) p(i, x)$ $\forall i \in [0]$ and $p(i, x') < (>) p(i, x)$ $\forall i \in [1]$. Moreover,*

[20] To understand this terminology, note that $A(i)$ is the total derivative pertaining to (4) with respect to x, normalized by $g(i)$.

[21] A function $\frac{h_T(i,p,T)}{h(i,p,T)}$ with $h(\cdot) > 0$ is invariant to i if and only if $h(i, p, T) = h_1(i, p)h_2(T)$. Note that multiplicative separability includes the case where $h(i, p, T)$ does not depend, e.g., on T at all.

if $p(\cdot, x)$ is of class I and R is uniformly positive (negative) at $x_0 \in X$, then $dx > 0$ induces a local OR (IR) of $p(\cdot, x_0)$.

The first statement in Theorem 3 likewise applies to class I and II densities, and shows how the extreme tails of the distribution evolve for $dx \neq 0$. An immediate corollary is that the distributional pattern of the two-types case is entirely characterized, at least locally, as a consequence of Theorems 2 and 3. This follows because in the two-types case $R_{01} \geq (>)0$ iff $R_{10} \leq (<)0$, meaning that R is either uniformly positive (negative) or $R = 0$. Moreover, it follows from (the proof of) Theorem 3 that if R is globally uniformly positive (or negative) in the three-types case, then *any* $x > x_0$ induces an OR (IR) of $p(\cdot, x_0)$, because the behavior of the "middle group" does not matter.

While the indirect effects ("k" in (14)) are not decisive whether or not dx induces a rotation (given a uniform positive or negative R), the value of k has implications for the structure of the distributional pattern, as we highlight the next two results. Similar to Definition 3, we say that $k(x_0)$ is uniformly larger (smaller) than one if $k(x_0) \geq (\leq)1$ $\forall i, j \in [0, 1]$.

Corollary 3 *Suppose that $p(\cdot, x)$ is class I, and let R be uniformly positive (negative) at $x_0 \in X$. If k is uniformly larger (smaller) than one, there is $\delta > 0$ such that*

$$p(i, x') > p(i, x) \quad \Rightarrow \quad \frac{p(i, x')}{p(i, x_0)} > (<) \frac{p(j, x')}{p(j, x_0)} \quad \forall j \rhd i \tag{16}$$

if $x' \in (x_0, x_0 + \delta)$. If k is uniformly smaller (larger) than one, there is $\delta > 0$ such that

$$p(i, x') < p(i, x) \quad \Rightarrow \quad \frac{p(i, x')}{p(i, x_0)} > (<) \frac{p(j, x')}{p(j, x_0)} \quad \forall j \rhd i \tag{17}$$

if $x' \in (x_0, x_0 + \delta)$.

Given that R is uniformly positive, (16) says that among the agents gaining market share, the stronger an agent is (lower index i), the more the agent gains in relative terms. Similarly, (17) says that among the loosing agents, the weaker the agent is, the more she looses in relative terms.

If k is uniformly equal to one, both statements of Corollary 3 apply, which suggests a complete order across equivalence classes in terms of how the relative market shares change. This applies likewise to class I and class II densities. As Theorem 4, the second main result of this section, shows, this is indeed the case, but there is more. If k is uniformly equal to one, then the possible rotational pattern induced by $dx > 0$ are globally valid for any $x > x_0$ by means of the monotone ratio property.

Theorem 4 (Global rotation condition) *If $k(x) = 1$ $\forall i, j \in [0, 1]$ and any $x \in X$, and R is globally uniformly positive (negative), then $p(i, x)$ has monotone ratios, and $p(i, x)$ is OR (IR) of $p(i, x_0)$ for any $x > x_0$.*

Theorem 4 applies to class I and II densities likewise. The additional power of the global rotation condition stated in Theorem 4 is that, because of monotone ratios, rotations satisfy a global transitivity condition. If R is globally uniformly positive (negative), then for any $i \in [0, 1]$, the relative market share $\frac{p(i,x)}{p(j,x)}$, $j \rhd i$, is strictly increasing (decreasing) in x, meaning that the market shares will be distributed less and less equally (more and more equally) as x increases.

Inspection of (14) shows that $k(x) = 1$ uniformly, and thus $p(i, x)$ has monotone ratios, if both functions $g(\cdot)$ and $\varphi(\cdot)$ have agent-independent p-elasticities, i.e., if

$$g(i, p, T, x) = \hat{g}(i, x, T)p^{\xi(x,T)} \qquad\qquad \varphi(i, p, T) = \hat{\varphi}(i, T)p^{\zeta(T)}, \tag{18}$$

where the exponents $\xi(x, T), \zeta(T)$ are real-valued functions.[22] Because the p-derivative of a p-power function $f(p) = \alpha p^{\gamma}$ is again a p-power function, the fact that with agent-independent p-elasticities we obtain global rotations with the interesting probabilistic structure of monotone ratios, makes for a strong case to model $B(\cdot)$ and $\Phi(\cdot)$ as power functions of p. Power functions indeed constitute a quite flexible functional class, particularly because in (18) the exponents need not be constant. As we shall see in section 4, many important examples from several subfields of economics indeed involve power functions and, by Theorem 4, will therefore produce strong distributional comparative-static predictions.

3.2.3 General determinants of the direct-aggregative effect

Since the direct-aggregative effect is decisive for the distributional pattern of $p(\cdot)$ (Theorems 2 and 3), we ought to learn more about the main principles that determine $sign(R)$. We assume that $\varphi(i) = \varphi(i, p)C(T)$ throughout the remainder of this section. Many applications indeed

[22]Note that this includes the case, where marginal revenue does not depend on p at all ($\xi(x, T) = 0$).

imply this type of cost function (see section 4). If $\varphi(i) = \varphi(i,p)C(T)$, then $sign\,(A(i) - A(j))$ depends only on $g(\cdot)$. The next proposition states general principles that determines whether R is (uniformly) positive or negative.

Proposition 4 *Suppose that $\varphi(i) = \varphi(i,p)C(T)$, let $g(\cdot)$ be a C^2-function and define $h(i,p,T,x) = Ln\,(g(i,p,T,x))$. If $h_x(i,p,T,x_0) \geq h_x(j,p,T,x_0)$, $h_T(i,p,T,x_0) \geq h_T(j,p,T,x_0)$ for all $i,j \in (0,1)$ with $j \rhd i$, and $h_T(i,p',T,x_0) \geq h_T(i,p,T,x_0)$, $h_x(i,p',T,x_0) \geq h_x(i,p,T,x_0)$ for any $i \in (0,1)$ and $p' > p$, where at least one of the above inequality is strict, then R is uniformly positive at x_0. If all inequalities are reversed (and one strict so), then R is uniformly negative at x_0. Moreover R is globally uniformly positive (negative) if these inequalties hold $\forall x \in X$.*

We briefly explore the formal conditions in Proposition 4; further applications are in section 4. A shock $dx > 0$ induces all agents to seek for a larger market share which, by Lemma 1, increases the equilibrium T. Incentives to increase market shares are relatively stronger for strong agent types if and only if marginal benefits increase proportionally more for these types, i.e., iff $dg(i)/g(i) > dg(j)/g(j)$ holds for $j \rhd i$. This is why a *uniform* change in marginal benefits ($dg(i) = dg(j)$) together with heterogeneity $g(i) > g(j)$ causes an IR ($R < 0$) of $p(i)$, while in case of a *proportional* change $dg(i)/g(i) = dg(j)/g(j)$ there are no distributional effects. The possibly differential incentive effects of $dx > 0$ are either reinforced or weakened by the equilibrium change in aggregate effort. Particularly, if $g_T > (<)0$, then $R > 0$ is more likely to result if the increase (decrease) in marginal benefits triggered by $dT > 0$ affects the strong agent relatively more (less). In the special case, where marginal benefits are a non-idiosyncratic p-power function,

$$g(i,p,T,x) = u(i)v(T,x)p^{\xi(T,x)},$$

Proposition 4 implies that R is determined solely by the elasticity function $\xi(T,x)$. In particular, R is uniformly positive if $\xi_T, \xi_x \geq (\leq)0$ with one strict inequality, and $R = 0$ uniformly if ξ is constant.

3.2.4 Idiosyncratic shocks

While the above analysis dealt with shocks dx that affect all agents, it can easily be adopted to the case of a pure idiosyncratic effect of dx, which we illustrate in a simple example (see section 4.1.2 for another example). Let $p(\cdot,x)$ be of class I, and let $g(i) = g(i,x)$ and $\varphi(i) =$

$c(i)p(i)^\eta h(T)$, such that $k_{ij} = 1$ everywhere. Suppose that $dx > 0$ only for agents $i \in [m]$, where $m \in (0, 1)$. That is, $g_x(i, x_0) > 0$ $\forall i \in [m]$ and $g_x(i, x_0) = 0$ else.[23] Then $R_{mj}(x_0) > 0$ $\forall j \rhd m$, $R_{im}(x_0) < 0$ $\forall m \rhd i$, and $R_{ij}(x_0) = 0$ $\forall i, j \notin [m]$. It then follows from (14) that $\Delta_i(x_0) > 0$ $\forall i \in [m]$ while $\Delta_j(x_0) < 0$ $\forall j \notin [m]$. Hence we obtain the intuitive result that type m gains market shares while all other types lose market shares. Moreover, because we must have that $\Delta_i = \Delta_j$ for $i, j \notin [m]$ the other types lose market share by approximately the same proportion.

4 Applications

In this section we analyze the distributional comparative statics in an array of examples. We must restrict the analysis to a subset of all interesting comparative-statics, and some of them may be worth exploring further in stand-alone contributions.

4.1 Monopolistic Competition

4.1.1 Competition with CES-Demand

Suppose that there is a continuum $[0, 1]$ of consumers, indexed by ι, each endowed with CES utility of the form

$$U(\iota) = \int_0^1 r_s q_s(\iota)^\sigma ds, \qquad \iota \in [0, 1], \tag{19}$$

where $s \in [0, 1]$ is a differentiated product supplied by a single firm, also labeled by s. Further, $r_s > 0$ measures the importance of product s to any consumer,[24] $q^s(\iota) \geq 0$ is the respective quantity demanded by ι, and $\sigma \in (0, 1)$ is the elasticity of substitution. Each consumer is endowed with disposable income $I(\iota) > 0$, and chooses each $q_s(\iota)$, $s \in [0, 1]$, to maximize (19) subject to $\int P_s q_s(\iota) ds = I(\iota)$, where $P_s > 0$ is the price of product s. Setting $\eta \equiv \frac{1}{1-\sigma} > 1$, and summing up consumer demand $q_i(\iota)$ for product $i \in [0, 1]$ yields

$$q_i = \frac{I r_i^\eta P_i^{-\eta}}{\int r_s^\eta P_s^{1-\eta} ds}, \qquad I \equiv \int I(\iota) d\iota > 0. \tag{20}$$

To write firm profits $\Pi(i)$, $i \in [0, 1]$, in a way consistent with our framework, let $T \equiv \int r_s^\eta P_s^{1-\eta} ds$, $p(i) \equiv \frac{r_i^\eta P_i^{1-\eta}}{T}$ and for supply costs $\Phi(i, q_i) = c(i) q_i^{\gamma_i}$, $c(i) > 0$, $\gamma_i \geq 1$. This includes the standard

[23] If $[m] \neq [0]$ we additionally assume that $dx > 0$ is such that it does not change the order of market shares (i.e. $p(\cdot, x')$ remains a decreasing density).

[24] For example, r_s could be the (relative) quality of variety s.

case of constant marginal production costs. Then

$$q_i = \frac{I}{r_i^{\frac{\eta}{\eta-1}}} p(i)^{\frac{\eta}{\eta-1}} T^{\frac{1}{\eta-1}} \tag{20'}$$

and

$$\Pi_i = P_i q_i - c(i) q_i^{\gamma_i} = I p(i) - I^{\gamma_i} \frac{c(i)}{r_i^{\frac{\gamma_i \eta}{\eta-1}}} p(i)^{\frac{\gamma_i \eta}{\eta-1}} T^{\frac{\gamma_i}{\eta-1}}, \tag{21}$$

where $p(i)$ is the market share of total income that firm i obtains. From a formal viewpoint, competition in the CES-model is akin to a *contest for expenditure shares* (see section 4.3). Each firm chooses its market share $p(i)$ (by choosing its price P_i given own demand (20)) to maximize (21), which yields the FOC

$$\left(\frac{\eta - 1}{\eta \gamma_i}\right)^{\eta-1} I^{(1-\gamma_i)(\eta-1)} = \frac{c(i)^{\eta-1}}{r_i^{\gamma_i \eta}} p(i)^{\eta(\gamma_i - 1)+1} T^{\gamma_i}. \tag{22}$$

We discuss the distributional comparative statics of the CES model by considering separately the standard case of homogeneous and the more challenging case of heterogeneous production elasticities. One major upshot of our tools from section 3 is that we can study the distributional comparative statics without the need to explicitly solve the model (which in general is not possible here).

Homogeneous production elasticities Suppose that $\gamma_i = \gamma \geq 1$, $\forall i \in [0, 1]$. Then (22) implies that $k_{ij} = 1$ $\forall i, j \in [0, 1]$. Therefore, by Theorem 4, equilibrium market shares $p(i)$ must have the monotone ratio property whenever the direct-aggregative effect R is either globally uniformly positive or negative. The power-function property of $\varphi(i, p(i), T)$ is implied by the constant substitution elasticity σ. Suppose that $c(i) \leq c(j)$ and $r(i) \geq r(j)$ whenever $i < j$, such that $p(i) \geq p(j)$ in equilibrium.[25] Moreover, let $c(i) < c(j)$ or $r(i) > r(j)$ for some $i, j \in [0, 1]$ such that $p(i)$ either is class I or II. Our first result states that equilibrium expenditure shares become more unequal the stronger substitutes the products are or the less elastic the cost function is, while there is no such effect of income, quality or production efficiency.

Proposition 5 *The CES-model with homogeneous production elasticities has the following distributional comparative-statics:*

[25]It is easy to verify that the CES verifies (A1) and (A2). Thus, by Theorem 1, a unique equilibrium exists.

	$p(\cdot)$	$\frac{p(i)}{p(j)}$	$\frac{\Pi(i)}{\Pi(j)}$	$\frac{P(i)}{P(j)}$	$\int \Pi(i)di$
$d\sigma > 0$	OR	$+$	$+$	$\gamma > 1$: $sign\left(\frac{r(i)}{r(j)} - \frac{c(i)}{c(j)}\right)$	-
$d\gamma > 0$	IR	-	-	$\gamma > 1$: $sign\left(\frac{r(i)}{r(j)} - \frac{c(i)}{c(j)}\right)$	$+$
$dI > 0$	0	0	0	0	$+$
$dc, dr > 0$	0	0	0	0	0

Table 1: Comparative statics: CES

The last row of Table 1 considers the case of either a common cost (or quality) shock, where $c(i) = c$ *(*$r(i) = r$*) and* $dc > 0$ *(*$dr > 0$*).*

Equilibrium market shares $p(i)$ are invariant to dI, dc, dr, and it follows from the proof of Proposition 5 that $\Pi(i)$ is invariant to dc, dr as well. Stronger substitutes ($d\sigma > 0$) makes competition more intense which, intriguingly, lets already rich firm expand their market size and may marginalize small firms further. An interesting observation pertains to relative prices. In general, $P_i \gtreqless P_j$ is possible, meaning that relative prices can move quite independently from market shares and relative profits (and vice-versa) in this model. With linear production costs ($\gamma = 1$), we obtain that $\frac{P_i}{P_j} = \frac{c(i)}{c(j)} \leq 1$, showing that in this case relative prices do not depend on the substitution parameter σ at all. Similarly, prices are invariant to the distribution of quality $r(i)$. Moreover, if heterogeneity is entirely driven by quality ($c(i) = c(j)$), then even $P_i = P_j$ for any quality distribution. This changes with nonlinear production ($\gamma > 1$), because then $P_i > P_j$ whenever $c(i) = c(j)$ but $r(i) > r(j)$, and the change in relative price given $d\sigma$ depends entirely on the quality-cost ratios.

Heterogeneous production elasticities We now consider the more challenging case, where heterogeneity originates from different cost elasticities γ_i, and discuss separately the cases of a common efficiency shock[26] ($dc < 0$) and a positive income shock ($dI > 0$). Contrary to the homogeneous elasticity case, neither income nor efficiency are neutral as both shocks now imply an OR of $p(i)$ as well as an increase in relative profits and prices. Moreover, $dc < 0$ generates winners ($d\Pi(i) > 0$) and losers ($d\Pi(j) < 0$), and increasing income yields quantity growth together with a growing inequality of market shares.

[26] An alternative interpretation is that each q_i is produced with a single-input technology, where c is the competitive factor cost.

For simplicity, we set $r(i) = 1$ and $c(i) = c > 0$ $\forall i \in [0,1]$, and assume that $\gamma(\cdot) \geq 1$ is an increasing, non-trivial finite step function. Moreover, we assume parameters such that $q(i) \geq 1$ $\forall i \in [0,1]$,[27] which by optimality (22) and (20') then assures that $p(i) > p(j)$ for $j \rhd i$, such that $p(i)$ is class I. This also implies that $P_i < P_j$ and $q_i > q_j$ for any $j \rhd i$. Because for any given $T > 0$, (22) implies that $\frac{\partial p(i;T)}{\partial I}, \frac{\partial p(i;T)}{\partial c} < 0$, $T'(I), T'(c) < 0$ follow. Evaluating (15) for (22) and $x = c$ analogously gives

$$sign(R_{ij}) = sign\left(A(i) - A(j)\right) = sign\left((\gamma_j - \gamma_i)\frac{T'(c)}{T}\right), \qquad j \rhd i,$$

showing that $dc < 0$ together with heterogeneous elasticities causes an OR of $p(\cdot)$ by Theorem 3. Using (20') in (22) and rearranging yields

$$\frac{\eta - 1}{\eta}\left(\frac{I}{T}\right)^{\frac{1}{\eta}} = \gamma_i c q_i^{(\gamma_i - 1) + \frac{1}{\eta}} \qquad \text{and} \qquad \gamma_i q_i^{(\gamma_i - 1) + \frac{1}{\eta}} = \gamma_j q_j^{(\gamma_j - 1) + \frac{1}{\eta}}. \tag{23}$$

Because $dc < 0$ triggers an OR and $dT > 0$, (20') implies that $dq_i > 0$ for some i and, by the 2nd equation in (23), hence $dq_i > 0$ $\forall i \in [0,1]$. But $dq_i > 0$ further implies, again by (20'), that $\eta\Delta_i + \Delta_T > 0$, hence also $d(p(i)T) > 0$, which by $p(i)T = P_i^{1-\eta}$ assures that $dP_i < 0$. This shows that $dc < 0$ increases all quantities supplied and decreases all equilibrium prices, in a way that relative quantities $\frac{q_i}{q_j}$ (use (20')) and relative prices $\frac{P_j}{P_i}$ must increase. Because $\Pi(i) = p(i)I\left(\frac{\gamma_i\eta - \eta + 1}{\gamma_i\eta}\right)$ it follows that $\frac{\Pi(i)}{\Pi(j)}$ increases, too. Since, by the OR, $p(i)$ increases for $i < i_0$ and decreases for $i > i_0$, there must be winners and losers, where firm i wins iff its equilibrium market share increases. This occurs, in contrast to the homogeneous elasticity case, because $dc < 0$ allows large firms to decrease their prices by relatively less (P_i/P_j increases), reflecting their relative advantage in the production process, while their quantity q_i expands by relatively more. It follows that in such a CES economy, the most dominant firms would also have the strongest incentive to innovate, if this allows to reduce c, even if the innovation spills over to their competitors, while laggards are always hurt by such innovations. The case $dI > 0$ is more intricate, and we summarize our main result as a proposition.

Proposition 6 *An increase in income $dI > 0$ triggers an OR of $p(\cdot)$, increases all quantities q_i and also increases relative quantities $\frac{q_i}{q_j}$, relative profits $\frac{\Pi_i}{\Pi_j}$, relative market shares $\frac{p(i)}{p(j)}$ and*

[27] This can be assured, e.g., by assuming that I is large (or c small) enough.

relative prices $\frac{P_j}{P_i}$ *for any* $j \rhd i$.

4.1.2 Competition with Logit-Demand

In this section we analyze the distributional comparative statics implied by the Logit model with heterogeneous quality. In this model, expected demand from a given consumer is a choice probability system (Anderson et al., 1992). Specifically, we consider the Logit demand-system of a single consumer (or a unit mass of identical consumers) with linear utility and Logit noise parameter $\lambda > 0$, single-product suppliers, and no outside option.[28] As in the last section, firms $i \in [0,1]$ offer possibly different qualities, parametrized by $a(i) \geq 0$. A firm's market share in this model is its selling chance

$$p(i) = \frac{e^{\lambda(a(i)-P_i)}}{\int e^{\lambda(a(s)-P_s)}ds}, \tag{24}$$

where P_i is the price of product i. The market share $p(i)$ depends negatively on P_i and positively on $a(i)$. Assuming risk neztral firms, a constant production cost $c(i)$ of the good, and setting $T \equiv \int e^{\lambda(a(s)-P_s)}$, the (expected) profit of firm i is

$$\Pi_i = P_i p(i) - c(i)p(i) = \left(a(i) - \frac{Ln(p(i)T)}{\lambda}\right) p(i) - c(i)p(i). \tag{25}$$

Let $a(\cdot)$ be decreasing and $c(\cdot)$ increasing, such that $p(i)$ is either of class I or II. Each firm chooses its market share (its price subject to demand (24)) to maximize profits. The FOC of this problem are[29]

$$a(i) - \frac{Ln(p(i)T)}{\lambda} - \frac{1}{\lambda} = c(i) \tag{26}$$

from which

$$p(i) = \frac{e^{\lambda(a(i)-c(i))}}{eT} = \frac{e^{\lambda z(i)}}{eT}, \qquad z(i) \equiv a(i) - c(i). \tag{27}$$

Hence

$$\frac{p(i)}{p(j)} = e^{\lambda(z(i)-z(j))} \tag{28}$$

[28]In Online Appendix B.2 we show that the distributional comparative-statics of the Logit model with outside option are similar. Moreover, if there is a measure $L > 0$ of iid consumers, then $p(i)L$ would be the fraction of consumers served by firm i. With respect to the distributional outcome, setting $L = 1$ is wlog.

[29]Exponentiating (26), one can easily see that this model verifies (A1) and (A2), and hence a unique equilibrium exists by Theorem 1.

and

$$\frac{\partial}{\partial\lambda}\frac{p(i)}{p(j)} = e^{\lambda(z(i)-z(j))}(z(i)-z(j)),$$

where the assumptions made on $a(i), c(i)$ assure that $z(i)-z(j) > 0$ for any $j \rhd i$. This shows that the ordered ratio property holds with the Logit, and an increase of λ yields an OR of $p(\cdot)$. The parameter $\lambda > 0$ controls the noise in the logit. In the degenerate case where $\lambda = 0$, the choice process is purely random in such that neither price nor quality have any influence on choice probabilities, and uniform market shares ($p(i) = 1$) result in any equilibrium. An increase in λ means that the price, i.e., the deterministic part of utility, becomes more decisive, which always is associated with an increasing inequality of the market shares. This finding is similar to the CES-case, where an increase in the substitution elasticity (also making prices "more decisive") leads to an OR. It follows from (26) that $P_i = \frac{1}{\lambda} + c(i)$, showing that prices always decrease in λ, and

$$sign\ \left(\frac{\partial}{\partial\lambda}\frac{P_i}{P_j}\right) = sign\ (c(i) - c(j)) \leq 0$$

Because $\Pi(i) = \frac{p(i)}{\lambda}$, relative payoffs increase in λ while industry profits $\int \Pi(s)ds = \frac{1}{\lambda}$ decrease; hence $d\lambda > 0$ always generates losers ($d\Pi(i) < 0$ must hold for a positive-measure set of firms). The interesting converse is that the weakest firms (in terms of low quality or high costs) have the strongest incentive to increase the "noisiness" in the choice procedure ($d\lambda < 0$), e.g., by resorting to obfuscation tactics (Ellison and Wolitzky, 2012; Hefti, 2016a).

Finally, we prove that $d\lambda > 0$ *can* generate winners, provided that there is sufficient firm heterogeneity. From

$$\Pi(i) = \frac{1}{\lambda \int e^{\lambda(x(s))}ds}, \qquad x(s) \equiv z(s) - z(i) \leq 0,$$

we obtain that

$$sign\ \left(\frac{\partial\Pi(i)}{\partial\lambda}\right) = sign\ \left(-\int e^{\lambda(x(s))}ds - \lambda\int x(s)e^{\lambda(x(s))}ds\right)$$

With symmetric firms we must have $\frac{\partial\Pi(i)}{\partial\lambda} < 0$ as then $x(s) = 0$. To see that profits of the best firms can increase in λ, consider the two-types case with $x(0) = 0$ and $x(1) = z(1)-z(0) = C < 0$. If $\alpha \in (0,1)$ is the fraction of strong types, then $\partial_\lambda\Pi(0) > 0$ iff $\alpha < -(1-\alpha)(1+\lambda C)e^{\lambda C}$. We can always find $\alpha \in (0,1)$ small enough, such that this inequality is satisfied, provided that

$\lambda C < -1$ (enough heterogeneity). Note that because $\partial_\lambda \int \Pi(i)di < 0$, the losses of the poor must always outweigh the gains of the rich.

Import competition and taxation In this variation, we study the distributional consequences that an import tax has in the domestic market for home firms and importers using the Logit framework.[30] Suppose that the domestic market for the products is composed of home ("H") and foreign ("F") firms, that export their products into home at an import tax of τ. Let $i_H \in [0,1]$ and $i_F \in [0,1]$ index a home and foreign firm, respectively. Then the market share of firm i_χ in the home market is

$$p(i_\chi) = \frac{e^{\lambda(a(i_\chi)-P_s^\chi)}}{\int e^{\lambda(a(s_H)-P_s^H)}ds_H + \int e^{\lambda(a(s_F)-P_s^F)}ds_F}, \qquad \chi \in \{H, F\},$$

where P_s^H (P_s^F) is the price of firm s_H (s_F), and $\int p(i)di = \int p(i_H)di_H + \int p(i_F)di_F = 1$. With $T \equiv \int e^{\lambda(a(s_H)-P_s^H)}ds_H + \int e^{\lambda(a(s_F)-P_s^F)}ds_F$ we obtain

$$\begin{aligned} \Pi(i_H) &= \left(a(i_H) - c(i_H) - \tfrac{Ln(p(i_H)T)}{\lambda}\right) p(i_H), \\ \Pi(i_F) &= \left(a(i_F) - (\tau + c(i_F)) - \tfrac{Ln(p(i_F)T)}{\lambda}\right) p(i_F), \end{aligned}$$

which have the same formal structure as (25). It follows that (26) - (28) logically apply also to this version of the model, where we replace $c(i)$ by $c(i_F) + \tau$ if $i_F > 1/2$.[31] Suppose that heterogeneity is such that $p(i)$ is class I, and consider a small tax increase $d\tau > 0$ which does not change the ranking expressed by $p(i)$. By (28) we see that such a change has no effects on the relative market shares $\frac{p(i_H)}{p(j_H)}$ of domestic firms nor on the relative domestic market shares of foreign firms $\frac{p(i_F)}{p(j_F)}$, but the relative market shares $\frac{p(i_H)}{p(i_F)}$ of domestic firms to foreign firms increases in τ (independent of whether $i_H \triangleright i_F$ or $i_F \triangleright i_H$). Since by (27) we must have $T'(\tau) < 0$, it follows again from (27) that $p(i_H)$ increases for any domestic firm and $p(i_F)$ decreases for any foreign firm. It follows that all domestic firms benefit from the import tax ($d\Pi(i_H) > 0$), but in a way that leaves relative market shares and relative profits of domestic firms unaltered. This essentially follows because the tax allows domestic firms to expand their quantities. From (22) one can infer that all domestic prices P_i^H remain constant, while foreign prices P_i^F increase

[30] A similar analysis applies with CES-demand.

[31] Note that the following analysis could also be reformulated as finding the effects of an idiosyncratic change of costs, in the spirit of section 3.2.4, for a proper subset of all firms (no H-F-distinction), where $d\tau$ quantifies a non-common cost innovation or regulation.

isometrically with τ.

Finally, we note that $d\lambda \neq 0$ has similar effects as before. An additional interesting insight here is, that if importers (F-firms) are comparably strong (meaning that $z(i_F) - \tau \gg z(i_H)$), such that importing firms gain from $d\lambda > 0$, then importers have an incentive to reduce the noisiness of consumer choice, e.g. by educating consumers. The opposite holds if importers are comparably weak; we would then expect importers to blur or complicate consumer perception.

4.2 Perfect competition and general equilibrium

This section demonstrates how the distributional methods and results from previous sections can be applied to general equilibrium theory, where the distributions of firm-side and consumer-side market shares are jointly determined. We begin by studying perfect competition with price-taking firms that face and exogenous downward-sloping demand.

4.2.1 Perfect competition with exogenous demand

Suppose that price-taking firms face a given inverse demand function $P = P(\int q(i)di, x)$ with $P_1 < 0$ and $P_2 > 0$, where $q(i) \geq 0$ is firm i's quantity supplied, and x is a demand shifter. Further, $\Phi(i, q(i))$ is firm i's cost function, which is strictly concave in $q(i)$. Defining $T \equiv \int q(i)di$ and the market share $p(i) = q(i)/T$ yields the profit function $\Pi(i) = Pp(i)T - \Phi(i, p(i)T)$, with associated FOC $P(T, x) = \varphi(i, p(i)T)$. Because price-taking behavior implies an identical direct effect of $dx \neq 0$ for all firms, we have

$$A(i) = -\frac{\varphi_T(i)}{\varphi(i)} T' = -\frac{\varphi_q(i, p(i)T)p(i)T}{\varphi(i, p(i)T)} \frac{T'}{T}. \tag{29}$$

Proposition 7 *Equilibrium price P, quantities $q(i)$ and profits $\Pi(i)$ are strictly increasing in x. If $\Phi(i, q)$ is a power function of q with common and constant exponent η, then market shares, relative profits and relative quantities all are invariant to x, but $\Pi(i) - \Pi(j)$ and $q(i) - q(j)$, $j \rhd i$, both increase in x. If $\Phi(i, q) = q^{\eta(i)}$, where $\eta(i) > 1$ is an increasing finite step function and $P(T(x), x) > \eta(1)$, then $dx > 0$ induces a local OR of $p(i)$, jointly with an increase in relative profits and relative quantities.*

If $\Phi(i, q) = q^{\eta(i)}$ with heterogeneous exponents, the fact that $k_{ij} = \frac{\eta(j)-1}{\eta(i)-1} > 1$ implies, by Corollary 3, that $\frac{p(i,x)}{p(j,x)}$ increases (locally) in x for any $j \rhd i$, provided that $p(i, x)$ increases in

x. The case of iso-elastic (but possibly heterogeneous) costs is very relevant. For example, if the output $q(i)$ is produced with an input vector $x(i)$, acquired on competitive factor markets, according to a production function $f(x(i))$ which has (decreasing) returns to scale, then the corresponding cost function is of the form $\Phi(i, q(i)) = q(i)^{\eta(i)}w$. Then different $\eta(i)$ amount to differences in the scale technology, where a lower $\eta(i) < \eta(j)$ means that i produces at higher returns to scale than j. Note that in such a case we could never expect to analytically solve the equilibrium system of equations, even if $P(T, x)$ were linear. Nevertheless, our tools provide us with a clear comparative-static distributional prediction.

We now show that a different class of technology, the family of exponential costs, yields an IR of $p(i)$. Let $\Phi(i, q) = e^{c(i)q}$, $c(i) < c(j)$, such that $P(T, x) = c(i)e^{c(i)p(i)T}$ and $p(i) > p(j)$, as well as $c(i)p(i) > c(j)p(j)$. Then

$$A(i) - A(j) = T' \left(c(j)p(j) - c(i)p(i)\right) < 0,$$

showing that $dx > 0$ triggers an IR of $p(\cdot)$. It is easy to check that if $\Phi(i, q) = c(i)e^{q}$, the same conclusion results.

A second example of costs with which $dx > 0$ yields an IR are polynomial costs. Suppose that

$$\Phi(i, q(i)) = c(i) \sum_{s=1}^{m} a_s q(i)^s, \qquad m > 1,\, a_s \geq 0, \tag{30}$$

such that costs are scaled polynomials of each other. Note that if $a_\sigma > 0$ for a single $\sigma > 1$ and $a_s = 0\ \forall s \neq \sigma$, then (30) is monomial, and $dx > 0$ has no effect on $p(i)$ by Proposition 7. The following proposition shows that in all other cases, $dx > 0$ yields an IR of $p(i)$.

Proposition 8 (Polynomial costs) *If $\Phi(i, q(i))$ is given by (30), $c(i)$ is an increasing step function and $a_s, a_{s'} > 0$ for at least two $s \neq s'$, then $dx > 0$ induces a local IR of $p(i)$.*

4.2.2 Taxation effects in a simple partial equilibrium model

Suppose that each consumer $i_c \in [0, 1]$ has quasilinear utility $U(i_c) = m(i_c) + V(i_c, q(i_c)^d)$, where $m(i_c)$ is the quantity of the numeraire ("money"), and $q(i_c)^d \geq 0$ is the quantity of the consumption good with price P. Moreover, $V(i_c, \cdot)$ is a strictly increasing and strictly concave C^2-function, and $v(i_c, q) \equiv \partial_q V(i_c, q)$. To avoid boundary problems, we let $m(i_c) \in \mathbb{R}$. Each consumer is endowed with $\omega_{i_c} > 0$ of money, where $\omega = \int \omega_{i_c} di_c$. Each firm $i \in [0,1]$ produces output $q(i)$ with a technology $q(i) = f(i, z(i))$, where $z(i)$ is the amount of the numeraire used as input, acquired on a competitive factor market, and $f(i,\cdot)$ is a C^2-function with $f(i,0) = 0$, $f_q(i,q) > 0$ and $f_{qq}(i,q) < 0$ (strict concavity), with (q-)inverse $f^{-1}(i, f(i,z)) = z$. Hence $\Pi(i) = Pp(i)T - \Phi(i, p(i)T)$ as in the last section, where $\Phi(i, q_i) = f^{-1}(i, z(i))$ is strictly increasing and strictly convex, and $T = \int q(i)di$. Each firm maximizes its profits, and each consumer maximizes her utility, subject to $Pp^d(i_c)T^d + m(i_c) = \omega_{i_c}$, where $T^d = \int q(i_c)^d di_c$, and $p^d(i_c) = q^d(i_c)/T^d$ is consumer i_c's share of total market demand.[32] An equilibrium consists of two densities $p(i), p^d(i) > 0$ with $\int p(i)di = \int p^d(i)di = 1$ and a quantity $T = T^d$ such that the following system of FOC's is satisfied:

$$\begin{aligned} P &= v(i_c, p^d(i_c)T) \qquad & i_c \in [0,1] \\ P &= \varphi(i, p(i)T) \qquad & i \in [0,1] \end{aligned} \tag{31}$$

Because all agents are price-takers, a shock to one market side affects the market shares of the other side only over the indirect-aggregative effect. We illustrate the consequences of this by means of a tax example. Suppose that τ is a quantity tax levied on the supply side. Hence (31) becomes

$$\begin{aligned} P - \tau &= \varphi(i, p(i)T) \\ P &= v(i_c, p^d(i_c)T) \end{aligned}$$

Note that $T'(\tau) < 0$.[33] An introduction (or increase) of τ has the following distributional effects. For firms: Note that $A(i)$ is given by (29), hence the results from last section apply. The tax has differential implications for firm-side market shares only if costs are not a common-elastic power function. If $\varphi(i,q)$ has a q-elasticity which is increasing (decreasing) in i, then an increase in the tax causes an IR (OR) of $p(i)$, while a decrease of τ (or the introduction of a subsidy) has opposite effects. For consumers: The tax affects prices and equilibrium quantities, where

$$A^d(i_c) = \frac{T'(\tau)}{T}\left(-\frac{v_q(i_c, p(i_c)T)p(i_c)T}{v(i_c, p(i_c)T)}\right),$$

[32] By quasilinearity, we need not worry about the firm ownership, and therefore have not included corporate shares in the budget constraints.

[33] This intuitive result can formally be derived as follows. $T = \int \varphi^{-1}(i, P - \tau)di \equiv T(P,\tau)$, and $T^d = \int v^{-1}(i_c, P)di_c \equiv T^d(P)$, where $T_P(P,\tau) > 0$, $T_\tau(P,\tau) < 0$ and $T^d_P(P) < 0$. Because $T = T(P,\tau) = T^d(P) = T^d$ we have $P'(\tau) > 0$, and thus $T'(\tau) < 0$ because $T^d_P(P) < 0$.

and thus (note that $\eta^d_{i_c}(\cdot) < 0$ by concavity)

$$R^d_{i_c j_c} = \frac{1}{-\eta^d_{i_c}}(A^d(j_c) - A^d(i_c)) = \frac{1}{\eta^d_{i_c}}\left(\frac{v_q(i_c, p(i_c)T)p(i_c)T}{v(i_c, p(i_c)T)} - \frac{v_q(j_c, p(j_c)T)p(j_c)T}{v(j_c, p(j_c)T)}\right)\frac{T'(\tau)}{T}$$

Thus the tax has consequences for $p^d(i_c)$ only if $V(i_c, q)$ is not a common-elastic power function. A standard example in the quasilinear framework is Log-utility $V(i_c, q) = a_{i_c} Ln(1+q)$, $a_{i_c} < a_{j_c}$. In this case an increase in the tax yields an IR of $p^d(i)$, i.e., $d\tau > 0$ reduces consumption inequality of the good among consumers. A subsidy therefore tends to increase consumption inequality. A similar finding holds with an efficiency shock. Suppose that $\varphi(i, q) = c(i)\varphi(q)$, $c(i) \leq c(j)$, and now consider a process innovation leading to a lower cost coefficient $\hat{c}(i) < c(i)$ $\forall i \in [0, 1]$. Because $P \int \frac{1}{c(i)} di = \int \varphi(p(i)T)di \equiv G(T)$, where $G'(T) > 0$, we have $T = G^{-1}\left(P \int \frac{1}{c(i)} di\right)$. Therefore, the aggregate supply increases in the innovation, and we have $\hat{T} > T$ in equilibrium because aggregate demand decreases in P. It follows that, with Log-utility, the innovation leads to an OR of $p^d(i)$, i.e., an increase in the consumption inequality, similar to a subsidy.

4.2.3 General equilibrium: two applications

Decreasing resources Consider a single input-output private ownership economy; firms and consumers are indexed as before. Consumers sell their production resources to firms and acquire the consumption good at a price P with their income.[34] All markets are competitive. Let $\omega(i_c) > 0$ denote the resource endowment of consumer i_c, and $S(i_c) = s(i_c)\Pi$ is consumer i_c's share of aggregate profits $\Pi = \int \Pi(j)dj$. Further $\omega = \int \omega(i_c)di_c$ and $\int s(i_c)di_c = 1$, where both $\omega(\cdot)$ and $s(\cdot)$ are (weakly) decreasing functions. This allows us to distinguish between income inequality caused either by differences in resource or capital endowment. Every consumer i_c uses his entire income for consumption, yielding a demand ($q(i_c) = p^d(i_c)T^d$)

$$p^d(i_c)T^d = \frac{\omega(i_c) + S(i_c)}{P}. \tag{32}$$

Aggregate demand is $T^d = (\omega + \Pi)/P$. Further, the production process of each firm is described by a cost function as in section 4.2.2. Because firms behave as price-takers, firm FOC are given by (31) (2nd equation), and $A(i)$ by (29). The difference to the partial equilibrium model

[34] Consumers care only about the consumption good.

is that now demand and the (relative) consumption price depend on capital income (profit), which in turn depends on demand. In the following we consider the comparative statics if $d\omega(i_c) = d\omega < 0 \; \forall i_c \in [0,1]$, i.e., there is a uniform contraction of the available resources. We assume cost functions and $\omega(\cdot), s(\cdot)$ to be such that $p(\cdot), p^d(\cdot)$ are of class I.

Proposition 9 (Firms) *In equilibrium:* $T'(\omega), P'(\omega) > 0$ *and* $\partial_\omega \Pi(i), \partial_\omega q(i) > 0$. *If the q-elasticity of* $\varphi(i,q)$ *is strictly increasing (decreasing) over firm equivalence classes,* $d\omega < 0$ *induces a local IR (OR) of* $p(i)$. *If* $\Phi(i, q(i)) = c(i)q(i)^\eta$, *then* $p(i)$ *is invariant to* ω.

While all firms supply less to the market as the scarcity of the resource increases, it depends in a clear way on the production possibilities whether market concentration increases or decreases. Additionally, it can be shown that if production functions are heterogeneous power functions (such that $\eta(i)$ is increasing), the resulting IR caused by $d\omega < 0$ is passed on to the shares of the firms in the factor market. Specifically, $d\omega < 0$ induces an IR of the firm's relative shares in the factor market.

The next proposition shows how consumption inequality depends on ω. We consider the two extreme regimes, where consumption inequality originates from i) the resource endowment allocation or ii) from the share allocation. The main result is that if consumption inequality is driven by distribution of the resource endowment, then a resource contraction increases consumption inequality. If instead inequality originates mainly from the dispersion of capital holdings, then the distributional pattern depends on aggregate profits and thus on aggregate production possibilities.

Proposition 10 (Consumption) *(i) If* $s(i_c) = 1 \; \forall i_c$, *then* $dw < 0$ *induces a global OR of* $p^d(\cdot)$ *with monotone ratios, and relative consumption* $q^d(i_c)/q^d(j_c)$ *increases. (ii) If* $\omega(i_c) = \omega \; \forall i_c$, *then* $dw < 0$ *induces a global OR (IR) of* $p^d(\cdot)$ *with monotone ratios, if and only if* $\Pi(\omega)$ *is strictly concave (convex). Moreover,* $q^d(i_c)/q^d(j_c)$ *increases (decreases) if and only if* $d\omega < 0$ *leads to an OR (IR) of* $p^d(\cdot)$.

Note that if income inequality originates from the share distribution and production is best described by common-elasticity power functions, consumption shares, relative consumption, firm market shares and relative profits all become invariant to ω.

Corollary 4 *If $\omega(i_c) = \omega$ $\forall i_c$ and $\Phi(i, q(i)) = c(i)q(i)^\eta$, then $p^d(\cdot)$ and $q^d(i_c)/q^d(j_c)$ are invariant to ω, but $dq(i_c) < dq(j_c) < 0$, i.e., absolute consumption inequality decreases.*

Note that all propositions apply analogously (with "reversed signs") in case of a resource expansion $d\omega > 0$.

Labor-consumption decisions Consider a perfectly competitive private ownership economy as before, where a consumption output is produced by firms using only labor as input. Each consumer $i_c \in [0,1]$ owns a unit of perfectly divisible labor and decides between consumption $q(i_c) \geq 0$ and leisure $f(i_c) \in [0,1]$ according to utility $u(i_c) = x(i_c)^\alpha f(i_c)^{1-\alpha}$. The parameter $\alpha \in (0,1)$ measures how important consumption is relative to leisure. The consumer budget constraint is $q(i_c) = (1 - f(i_c))w + S(i_c)$, where w is the real wage and $S(i_c) = s(i_c)\Pi \geq 0$ is capital income of corporate shares. As before $s(\cdot) \geq 0$ is decreasing with $\int s(i_c)di_c = 1$, and $\Pi = \int \Pi(i)di$ are aggregate profits. It follows that optimal consumption and leisure choices are given by $q(i_c) = \alpha(w + S(i_c))$ and $f(i_c) = \frac{\alpha}{1-\alpha}\frac{q(i_c)}{w}$, respectively.[35] Let $p^d(i_c) = q(i_c)/T^d$, $T^d = \int q(i_c)di_c$ deonte i_c's share of total consumption. Each firm $i \in [0,1]$ produces $y(i) \geq 0$ units of the good using only labor as input according to a strictly concave technology, yielding a corresponding strictly convex cost function $\hat{\Phi}(i, q(i)) = w\Phi(i, q(i))$ with $\Phi(i, 0) = 0$. We suppose that $\Phi(\cdot, q)$ is such that $p(\cdot)$ is of class I (while $p^d(\cdot)$ could be of class I or II). The profit is $\Pi(i) = p(i)T - w\Phi(i, p(i)T)$, with FOC $1 = w\varphi(i, p(i)T)$. Because the wage is exogenous to each firm, $A(i)$ must be of the type (29). Consequently, the type of rotation of $p(i)$ is determined entirely by the indirect-aggregative effect of a common shock, and $R_{ij} > (<)0$ if $dT > 0$ and the q-elasticity of $\varphi(i, q)$ is strictly increasing (decreasing) in q. Hence the distributional comparative-statics of firm-side variables obey essentially the same laws as with an exogenous demand (section 4.2.1).

We now seek to analyze how the distribution of consumption and leisure across consumers depends on i) the (common) state of technology in the economy and ii) the importance of leisure as parametrized by α. To this end, we begin by setting $\Phi(i, q(i)) = c(i)q(i)^\eta$, $\eta > 1$, $c(i) > 0$, where $c(\cdot)$ is increasing. A common positive technology shock means $dc(i) < 0$ $\forall i$, i.e., production efficiency increases for all firms.[36] With this technology $R_{ij} = 0$, showing that $p(\cdot)$

[35] This interior solution requires that $S(i_c) \leq \frac{\alpha}{1-\alpha}w$ which, for simplicity, we shall assume to hold. Corresponding precise conditions can be derived, e.g., in the context of our parametric example. Moreover, the main results carry through if the boundary condition $f(i_c) \leq 1$ becomes binding (and thus $q(i_c) = S(i_c)$) for some consumers.

[36] More precisely, we mean by $dc(i) < 0$ a downward shift of the function $c(\cdot)$ in the sense that $c(i) = \hat{c}(i) + \varepsilon$

and relative profits do not depend on the state of technology or α. We summarize the main distributional results in the following proposition, where always $j \rhd i$ and $j_c \rhd i_c$.

Proposition 11 *(i) A common increase in efficiency ($dc(i) < 0$) increases all equilibrium consumption levels $q(i_c)$, the wage and all profits $\Pi(i)$, while $p^d(i_c)$, $p(i)$, $\frac{q(i_c)}{q(j_c)}$, $f(i_c)$ (and labor supply) and relative profits all are invariant to the state of technology, but $dq(i_c) > dq(j_c)$ and $d\Pi(i) > d\Pi(j)$, showing that absolute consumption and profit inequality increases in efficiency. (ii) $d\alpha > 0$ induces an OR of $p^d(i_c)$, with monotone ratios, and increases $\frac{q(i_c)}{q(j_c)}$ and $\frac{f(i_c)}{f(j_c)}$ as well as all profits, aggregate consumption and aggregate labor supply, while wages fall. Firm market shares $p(i)$ and relative profits are invariant to α. Finally, $d\Pi(i) > d\Pi(j)$, and $dq(i_c) > dq(j_c)$ whenever $dq(i_c) > 0$.*

The intuition in case of an efficiency increase is that this allows firms to produce at lower costs (equivalently: more from a given input), yielding higher supply, higher profits, higher real wages and more consumption. As profits and wages increase proportionally, the incentives to work more and benefit from the higher wage or to rather enjoy more leisure and finance consumption from the higher capital income counterbalance each other, leading to a constant labor supply and constant consumption shares. Because consumption levels increase, $dq(i_c) > dq(j_c)$ follows.

The intuition for $d\alpha > 0$ is that if consumption is more important, consumers want to supply more labor to afford more consumption, which reduces real wages and increases profits, which benefits capital owners and therefore increases consumption inequality. Additionally, one can show that because real wages plunge the poorest may even end up with a *lower* consumption level despite that $d\alpha > 0$ increases the propensity to consume more.[37]

We now show that the result on $d\alpha$ generalizes beyond the case where costs are common-elasticity functions. Suppose that $\Phi(i, q(i))$ is not restricted beyond the assumption stated in the beginning of this paragraph, and consider $d\alpha > 0$. Then the results on $p^d(i_c)$, $q(i_c)/q(j_c)$, $f(i_c)/f(j_c)$, w, T and aggregate labor supply in Proposition 11 remain valid provided that $T \leq w$.[38] To see this, write firm FOC as $p(i)T = \varphi^{-1}(i, 1/w)$. Integration yields

$$T = \int \varphi^{-1}(i, 1/w) di \equiv H(w),$$

and $d\varepsilon < 0$.

[37]If $\Phi(i, q(i)) = c(i)q(i)^\eta$ and a positive mass of consumers holds no shares at all($s(i) = 0$) this is the case iff $\alpha(\eta - 1) > 0$

[38]If $\Phi(i, q(i)) = c(i)q(i)^\eta$ it can be shown that indeed $T \leq w$ must hold in equilibrium.

with $H'(w) < 0$. Setting $T^d = T$ implicitly defines $w(\alpha)$ by

$$H(w(\alpha)) = \alpha(w(\alpha) + \Pi(w(\alpha))). \tag{33}$$

By the Envelope Theorem $-\partial_w \Pi(i) = \Phi(i, q(i)) < q(i)\varphi(i, q(i)) = q(i)/w$, where the inequality follows from the strict concavity of $\Phi(i, \cdot)$ and $\Phi(i, 0) = 0$, and the equality from firm FOC. Integration yields $-\Pi < T/w$. By (33) we have

$$w'(\alpha) = \frac{w + \Pi}{H'(w) - \alpha(1 + \Pi'(w))}.$$

The denominator is negative if $T \leq w$, because then $1 + \Pi'(w) \geq 0$, which assures that $w'(\alpha) < 0$. Then $T'(\alpha) > 0$, and the remaining claims follow from the proof of Proposition 11.

4.3 Contests

Many real-world competitions can best be described as *contests* for scarce goods. In a contest, a number of agents can invest efforts of some kind to win a prize. The prize could be, e.g., obtaining a research grant, winning a political election, or winning a sport championship. In (imperfectly discriminatory) contests, individual effort levels and winning chances are positively correlated but, other than in auctions, the highest effort level does not win the contest for sure (Konrad, 2009). Further, there are some cases where we would expect the value of the prize itself to depend on the winning or aggregate effort.

In this section we seek to study how the reward scheme in contests affects the equilibrium dispersion of success chances and related measures. A general formulation of a contest payoff function is

$$\Pi(i) = \pi\left(t(i), T\right) V\left(t(i), T, x\right) - \Phi\left(i, t(i)\right), \tag{34}$$

where $\pi(t(i), T)$, $T = \int t(i)di$, is a Contest Success Function (CSF), and $V(t(i), T, x)$ is the prize function. The CSF captures how individual efforts to seize the prize translate into winning odds, where we always assume that $\partial_t \pi(t, T) > 0$ on the relevant range. A natural benchmark assumption is that $\pi(\cdot)$ is zero-homogeneous in (t, T), i.e., doubling all effort level leaves individual success chances unchanged. Then, for $T > 0$, we let $\pi(t, T) = \hat{\pi}(t/T)$ wlog. Regarding the prize function, we assume that $V(\cdot) > 0$ is a C^2 function. The parameter x is an exogenous prize shifter with $V_x > 0$ and $V_{xx} \geq 0$. Note that this includes the standard case where $V > 0$ is a fixed prize value. We take all functions involved in (34) to be such that Assumption 1 is satisfied. A standard example is the Tullock fixed-prize contest with linear effort costs

$$\Pi(i) = \frac{\hat{t}(i)^{1/\eta}}{\int \hat{t}(s)^{1/\eta} ds} V - c(i)\hat{t}(i),$$

where $\eta \in (0, 1)$ quantifies the degree of noise in the Tullock CSF (higher η means more randomness in the CSF). Using the monotone transformation $t(i) \equiv \hat{t}(i)^{1/\eta}$, we obtain the equivalent representation of a linear Tullock CSF with iso-elastic costs

$$\Pi(i) = \frac{t(i)}{T} V - c(i)t(i)^{\eta}. \tag{35}$$

The distributional comparative-statics of the contest model (34) can be analyzed by the framework of this article, using the transformation $p(i) = \hat{\pi}(t(i)/T)$. Then

$$\Pi(i) = p(i)V\left(\hat{\pi}^{-1}(p(i))T, T, x\right) - \Phi\left(i, \hat{\pi}^{-1}(p(i))T\right) \tag{34'}$$

There are two conceivable interpretations of $p(i)$. First, $p(i)$ is agent i's probability to seize a single prize worth $V(\cdot)$. Second, $p(i)$ is the market share of agent i, and $V(\cdot)$ is the value of the market share to the agent. The two models are formally equivalent, while the notion of a "winner" makes sense in the first interpretation.

Our first result shows that a prize shift induces distributional effects only if $V(\cdot)$ depends on i (through $p(i)$) or $\Phi(i, t(i))$ is not a common-elastic power function.

Proposition 12 *Let $\pi(t(i), T)$ be a zero-homogeneous CSF. If the prize function is $V(T, x)$ the distributional patterns of $p(i)$ are determined by the t-elasticity of marginal costs alone. If additionally $\Phi(i, t) = c(i)t^{\eta}$, then $p(i)$, $\Pi(i)/\Pi(j)$ as well as $t(i)/t(j)$ are invariant to x.*

As with quantity competition, if the q-elasticity of marginal costs increases (decreases) over agent equivalence classes, $dx > 0$ induces an OR (IR) of $p(i)$. Because the Tullock fixed-prize contest (35) is a special case of Proposition 12, an exogenous change of the prize dV has no distributional effects other than $dt(i) > dt(j) > 0$ as well as $d\Pi(i) > d\Pi(j) > 0$. The derivation of (34') shows that η parametrizes the noise of the CSF in a Tullock contest with an agent-independent prize, and an increase in noise always induces an IR of $p(i)$.

Corollary 5 *In the Tullock contest with* $\Pi(i) = p(i)V(T) - c(i)p(i)^{\eta}T^{\eta}$, *where* $c(\cdot)$ *is such that* $p(\cdot)$ *is of class I or II,* $d\eta > 0$ *induces a global IR of* $p(i)$, *with monotone ratios, and* $\Pi(i)/\Pi(j)$ *as well as* $t(i)/t(j)$ *both increase in* η.

Hence an increase in randomness of the contest tends to benefit weak contestants more, which is reminiscent of the finding with Logit demand in section 4.1.2.

In some circumstances the market value $V(\cdot)$ may be thought of depending positively or negatively on the agent's effort level. For example, the intensity of political lobbying (Konrad, 2009), litigation expenditures (Posner, 1992), salary negotiations (Amegashie, 1999) or money invested to obtain a monopoly franchise (Chung, 1996) can influence the final conditions of the winning agent. Similarly, $V(\cdot)$ can express the market value of limited consumer attention to a firm (Hefti, 2016a). To unshroud the effects of the $t(i)$-sensitivity of $V(\cdot)$, we let $V(i) = \alpha t(i) + \beta$. In the advertising example, $\alpha \neq 0$ means that advertising intensity $t(i)$ determines market share, e.g., the fraction of consumers paying attention to i, but $t(i)$ further affects the willingness to pay of attentive consumers, e.g., by strengthening the attachment or joy experienced by consuming a brand ($\alpha > 0$),[39] or by increasing the nuisance or intrusion felt by ad exposure ($\alpha < 0$) (Johnson, 2013; Hefti and Liu, 2016).

Proposition 13 *Suppose that* $V(t(i)) = \alpha t(i) + \beta$, *where* $\beta > 0$ *and* $\alpha \gtreqless 0$. *Let* $\pi(t(i), T)$ *be a zero-homogeneous CSF, and* $\Phi(i,t) = c(i)t^{\eta}$, $\eta > 1$, *where* $c(\cdot)$ *is such that* $p(\cdot)$ *is class I. Let* $z(p) \equiv \hat{\pi}^{-1}(p)$. *Then* $d\alpha > 0$ *induces a local OR (IR) if* $\forall p > 0$: $\frac{z''(p)p}{z'(p)} > (<) - 2$. *Further,* $d\beta > 0$ *induces a local OR if either* $\alpha > 0$ *and* $\frac{z''(p)p}{z'(p)} < -2$ *or if* $\alpha < 0$ *and* $\frac{z''(p)p}{z'(p)} > -2$.

A corollary is that in the Tullock contest ($z''(p) = 0$), $d\alpha > 0$ induces an OR with monotone ratios (and an increase in relative efforts and payoffs), while $d\beta > 0$ leads to an OR if $\alpha < 0$. Because $T'(\alpha) > 0$, Proposition 13 reveals an interesting efficiency-equity trade-off. For example, if α is a design variable, say in a sport tournament, and the CSF is best described by the Tullock formula, then increasing the effort sensitivity of the prize function induces more unequal winning chances (the same teams tend to win all the time), but also increases aggregate effort.[40] Conversely,

[39] In such a case, advertising has a persuasive or complementary nature (Bagwell, 2007).

[40] In practice, this could mean making the prize function more or less expenditure-dependent, because individual efforts typically are not directly observable.

if the goal is to make the competition less predictable (IR of $p(\cdot)$),[41] then a reduction of total effort is unaviodable.

In terms of real-world prediction, in the advertising example we expect to observe more market concentration at higher levels of advertising affinity ($\alpha > 0$) and less market concentration with annoying advertising ($\alpha < 0$) provided that $\hat{\pi}^{-1}(p)$ is not too concave (otherwise the reverse prediction applies). The fact that the curvature of $\hat{\pi}^{-1}(p)$ plays a role is quite intuitive as we exemplify in case of $\alpha > 0$. An increase in α increases marginal revenues of advertisers, but more so for high intensity advertisers ($t(i)$ high). If $\hat{\pi}^{-1}(p)$ is convex, then marginally increasing the market share is easier the higher the current level of market share is. Hence $d\alpha$ has a stronger incentive effect to increase the market share (advertising) for already strong advertisers, leading to an OR of $p(\cdot)$.

Two-prize contests As a final variation we show that introducing (or increasing) a second prize in a contest tends to make the chances to win the first prize more equal. Contests with multiple prizes have received attention in the contest architecture literature, but typically only the case of symmetric agents is studied.[42] Consider a contest with two fixed prizes $V_1 > 0$ and $V_1 \geq V_2 \geq 0$. Suppose that there is a number n of atomistic agents.[43] Expected profit of i, given an effort profile $(t(1), ..., t(n))$, is

$$\Pi(i) = \frac{t(i)}{T} V_1 + \frac{t(i)}{T} \sum_{j \neq i} \frac{t(j)}{T - t(j)} V_2 - \frac{1}{\eta} c(i) t(i)^{\eta} \tag{36}$$

Then $p(i) = t(i)/T$ is the probability to win the first prize. It can be verified by our standard approach that the overall prize composition (V_1, V_2) has no distributional effects on $p(i)$, provided that there are only two cost types. It turns out that this is not valid in general. Already with three cost types it is possible to construct examples where $p(i)$ varies with the prize composition.

The general difficulty with (36) is that if there are more than two cost types, (36) is a function not only of $t(i)$, but also of all $t(j)$ with $j \neq i$. By imposing the simplifying assumption

[41] In sports economics, the idea of increasing the competitive balance between two agents, i.e. reducing $p(i)/p(j)$, is a very important concern of optimal tournament designer (Szymanski, 2003).

[42] For example Clark and Riis (1998) consider the case of a multi-prize contest with symmetric contestants. Their main interest is aggregate effort, and they find that highest aggregate effort requires to award only one prize.

[43] It is easier to set up the problem with discrete agents. The respective optimality conditions can then be easily given a "continuum interpretation" such that our distributional tools can be applied.

of sampling with (instead of without) replacement for the second prize, the two-prize contest fits into the present framework, and (36) becomes

$$\Pi(i) = \frac{t(i)}{T} V_1 + \frac{T - t(i)}{T} \frac{t(i)}{T} V_2 - \frac{1}{\eta} c(i) t(i)^\eta, \tag{37}$$

where $\frac{T-t(i)}{T}\frac{t(i)}{T}$ is the chance of i to win the second prize.[44] Replacing $t(i)$ by $p(i)T$ in (37) and differentiating with respect to $p(i)$ yields the FOC

$$V_1 + (1 - 2p(i))V_2 = c(i)p(i)^{\eta-1}T^\eta. \tag{38}$$

It turns out that with (38) we can obtain a characterization of how changes in the prize scheme (V_1, V_2) affect the distribution of the first-prize winning chances.

Proposition 14 *If* $\frac{dV_1}{V_1} > (<) \frac{dV_2}{V_2}$ *a local OR (IR) results. If* $\frac{dV_1}{V_1} = \frac{dV_2}{V_2}$*,* $p(i)$ *remains constant.*

Proposition 14 suggests that an unilateral change of a single prize in presence of another prize has distributional consequences. An unilateral increase of the first (second) prize implies more inequality (equality) in the first-prize winning chances $p(i)$, and if the overall prize money $V = V_1 + V_2$ is increased, both prizes should be increased proportionally if $p(i)$ is to be kept constant. Proposition 14 further provides insights on how changing the composition of a given overall prize sum $V = V_1 + V_2 > 0$, $V_1 \geq V_2 \geq 0$, affects $p(i)$ and thus relative efforts $t(i)/t(j)$. The fact that $dV_1 < 0$ and $dV_2 > 0$ both imply an IR suggests that the case of an even prize split $V_1 = V_2 = V/2$ generates most equality, while $V_1 = V$ (a single-prize contest) induces inequality. Moreover, there is an interesting effort-equality trade-off because at the same time aggregate effort $T(V_1, V_2)$ decreases if $V_2 = V - V_1$ increases.[45]

Analyzing how the distribution of equilibrium payoffs depends on the prizes turns out to be tricky in general. However, in the special case where $\eta = 2$ it is possible to show that $\frac{\Pi(i)}{\Pi(j)} = \frac{p(i)}{p(j)}$. It follows that relative payoffs increase (decrease) whenever $\Delta_i > \Delta_j$. Since $dV_1 > 0$ causes an OR, it follows that $\frac{\Pi(0)}{\Pi(1)}$ increases, while this ratio decreases if $dV_2 > 0$. Hence moving from a single prize two two equal prizes tends to squish payoffs together. Finally, one could also ask

[44] We would expect these sampling chances to be close to those in (36) for a large number of agents. For example, if $t(j) = t > 0$ $\forall j$ then the absolute difference between the two is of order $1/n^2$. The approximation comes at a cost, however. It follows from Proposition 14 that a change of (V_1, V_2) may have distributional effects already in the two-types case.

[45] To see that $T(V_1, V - V_1)$ decreases in V_1, use $V_2 = V - V_1$ in (38) and note that $\frac{\partial}{\partial V_1} p(i) > 0$.

how the overall chances to win *any* prize depend on (V_1, V_2). The chance of agent i to win a prize is $s(i) \equiv p(i) + (1-p(i))p(i)$, and thus $ds(i) = 2dp(i)(1-2p(i))$. Assuming that there is not so much asymmetry that $p(i) \geq 1/2$, $s(i)$ behaves like $p(i)$ since $sign\, ds(i) = sign\, dp(i)$.

4.4 Relation to Nash equilibrium

In the previous sections we analyzed a contest model under the assumption that individual agents take the aggregate $T = \int t(i)di$ as given when choosing their effort (their equilibrium market share), while T is still endogenous to the model. One justification is that contestants have a good intuition about the average (or aggregate) effort imputed in equilibrium while they do not know the cost functions (the strategies) of the competitors.[46] We now demonstrate, by means of the contest example, that our distributional tools can be successfully applied to the concept of Nash equilibrium.[47]

Consider first a fixed-prize contest with n atomistic agents and payoffs

$$\Pi(i) = \pi\left(\frac{t(i)}{\sum t(s)}\right)V - \Phi(i, t(i)) \tag{39}$$

Define the market share $p(i) = \pi(t(i)/T)$ as before. The only difference is that each agent i now takes into account its own effect on the aggregate. Let $T_i \equiv \sum_{s\neq i} t(s)$ and $z(p(i)) \equiv \pi^{-1}(p(i))$. Because $T = T_i + t(i)$ and $t(i) = z(p(i))T$ we obtain $t(i) = \frac{z(p(i))}{1-z(p(i))}T_i$. Thus we can restate (39) in terms of own market share as

$$\Pi(i) = p(i)V - \Phi\left(i, \frac{z(p(i))}{1-z(p(i))}T_i\right) \tag{39'}$$

A Nash equilibrium is a probability vector $(p(1), ..., p(n))$ and an aggregate $T > 0$ such that $T_i = (1 - z(p(i)))T$ and $p(i)$ maximizes (39'). It follows that any interior Nash equilibrium satisfies the FOC system

$$V = \frac{\varphi(i, z(p(i))T)}{1-z(p(i))}z'(p(i))T, \tag{40}$$

showing that we can apply our rotation tools to study the distributive comparative-statics as

[46] Another justification is that with continuum agents $T = \int(t(s)ds$ does not depend on $t(i)$. This type of argument was used by Dixit and Stiglitz (1977) and later by Melitz (2003) in the context of monopolistic competition with continuum firms. Hefti (2016b) shows that similar principles apply to Nash and aggregate-taking behavior in sum-aggregative games with respect to equilibrium uniqueness and stability.

[47] More generally, the following technique can be used in games with a sum-aggregative structure, such as the Cournot model.

in the earlier models. One advantage, however, of the previous contest model is that it yields a slightly more tractable structure.[48] From (40) it follows that the distributional effects of dV depend only on the elasticity of the cost function, similar to Proposition 12.

Proposition 15 *Let $\pi(t(i),T)$ be a zero-homogeneous CSF. The distributional patterns of $p(i)$ induced by dV are determined by the t-elasticity of marginal costs alone. Specifically, if $\Phi(i,t) = c(i)t^{\eta}$, then $p(i)$, $\Pi(i)/\Pi(j)$ as well as $t(i)/t(j)$ are invariant to x.*

Evaluated in the special case of a Tullock CSF with arbitrary noise parameter $\eta \geq 1$, Proposition 15 shows that the choice of V has no distributional impacts also in the Nash equilibrium. Proposition 12 is not valid if $V = V(T,x)$ and agents take into account own effects on the aggregate T. The reason is that $V(\cdot)$ then depends on $t(i)$. From Proposition 13 we know that an idiosyncratic prize function may induce rotations of $p(i)$. Nevertheless, nothing prevents us from applying the previous methods also in this case. As a final illustration, suppose that $V(T,x)$ is affine-linear in T. Then a similar finding as in Proposition 13 results.[49]

Proposition 16 *Suppose that $V = \alpha \sum t(s) + \beta$, $\Phi(i,t) = c(i)t^{\eta}$, where $\beta > 0$ and $\pi(x) = x$ (Tullock CSF). If $\alpha \geq 0$, then $d\alpha > 0$ induces an OR of $p(i)$. If $\alpha < 0$, then $d\beta > 0$ induces an OR of $p(i)$.*

In the advertising interpretation, Proposition 16 thus suggests that if aggregate advertising has a strong positive externality among the advertisers, we can expect to see very concentrated markets.

5 Conclusion

The methods developed in this paper allow to analyze how the equilibrium distribution of agent market shares and related measures depends on fundamental parameters of the respective model. The distributional predictions are directly derived in term of output variables, such as (relative) market shares or relative payoffs, rather than by the strategic variables that generated them (e.g., effort levels in contests or prices in monopolistic competition). The advantage is that the

[48] The respective FOC is $V = \varphi(i, z(p(i))T)z'(p(i))T$. For given $p(i), T$ marginal costs are thus higher if the own effect on the aggregate are taken into account. This is intuitive, because an increase in $t(i)$ also increases T which, ceteris paribus, decreases $t(i)/T$.

[49] In the following proposition we assume parameters such that payoffs are strictly quasiconcave in $p(i)$.

former is more likely to be available in empirical data. Obtaining robust comparative-static predictions, which do not hinge on ex ante restricting heterogeneity, e.g., to just two types or to a specific type of distribution may be particularly valuable to applied work, because real-world agents barely are symmetric and the extent of the ex ante heterogeneity may be unobservable. While we have applied our tools to several different models, this paper is far from comprehensive, and some of our applications may be worth a detailed separate consideration.

Our approach could provide a valuable instrument for studying questions related to distribution and welfare. For example, a planer may aim to implement a set of instruments to obtain a certain distributional outcome, e.g., because of fairness concerns or other considerations. While we did not study normative questions directly, understanding the distributional comparative-static effects of various possible instruments in a certain economic context most likely constitutes a central milestone in addressing normative distributional questions.

References

Amegashie, J. A. (1999). The number of rent-seekers and aggregate rent-seeking expenditures: an unpleasant result. *Public Choice*, 99(1-2):57–62.

Anderson, S. P., De Palma, A., and Thisse, J. F. (1992). *Discrete choice theory of product differentiation*. MIT press.

Bagwell, K. (2007). The economic analysis of advertising. In Armstrong, M. and Porter, R., editors, *Handbook of Industrial Organization*, volume 3. Amsterdan: Elsevier B.V.

Chung, T. Y. (1996). Rent-seeking contest when the prize increases with aggregate efforts. *Public Choice*, 87:55–66.

Clark, D. J. and Riis, C. (1998). Competition over more than one prize. *The American Economic Review*, 88(1):276–289.

Dixit, A. K. and Stiglitz, J. E. (1977). Monopolistic competition and optimum product diversity. *The American Economic Review*, 67:297–308.

Ellison, G. and Wolitzky, A. (2012). A search cost model of obfuscation. *The RAND Journal of Economics*, 43(3):417–441.

Gervais, A. (2015). Product quality and firm heterogeneity in international trade. *Canadian Journal of Economics*, 48(3):1152–1174.

Hefti, A. (2016a). Limited attention, competition and welfare. *Working paper*.

Hefti, A. (2016b). On the relationship between uniqueness and stability in sum-aggregative, symmetric and general differentiable games. *Mathematical Social Sciences*, 80:83–96.

Hefti, A. and Liu, S. (2016). Targeted advertising and limited attention. *Working Paper*.

Jackson, M. O. and Yariv, L. (2016). The non-existence of representative agents. *Working Paper*.

Johnson, J. P. (2013). Targeted advertising and advertising avoidance. *The RAND Journal of Economics*, 44(1):128–144.

Konrad, K. (2009). *Strategy and Dynamics in Contests.* Oxford University Press.

Melitz, M. J. (2003). The impact of trade on intra-industry reallocations and aggregate industry productivity. *Econometrica*, 71(6):1695–1725.

Mrazova, M., Neary, P. J., and Parenti, M. (2016). Sales and markup dispersion: theory and emprics. *Working paper.*

Posner, R. A. (1992). Economic analysis of law. *Boston, Toronto and London: Little, Brown and Company.*

Redding, S. (2011). Theories of heterogeneous firms and trade. *Annual Review of Economics*, 3:77–105.

Szymanski, S. (2003). The economic design of sporting contests. *Journal of economic literature*, 41(4):1137–1187.

Topkis, D. M. (1998). *Supermodularity and Complementarity.* Princeton: University Press.

A Appendix

A.1 Continuum representation for atomistic agents

We illustrate that the equilibrium distribution in case of $n \in \mathbb{N}$ atomistic ("discrete") agents can be identified with our finite step density model. The following argument considers the case, where heterogeneity enters the model through a cost coefficient function as in (5). This should suffice to make evident that the representation result applies similarly to other cases as well. Consider a population consisting of $n \in \mathbb{N}$ atomistic (or "discrete") agents, indexed by $\{1/n, 2/n, ..., 1\}$. Suppose that the agents differ in their cost coefficient $c(i)$, $i \in \{1/n, 2/n, ..., 1\}$. Then, the agents can be partitioned into $1 \leq K \leq n$ groups of identical agents, with group size n_k, $\sum_k n_k = n$. This partition gives $1 \leq K \leq n$ equivalence classes (groups) of sizes $n_1, ..., n_K$, $\sum_k n_k = n$. We identify each group by a "representative" agent i_k. In equilibrium every agent (i/n) chooses $p^d(i/n)$ (d for "discrete") to maximize her payoff $\Pi(i)$, where $p^d(i/n)$ must satisfy $\sum_{i=1}^{n} p^d(i/n) = 1$. Let $p(i)$ denote the (step) density function that characterizes our (continuum) equilibrium from definition 1 with the corresponding cost step function $c(i) = c(i_k)$ on $[i_k, i_{k+1})$, and group measures $\gamma_1, ..., \gamma_K$ satisfying $\gamma_k = n_k/n$. We now establish the formal equivalence between the discrete equilibrium probability distribution $\left\{p^d(1/n), ..., p^d(1)\right\}$ and the equilibrium step density $p(i)$.

Theorem 5 (Continuum Representation) *Let $n \in \mathbb{N}$ and suppose that agents are partitioned in K cost groups. If $\{p^d(i/n)\}$ corresponds to the discrete equilibrium and $p(i)$ is the equilibrium (step) density of the respective continuum problem, then $p^d(i/n) = \frac{1}{n}p(i/n)$ is satisfied for all $i \in \{1, ..., n\}$*

Proof: In the continuum case we only have to solve the optimization problem for a representative agents i_k. In the discrete equilibrium $1 = \sum_{i=1}^{n} p^d(i/n) = \sum_{k=1}^{K} p^d(i_k)n_k$. The claim now is that $\frac{1}{n}p(i_k) = p^d(i_k)$ for $k = 1, ..., K$. But because in the continuum equilibrium we must have

$$1 = \int_0^1 p(i)di = \sum\nolimits_{k=1}^{K} p(i_k)\gamma_k = \sum\nolimits_{k=1}^{K} \left(\frac{1}{n}p(i_k)\right) n_k$$

the claim follows from the uniqueness of equilibrium. ■

Hence the continuum step-function case and the atomistic case are equivalent up to the multiplicative constant $1/n$ (independent of group composition), which means that we can work with either model, and justifies our procedure of the main text. It then also follows that $p(i_k)\gamma_k = p^d(i_k)n_k$ corresponds to the market share of a member of group k, illustrating why we used the notion of a "representative" agent.

Because Theorem 5 remains valid as n grows arbitrarily large, this provides the following justification for using strictly increasing cost coefficient functions (class II) as an approximation for the case of many different agents. Suppose that $c(i)$ is a class II function defined on $[0, 1]$ (e.g., $c(i) = 1+i$), and let $p(i) = p(c(i))$ denote the corresponding equilibrium density (a strictly decreasing, continuous function). Then, because $c(i)$ is continuous on a compact interval, for $n \in \mathbb{N}$ the sequence of step functions defined by $c_n(i) = c(i)$ if $i \in \{0, 1/n, 2/n, ..., 1\}$ and $c_n(i) = c(s/n)$ for $i \in (s/n, (s+1)/n)$, $s \in \{0, 1, ..., n-1\}$, converges (uniformly) to $c(i)$ as $n \to \infty$.[50] Consider the atomistic equilibrium distribution $p^d(i/n)$ induced by $c(0), c(1/n), c(2/n), ..., c(1)$. By Theorem 5, $np^d(i/n) = p(i/n)$, where $p(i/n)$ is the step-density version of $p^d(i/n)$. More precisely, for a given $n \in \mathbb{N}$ this density is a decreasing finite step function with $p_n(i) = p(c_n(i))$, where $c_n(i)$ is as defined above. Because $c_n(i) \to c(i)$ and $p(i)$ is continuous, we have $p(i) = p(c(i)) = p(\lim c_n(i)) = \lim p(c_n(i)) = \lim p_n(i)$. This shows that while, of course, the atomistic $p^d(i/n)$ becomes arbitrarily close to zero as n grows large, the "scaled" distribution law as captured by the step-density version $p(i/n)$ approaches $p(i)$.

[50] Such approximations of continuous functions by a sequence of step functions are a standard result in real analysis and integration theory.

A.2 Proofs

Proof of Theorem 1 The proof consists of two steps. i) Fix $i \in [0,1]$ and $T > 0$ arbitrarily. (A1) assures that the equation (4) has a unique solution $p(i;T) > 0$, and that this solution indeed maximizes (3) given T. Now, consider the function $p(i,T) \equiv p(i;T)$, noting that $p(i,\cdot)$ is a strictly decreasing C^1-function on $(0,\infty)$ as a consequence of the Implicit Function Theorem, the strong quasiconcavity assumption in (A1), and the last assumption of (A2). Moreover, $p(\cdot,T)$ is a decreasing function because of (A3) and, hence, $p(\cdot,T)$ is integrable over $[0,1]$, so let $G(T) \equiv \int_0^1 p(i,T)di$, noting that G is differentiable. ii) We show: $\exists!\ T > 0$: $G(T) = 1$. Fix $i \in [0,1]$. By (A3) there must exist $T_i > 0$: $g(i,1,T_i) = \varphi(i,1,T_i)$. Therefore, $\exists T_0 > 0$ such that $p(0,T_0) = 1$. Because $p(i,\cdot)$ strictly decreasing, it follows that $p(0;T) < 1$ for $T > T_0$. Since $p(\cdot,T)$ is decreasing, we must have $p(i,T) < 1$ for any $i \in [0,1]$ and $T > T_0$, which implies that $\lim\limits_{T\to\infty} G(T) < 1$. Similarly, it follows that $\exists T_1 > 0$ such that $p(1;T_1) = 1$. Thus $p(i,T_1) > 1$ for $i \in [0,1]$ and $T < T_1$, hence $\lim\limits_{T\to 0} G(T) > 1$. As $G(\cdot)$ continuous, $\exists T > 0$ such that $G(T) = 1$, and uniqueness follows from the fact that, for each $i \in [0,1]$, $p(i;T)$ and hence $G(T)$ is strictly decreasing in T. Finally, $\Pi(i) > 0$, because $p(i) = p(i;T) > 0$ is the unique maximizer and $\Pi(i)|_{p(i)=0} = 0$. ■

Proof of Corollary 1 To see that $p(i) > (\geq)p(j)$ as claimed, note that $g(i,p,T) \gtreqless \varphi(i,p,T)$ $\Leftrightarrow p(i) \gtreqless p$ because, by strong quasiconcavity (A3), $g(i,\cdot,T)$ must intersect $\varphi(i,\cdot,T)$ from above at $p(i)$ (see Figure 1). Further, in equilibrium

$$g(i,p(j),T) > (\geq)g(j,p(j),T) = \varphi(j,p(j),T) > (\geq)\varphi(i,p(j),T) \tag{41}$$

Hence $g(i,p(j),T) \geq \varphi(i,p(j),T)$ and thus $p(i) \geq p(j)$, where these two inequalities are strict if at least one inequality in (41) is strict. It also follows that $p(i) = p(j)$ if all inequalities in (41) are equalities, which proves the last claim of Corollary 1. The claims about payoffs holds because

$$\begin{aligned}\Pi(i) &= B(i,p(i),T) - \Phi(i,p(i),T) \\ &\geq B(i,p(j),T) - \Phi(i,p(j),T) > (\geq)B(j,p(j),T) - \Phi(j,p(j),T) = \Pi(j),\end{aligned}$$

where the first inequality follows from optimality, and the second type of inequality follows from (A3) and the additional presumption in Corollary 1. ■

Proof of Proposition 1 Define $g(i) \equiv p(i, x') - p(i, x)$, and note that $\int g(s)ds = 0$. Suppose that $g(0) \leq 0$ By presupposition, g is decreasing, right-continuous and, by SSD, $\exists i_0 \in (0, 1)$: $0 \geq g(0) > g(i)$, $\forall i \geq i_0$. Hence $\int g(s)ds < 0$, a contradiction. Therefore $g(0) > 0$, and a similar argument shows that $g(1) < 0$. Because g is decreasing, right-continuous and $g(0) > 0$, the set $\{i : g(i) > 0, i > 0\}$ is non-empty, and we let $i_0 = sup\{i > 0 : g(i) > 0\}$, noting that $i_0 \in (0, 1)$. It follows that $p(i, x') > p(i, x)$ on $(0, i_0)$, and $\int_0^{i_0} g(s)ds > 0$. Because g decreases and $\int g(s)ds = 0$, the set $\{i : g(i) < 0, i \geq i_0\}$ is non-empty, and we set $i_1 = inf\{i \geq i_0 : g(i) < 0\}$. If $i_0 < i_1$ then $g(i) = 0$ on (i_0, i_1), as g is decreasing and right-continuous. These facts together imply that $p(\cdot, x')$ is OR of $p(\cdot, x)$. ■

Proof of Proposition 2 Define $g(i) \equiv \frac{p(i,x')}{p(i,x)}$, and establish $g(0) > 1$, $g(1) < 1$ and the existence of $0 < i_0 \leq i_1 < 1$ such that $g(i) > 1$ if $i < i_0$, $g(i) = 1$ if $i \in [i_0, i_1)$ and $g(i) = 1$ for $i \geq i_1$ by exactly the same reasoning as in the proof of Proposition 1. ■

Proof of Corollary 2 We only show the OR case. Define $f(x; i, j) \equiv \frac{p(i,x)}{p(j,x)}$. If $p(\cdot)$ is class II and (11) is satisfied, then $f(x; i, j) > f(x_0; i, j)$ whenever $x > x_0$, and the claim follows from Proposition 2. If $p(\cdot)$ is class I, then $p(\cdot, x)$ is piecewise constant for any given $x \in X$, with a finite number of downward jumps. If (11) is satisfied for any two $i, j \in (0, 1)$ with $j \rhd i$ that are not jump points of $p(\cdot, x)$, we must have that $f(x; i, j) > f(x_0; i, j)$ for any such i, j and any $x > x_0$, proving the claim by Proposition 2. ■

Proof of Lemma 1 By (A1), equation (4) has a unique solution for given T, x, denoted by $p(i; T, x)$. Define $G(T; x) \equiv \int p(i; T, x)di$, note that in equilibrium $G(T; x) = 1$. Quasiconcavity (A1) and $g_x > 0$ imply that $p(i; T, x)$ is strictly increasing in x for a fixed T and any $i \in [0, 1]$. Because, by (A2), $G_T(T; x) < 0$ (see the proof of Theorem 1), applying the Implicit Function Theorem to the equilibrium equation $G(T(x), x) = 1$ yields $T'(x) > 0$. ■

Proof of Theorem 2 We proof the first claim by contradiction. Hence suppose that $R = 0$ $\forall i, j \in [0, 1]$ and any $x \in X$, but $\exists j \in (0, 1)$ such that $\Delta_j \neq 0$ (and hence $dp(j) \neq 0$). Because in equilibrium the integral condition

$$\int \frac{\partial p(s, x)}{\partial x} ds = 0 \tag{42}$$

must hold, we can suppose, wlog, that $\Delta_j > 0$ for some $j \in (0, 1)$. By (14) we must have $\Delta_i > 0$ for all $i < j$, and because of (42) $\exists j' \in (0, 1)$, $j' > j$, such that $\Delta_i < 0$ for all $i > j'$. Take $i < j$ and $i' > j'$. Then $\Delta_i > 0$ but also $\Delta_i = k\Delta_{i'} < 0$, contradiction. Turning to the second claim, note that if $R \neq 0$ for some i, j then $\Delta_i = 0$ $\forall i \in [0, 1]$ is impossible by (14). Hence $\forall x \in X$ $\exists i$: $\Delta_i(x) \neq 0$, or equivalently $\frac{\partial p(i,x)}{\partial x} \neq 0$, and therefore $\exists \delta > 0$ such that $p(i, x') \neq p(i, x)$ for $x' \in (x - \delta, x + \delta)$, thus we have $p(\cdot, x') \neq p(\cdot, x)$ on that interval. ■

Proof of Theorem 3 Step 1: We first prove the second claim, and restrict attention to the OR-case (the IR-case is similar). Because $R(x_0)$ is uniformly positive, $\exists i \in (0, 1)$: $\Delta_i(x_0) > 0$ by the proof of Theorem 2. By the integral condition (42), there then must also be $i' \in (0, 1)$: $\Delta_{i'}(x_0) < 0$. It then follows from (14) that $i_0 = \sup\{i \in [0, 1] : \Delta_i(x_0) > 0\} \in (0, 1)$, $i_1 = \inf\{i \in [0, 1] : \Delta_i(x_0) < 0\} \in (0, 1)$ and $i_0 \leq i_1$. For any $i < i_0$: $\Delta_i(x_0) > 0$ and hence $\frac{\partial p(i,x_0)}{\partial x} > 0$. This derivative condition implies that $\forall i < i_0$ $\exists \delta_i > 0$: $p(i, x) > p(i, x_0)$ $\forall x \in (x_0, x_0 + \delta_i)$.

Step 2: Because $p(\cdot, x)$ is class I, there is a finite number of equivalence classes to the left of i_0, and we only need to consider a single i, with corresponding δ_i, for each step of $p(\cdot, x)$ to the left of i_0. Let $\delta^0 > 0$ be the smallest value of these δ_i. We have thus shown that $\exists$ $i_0 \in (0, 1)$ such that for any given $x \in (x_0, x_0 + \delta^0)$ we have $p(i, x) > p(i, x_0)$ for $i < i_0$. A similar argument shows that we can find $\delta^1 > 0$ such that $\exists$ $i_1 \in (0, 1)$ such that $p(i, x) < p(i, x_0)$ for $i > i_1$ and any $x \in (x_0, x_0 + \delta^1)$. Let $\delta \equiv \min\{\delta^0, \delta^1\} > 0$. Summarizing, the arguments so far show that $\exists i_0, i_1 \in (0, 1)$, $i_0 \leq i_1$ such that for $x \in (x_0, x_0 + \delta)$ we have $p(i, x) > p(i, x_0)$ for $i < i_0$ and $p(i, x) < p(i, x_0)$ for $i > i_1$. If $\Delta_i \neq 0$ for any $i \in (i_0, i_1]$ we must have $i_0 = i_1$ and the proof is complete. Now suppose that $\exists m \in (i_0, i_1]$: $\Delta_m(x_0) = 0$. Then (14) implies that $\Delta_i > 0$ for any $m \rhd i$, and $\Delta_j < 0$ for any $j \rhd m$. But this shows that there can be at most one step of $p(\cdot, x)$ for which $\Delta_m(x_0) = 0$. It follows that independent of whether $p(m, x') >=< p(m, x)$ for $x \in (x_0, x_0 + \delta)$, $p(\cdot, x')$ must be OR of $p(\cdot, x_0)$. We now prove the first claim. By step 1 and the global uniform positivity of R, we must have $\Delta_0(x) > 0$ and thus $\frac{\partial p(0,x)}{\partial x} > 0$ for any $x > x_0$ (note that this result is valid also if $p(\cdot)$ is of class II), hence $p(i, x') > p(i, x_0)$ $\forall i \in [0]$. Similarly, $\Delta_1(x) < 0$ for all $x > x_0$, hence $p(i, x') < p(i, x_0)$ $\forall i \in [1]$. ■

Proof of Corollary 3 We only prove the first claim as the remaining claims are proved identically. Recall from the equivalence class argument in step 2 of the proof of Theorem 3 that there is a finite number of $\Delta_i(x_0) > 0$, possibly a single $\Delta_m(x_0) = 0$ and a finite number $\Delta_j(x_0) < 0$. Define $f(i, j, x) = \frac{p(i,x)}{p(j,x)}$. If $\Delta_{i'}(x_0) \geq 0$ then any i with $i' \rhd i$ has $\Delta_i(x_0) > \Delta_{i'}(x_0)$ by (14). Hence we must have $\frac{\partial f(i,i',x_0)}{\partial x} > 0$. If $\Delta_{i'}(x_0) < 0$ but $\Delta_i(x_0) > 0$, then obviously $\frac{\partial f(i,i',x_0)}{\partial x} > 0$ Thus for any pair (i, i') as described above $\exists \delta_{i,i'} > 0$ such that $f(i, i', x') > f(i, i', x_0)$ for all $x' \in (x_0, x_0 + \delta_{i,i'})$. The proof is completed by letting $\delta > 0$ be the smallest among these (finitely many) $\delta_{i,i'}$ and δ^0, δ^1 as identified in the proof of Theorem 3. ■

The claim follows immediately from Corollary 2 because, by (14), if $R(x)$ is globally uniformly positive (negative) and $k = 1$, then $\Delta_i(x) > (<) \Delta_j(x)$, for any $j \rhd i$ and any $x \in X$, and hence condition (11') holds. ■

Proof of Proposition 4 We only prove the uniformly positive case (the negative case is established under the same type of arguments). We need to show that for

$$A(i) = \frac{g_T(i)}{g(i)} T'(x) + \frac{g_x(i)}{g(i)}$$

we have $A(i) > A(j)$ whenever $j \rhd i$. So take any $j \rhd i$. First, $h_T(i, p', T, x_0) \geq h_T(i, p, T, x_0)$ and $h_T(i, p, T, x_0) \geq h_T(j, p, T, x_0)$ yield

$$h_T(i, p(i), T, x_0) \geq h_T(i, p(j), T, x_0) \geq h_T(j, p(j), T, x_0)$$

and because $T'(x) > 0$ by Lemma 1 we have

$$\frac{g_T(i)}{g(i)} T'(x) \geq \frac{g_T(j)}{g(j)} T'(x),$$

where the inequality is strict, whenever at least one of the initial inequalities is strict. Second, $h_x(i,p,T,x_0) \geq h_x(j,p,T,x_0)$ and $h_x(i,p',T,x_0) \geq h_x(i,p,T,x_0)$ yield

$$h_x(j,p(j),T,x_0) \leq h_x(j,p(i),T,x_0) \leq h_x(i,p(i),T,x_0)$$

and hence also

$$\frac{g_x(i)}{g(i)} \geq \frac{g_x(j)}{g(j)}$$

where, again, the inequality is strict if one of the previous inequalities is strict. This shows that $R(x_0) > 0$, and the global case follows immediately. ■

Proof of Proposition 5 Consider the first two rows of Table 1. If $\gamma_i = \gamma \geq 1$, $\forall i \in [0,1]$, then (22) implies

$$\frac{p(i)}{p(j)} = \left(\frac{c(j)}{c(i)}\right)^{\frac{\eta-1}{\eta(\gamma-1)+1}} \left(\frac{r(i)}{r(j)}\right)^{\frac{\gamma\eta}{\eta(\gamma-1)+1}},$$

from which $\frac{\partial}{\partial\eta}\frac{p(i)}{p(j)} > 0$ and $\frac{\partial}{\partial\gamma}\frac{p(i)}{p(j)} < 0$, and the first column in Table 1 follows from Corollary 2. The third and fifth columns follow from $\Pi(i) = p(i)I\left(\frac{\gamma\eta-\eta+1}{\gamma\eta}\right)$, because then $\frac{\Pi(i)}{\Pi(j)} = \frac{p(i)}{p(j)}$. From $p(i) = \frac{r_i^{\eta}P_i^{1-\eta}}{T}$ we obtain

$$\frac{P(i)}{P(j)} = \left(\frac{c(i)}{c(j)}\right)^{\frac{1}{1+(\gamma-1)\eta}} \left(\frac{r(i)}{r(j)}\right)^{\frac{(\gamma-1)\eta}{1+(\gamma-1)\eta}},$$

and the fourth column follows from differentiating this expression. The remaining two rows of Table 1 are obvious from the above derivations. ■

Proof of Proposition 6 The fact that $T'(I) < 0$ gives $\frac{\partial q_i}{\partial I} > 0$ $\forall i \in [0,1]$ by the left equation in (23). The right equation in (23) implies that

$$\frac{q_i^{(\gamma_i-1)+\frac{1}{\eta}}}{q_j^{(\gamma_j-1)+\frac{1}{\eta}}}$$

is independent of I. Because $q(\cdot) \geq 1$ and $\gamma_i < \gamma_j$, $j \rhd i$, this together with $dq_i, dq_j > 0$ further implies that $\frac{q_i}{q_j}$ is strictly increasing in I. Therefore $\frac{p(i)}{p(j)}$ increases in I by (20') which, by Corollary 2, implies an OR of $p(\cdot)$. Since $\Pi(i) = p(i)I\left(\frac{\gamma_i\eta-\eta+1}{\gamma_i\eta}\right)$, relative profits increase in

I, and because $p(i) = \frac{r_i^\eta P_i^{1-\eta}}{T}$ so do relative prices $\frac{P_j}{P_i}$. ■

Proof of Proposition 7 By FOC, $P(T,x) = \varphi(i, q(i))$, either $dq(i) > 0$ or $dq(i) \leq 0\ \forall i$. Then, $dx > 0$ implies that $q(i)$ increases; if $dq(i) \leq 0\ \forall i$, then $dP \leq 0$ by FOC but because also $dT \leq 0$, $P_T < 0$ and $P_x > 0$ this is impossible. A standard Envelope-theorem argument shows that $\Pi(i)$ increases in x. If $\Phi(i,q)$ is a power function with common and constant exponent η, so is $\phi(i,q)$, $R_{ij} = 0$ by (29) and $p(i)$ is invariant to x by Theorem 2. If $\Phi(i,q)$ is a power function for each i, then $\Pi(i) = P(T,x)p(i)T\frac{\eta_i - 1}{\eta_i}$. Hence if $\eta_i = \eta\ \forall i$, then x does not affect relative profits and quantities as long as $p(i)$ does not change. Profit and quantity differences increase in x by the remark following Proposition 2. Let $\Phi(i,q) = q^{\eta(i)}$ as stated by the proposition. Because $P(T(x),x) > \eta(1)$, we must have $q(i) > 1$ by the FOC, $\forall i$, and because also $\eta(i)q(i)^{\eta(i)-1} = \eta(j)q(j)^{\eta(j)-1}$, we must have $q(i) > q(j)$ and hence $p(i) > p(j)$ for any $j \rhd i$. Because $T'(x) > 0$, $sign(A(i) - A(j)) = (\eta(j) - \eta(i)) > 0$ by (29), hence an OR results, and the claims about relative profits and quantities follows immediately. ■

Proof of Proposition 8 Since $T'(x) > 0$, we have $sign(A(i) - A(j)) = sign\left(\frac{\varphi_T(j)}{\varphi(j)} - \frac{\varphi_T(i)}{\varphi(i)}\right)$, where where

$$\frac{\varphi_T(i)}{\varphi(i)} = \frac{\sum s(s-1)a_s p(i)^{s-1}T^{s-2}}{\sum s a_s (p(i)T)^{s-1}}.$$

Then

$$\frac{\partial}{\partial p}\left(\frac{\varphi_T(i)}{\varphi(i)}\right) > 0 \quad \Leftrightarrow \quad \sum s(s-1)^2 a_s x^{s-1} \sum s a_s x^{s-1} > \left(\sum s(s-1)a_s x^{s-1}\right)^2$$

We claim that the second inequality holds. Note that both sides of this inequality are polynomials of degree $2(m-1)$, hence it is of the form

$$u_1 x + u_2 x^2 + ... + u_{2(m-1)} x^{2(m-1)} > w_1 x + w_2 x^2 + ... + w_{2(m-1)} x^{2(m-1)}$$

We now claim that $u_k \geq w_k$, $k = 1, ..., m-1$, where the inequality is strict for some k. Let $s, s' \in \{2, ..., m\}$ be such that $s + s' = k$ for a given k. Then $A = s(s-1)^2 a_s s' a_{s'} + s'(s'-1)^2 a_{s'} s a_s$ is a summand in the calculation of u_k, while $B = 2s(s-1)a_s s'(s'-1)a_{s'}$ is the corresponding summand in the calculation of w_k. If $a_s = 0$ or $a_{s'} = 0$, then $A = B$, so let $a_s, a_{s'} > 0$. If $s = s'$ then again $A = B$, so let $s \neq s'$ (such s, s' exist by presumption). Claim: $A > B$. To see this is suffices to show that $s(s-1)^2s' + s'(s'-1)^2s > 2s(s-1)s'(s'-1)$, which is equivalent to $(s-1)^2 - 2(s-1)(s'-1) + (s'-1)^2 > 0$. Since $s, s' > 1$ the last inequality is satisfied. Because $p(i) > p(j)$ for any $j \rhd i$, this implies that $\frac{\varphi_T(i)}{\varphi(i)} > \frac{\varphi_T(j)}{\varphi(j)}$, hence $R_{ij} < 0$, showing that an IR results. ■

Proof of Proposition 9 Aggregation of firm FOC yields $T(P) = \int \varphi^{-1}(i,P)di$, and $T'(P) > 0$ because $\varphi_q(i,q) > 0$. $T^d = T$ and aggregate demand imply that $PT(P) = \omega + \Pi(P)$, from which $P'(\omega) = (T + PT'(P) - \Pi'(P))^{-1}$. Hotelling's Lemma gives $\Pi'(P) = \int \partial_P \Pi(i)di = T$. Therefore $P'(\omega) > 0$, and $T'(\omega) > 0$ follows. Then, $\partial_\omega q(i) > 0$ because, by firm FOC and $\varphi_q(i,q) > 0$, $\partial_P q(i) > 0$, and $\partial_\omega \Pi(i) > 0$ because $\partial_P \Pi(i) = q(i) > 0$. The remainder of the claim follows from Theorem 3 because $T'(\omega) > 0$ and $A(i)$ is given by (29). ■

Proof of Proposition 10 Note first that $p^d(i_c)$ has the monotone ratio property because (32) is linear in $p^d(i_c)$, and therefore an OR (IR) of $p^d(i_c)$ always implies that $q(i_c)/q(j_c)$ increases (decreases). Equation (32) implies that for $j_c \rhd i_c$ the ratio $p^d(i_c)/p^d(j_c)$ increases (decreases) in ω if

$$\frac{d\omega(i_c) + s(i_c)d\Pi}{\omega(i_c) + s(i_c)\Pi} > (<) \frac{d\omega(j_c) + s(j_c)d\Pi}{\omega(j_c) + s(j_c)\Pi} \tag{43}$$

(i) If $s(i_c) = 1\ \forall i_c$ is used in (43) together with $d\Pi = \Pi'(\omega) > 0$, we obtain that $p^d(i_c)/p^d(j_c)$ decreases in ω. Therefore $d\omega > (<)0$ induces a global IR (OR) of $p^d(i_c)$ by Theorem 4. (ii) Using $\omega(i_c) = \omega\ \forall i_c$ in (43) shows that $p^d(i_c)/p^d(j_c)$ decreases (increases) in ω if $\Pi - \Pi'(\omega)\omega > (<)0$. Because $\Pi'(\omega) > 0$ and $\Pi(0) = 0$, $\Pi - \Pi'(\omega)\omega > (<)0$ if $\Pi(\omega)$ is strictly concave (convex), and the claim follows from Theorem 4. ■

Proof of Corollary 4 With iso-elastic costs, $\Pi(i) = Pq(i)\frac{\eta-1}{\eta}$, and thus $\Pi = PT\frac{\eta-1}{\eta}$. Together with aggregated consumer FOC, this implies that $PT = \eta\omega$ and $\Pi = (\eta-1)\omega$. Because Π is linear in ω, $p^d(i_c)$ and $q(i_c)/q(j_c)$ must be invariant to ω by the proof of Proposition 10 (ii). Aggregated firm FOC together with $PT = \eta\omega$ imply that $\omega/P = \gamma\omega^{1/\eta}$, where $\gamma > 0$ is a constant. Because $q(i) = \omega/P(1 + s(i_c)(\eta-1))$ it follows that $\partial_\omega q(i) > 0$. Hence the facts that $q(i_c)/q(j_c)$ is constant but $q(i_c) > q(j_c)$ both decrease, imply the last claim. ■

Proof of Proposition 11 Invariance of $p(i)$ and $\frac{\Pi(i)}{\Pi(j)}$ to α and a common $dc(i)$ follow from the constant and common elasticity of costs as argued in the main text. Equilibrium profits are

$\Pi(i) = p(i)T\frac{\eta}{\eta-1}$. In equilibrium $T = T^d$, and summation of consumer FOC yields $T = \alpha(w+\Pi)$. Using $\Pi = \int \Pi(i)di$ in this equation implies that

$$T = \frac{\alpha\eta}{\alpha + (1-\alpha)\eta} w. \tag{44}$$

But because $S(i_c) = s(i_c)\Pi$, capital incomes change in equal proportions, and

$$\frac{dS(i_c)}{S(i_c)} = \frac{d\Pi(i)}{\Pi(i)} = \frac{dT}{T}$$

(i) Equation (44) additionally implies that $dT/T = dw/w$. Therefore

$$\frac{dw + dS(i_c)}{w + S(i_c)} = \frac{dw + dS(j_c)}{w + S(j_c)}$$

showing that $p(i_c)/p(j_c)$ and $q(i_c)/q(j_c)$ remain constant, proving that $p(i_c)$ is unaffected. Because $f(i_c) = (1-\alpha)(1 + S(i_c)/w)$, also $f(i_c)$ remains constant. Integration of the FOC of the firm problem yields

$$T = \left(\frac{1}{w\eta}\right)^{\frac{1}{\eta-1}} C, \qquad C \equiv \int c(i)^{\frac{1}{1-\eta}} di. \tag{45}$$

The aggregate effect of the exogenous common efficiency increase thus is entirely captured by $dC > 0$, and (44), (45) together imply that $T'(C), w'(C) > 0$, which further assure that $d\Pi(i) > 0$ and $dq(i_c) > 0$ for each firm and consumer, respectively. Finally, $dq(i_c) > dq(j_c)$ and $d\Pi(i) > d\Pi(j)$ follow from the above results and the remark after Proposition 2. (ii) Equations (44), (45) then imply that $w'(\alpha) < 0$ and $T'(\alpha) > 0$. By the latter also $\partial_\alpha \Pi(i) > 0$, and hence $\Pi'(\alpha) > 0$ as well as $dS(i) > 0$ (provided that $s(i) > 0$). Because aggregate output increases so does aggregate labor supply. To prove the OR of $p^d(i_c)$, it suffices to show that $p^d(i_c)/p^d(j_c)$ increases in α by Corollary 2, hence that

$$\frac{dw + dS(i_c)}{w + S(i_c)} > \frac{dw + dS(j_c)}{w + S(j_c)}.$$

This inequality holds by the fact that $S(i_c) = s(i_c)\Pi$, $dw < 0$ and $d\Pi > 0$. The monotone ratio property follows because the consumer FOC is linear in $p^d(i_c)$, which immediately also implies that $q(i_c)/q(j_c)$ and $f(i_c)/f(j_c)$ increases. It follows that $dq(i_c) > dq(j_c)$ if $dq(i_c) > 0$. ∎

Proof of Proposition 12 The FOC of (34'), evaluated for the case $V(T,x)$, is $V(T,x) = \varphi\left(i, \hat{\pi}^{-1}(p(i))T\right)$. As $V_x > 0$ also $T'(x) > 0$ by Lemma 1 and, using $t(i) = \hat{\pi}^{-1}(p(i))T$, thus

$$sign(R_{ij}) = sign\left(\frac{\varphi_t(j,t(j))t(j)}{\varphi(j,t(j))} - \frac{\varphi_t(i,t(i))t(i)}{\varphi(i,t(i))}\right),$$

which shows the first claim. The second claim then follows, because $R_{ij} = 0$ if all $\Phi(\cdot)$ are common-elastic power functions and $\Pi(i) = V(T,x)\frac{\eta p(i)z'(p(i)) - z(p(i))}{\eta z'(p(i))}$. ■

Proof of Corollary 5 The FOC imply

$$\frac{p(i)}{p(j)} = \left(\frac{c(j)}{c(i)}\right)^{\frac{1}{\eta-1}}.$$

Thus $d\eta > 0$ induces a global IR with monotone ratios by Corollary 2, (18) and Theorem 4, and the claims on relative payoffs and efforts follow from $\Pi(i) = p(i)V(T)\frac{\eta-1}{\eta}$ and $t(i) = p(i)/T$. ■

Proof of Proposition 13 The FOC are $\alpha T(z(p(i)) + p(i)z'(p(i))) + \beta = \eta c(i) z(p(i))^{\eta-1}T^{\eta}$. Evaluation of the conditions in Proposition 4 for $h = Ln\left(\alpha T(z(p) + pz'(p)) + \beta\right)$ and $x = \alpha$ shows that both $h_\alpha(i) > (<) h_\alpha(j)$ and $h_T(i) > (<) h_T(j)$ both are equivalent to $z(p(i)) + p(i)z'(p(i)) > (<) z(p(j)) + p(j)z'(p(j))$. Thus, if $\frac{z''(p)p}{z'(p)} > (<) -2 \; \forall p > 0$, such that $z(p) + pz'(p)$ is a strictly increasing (decreasing) function, we have from Proposition 4 that R is uniformly positive (negative), and the first claim follows from Theorem 3. Similarly, if $x = \beta$, then $h_\beta(i) > h_\beta(j)$ if either $\alpha > 0$ and $\frac{z''(p)p}{z'(p)} < -2 \; \forall p > 0$ or if $\alpha < 0$ and $\frac{z''(p)p}{z'(p)} > -2 \; \forall p > 0$. In both cases an OR results,[51] proving the second claim. ■

Proof of Proposition 14 Totally differentiating the LHS of (38) yields $dV_1 + (1-2p(i))dV_2$. By Proposition 4 the differential change dV_1, dV_2 causes an OR of $p(i)$ if

$$z(p) \equiv \frac{dV_1 + (1-2p)dV_2}{V_1 + (1-2p)V_2}$$

is increasing in p. The claim then follows from $sign(z'(p)) = sign\left(dV_1V_2 - dV_2V_1\right)$. ■

[51]The cases $\alpha > (<) 0$ and $\frac{z''(p)p}{z'(p)} > (<) -2$ cannot be signed unambiguously.

Proof of Proposition 15 Like in the proof of Proposition 12 we obtain that

$$sign(R_{ij}) = sign\left(\frac{\varphi_t(j, z(p(j))T)z(p(j))T}{\varphi(j, z(p(j))T)} - \frac{\varphi_t(i, z(p(i))T)z(p(i))T}{\varphi(i, z(p(i))T)}\right)$$

proving the first claim. With iso-elastic costs $\varphi(i, z(p(i)T) = \eta c(i)z(p(i))^{\eta-1}T^{\eta-1}$ from which $R_{ij} = 0$ follows. Hence $p(i)$ is invariant to dV, and the remaining claims follow from

$$\Pi(i) = V\frac{\eta p(i)z'(p(i)) - z(p(i))(1 - z(p(i))}{\eta z'(p(i))}$$

and $t(i)/t(j) = z(p(i))/z(p(j))$. ■

Proof of Proposition 16 Rewriting the payoff with the market share variable $p(i)$ gives

$$\Pi(i) = p(i)\left(\frac{\alpha T_i}{1 - p(i)} + \beta\right) - c(i)\left(\frac{p(i)}{1 - p(i)}\right)^{\eta} T_i^{\eta},$$

from which the equilibrium FOC $\alpha T + \beta(1 - p(i)) = \eta c(i)p(i)^{\eta-1}T^{\eta}$ can be deduced. For $h(i) \equiv Ln(\alpha(1 - p(i)) + \beta)$ we obtain $sign(h_T(i) - h_T(j)) = sign\,\alpha$, $h_\alpha(i) > h_\alpha(j)$ and $sign(h_\beta(j) - h_\beta(i)) = sign\,\alpha$. All claims then follow from Proposition 4. ■

B Online Appendix

The following proposition describes the equilibrium distribution function $F(i,x)$ in greater detail.

Proposition B.1 *Suppose that $\forall x \in X$ the decreasing density $p(\cdot,x)$ has the SSD property. Then:*

a) $F(i,x) > i$, $i \in (0,1)$, $x \in X$

b) *If $p(i,x')$ is OR of $p(i,x)$, then $F(\cdot,x)$ strictly stochastically dominates $F(\cdot,x')$, i.e. $F(i,x') > F(i,x)$, $\forall i \in (0,1)$.*

c) *$F(i,x)$ is strictly increasing, continuous and concave in i, $x \in X$. If $p(i)$ is strictly decreasing, then $F(i)$ is strictly concave.*

Proof a) Consider $i_0 \in (0,1)$. If $p(i_0) \geq 1$, then by SSD and because $p(\cdot,x)$ is a decreasing density $\exists\, \hat{i} < 1$: $p(i) < 1$ on $[\hat{i},1]$. Thus there must also be $\tilde{i} \in (0,i_0]$ such that $p(i) > 1$ on $[0,\tilde{i})$. Hence $F(i_0) > i_0$. If $p(i_0) < 1$ then, by SSD and the decreasing density property, $\exists i \in (0,i_0]$:

$$F(i_0) = \int_0^i \underbrace{p(s)}_{\geq 1} + \int_i^{i_0} \underbrace{p(s)}_{<1}$$

Then $\int_0^i (p(s)-1) = \int_i^1 (1-p(s)) > \int_i^{i_0} (1-p(s))$, where the inequality follows from the fact that $1-p(s) > 0$ for $s \in [i,1]$. Hence $F(i_0) > i_0$. b) Follows from a similar type of argument. If $i \in (0,i_0]$, then $p(i,x') > p(i,x) > 0$ for $i \in (0,i_0)$, and hence $F(i,x') > F(i,x)$. If $i \in (i_0,1)$, then

$$\int_0^{i_0} \big(p(s,x') - p(s,x)\big) = \int_{i_0}^1 \big(p(s,x) - p(s,x')\big) > \int_{i_0}^i \big(p(s,x) - p(s,x')\big)$$

where the inequality follows from $p(s,x') = p(s,x)$ on (i_0,i_1) and $p(s,x) > p(s,x')$ on $(i_1,1]$. Hence again $F(i,x') > F(i,x)$. c) $F(\cdot,x)$ strictly increasing follows directly from $p(\cdot,x) > 0$. Let $i,j \in [0,1]$ and $i < j$. Let $t \in [0,1]$. First, note that $tF(i) + (1-t)F(j) = F(j) - t\int_i^j p(s)$, and $F(ti+(1-t)j) = \int_0^j p(s) - \int_{i'}^j p(s)$ where $i' = ti + (1-t)j$. From these two expressions it follows that F is concave iff $t\int_i^j p(s) \geq \int_{i'}^j p(s)$, or equivalently if

$$t\int_i^{i'} p(s) - (1-t)\int_{i'}^j p(s) \geq 0 \qquad t \in (0,1), i < i' < j \tag{46}$$

Because $p(i)$ is decreasing, we must have $p(i) \geq p(i') \geq p(j)$ (with strict inequalities in the strictly decreasing case). Hence (46) is satisfied (strictly so in the strictly decreasing case) if

$$t \int_{i}^{i'} p(i') - (1-t) \int_{i'}^{j} p(i') \geq 0 \qquad t \in (0,1)$$

This inequality reduces to $t(i'-i) - (1-t)(j-i') \geq 0$, which, by construction of i', is satisfied. This verifies the (strict) concavity of $F(\cdot, x)$, and continuity of $F(\cdot, x)$ follows from concavity and the fact that $F(\cdot, x)$ cannot jump at the boundary points $\{0, 1\}$. ■

In words, a) means that the market share of one of the best $i\%$ agents always exceeds $i\%$. Further, a) - c) are consequences of the SSD property, and thus in general of agent heterogeneity and the no leap-frogging property of $p(i)$.

B.1 More on rotations

We first state the differential version of Proposition 1:

Corollary B.1 *Suppose that the presumptions of corollary 2 hold. If $\forall i < j$ with $j \notin G(i)$ and $x_0 \in Int(X)$ we have that*

$$\frac{\partial}{\partial x} \left(p(i, x') - p(j, x') \right) > 0 \qquad \forall x' \geq x_0 \tag{47}$$

whenever the derivative exists, then $p(i, x')$ is OR of $p(i, x)$ whenever $x' > x_0$. If the inequality in (11) is reversed, then $p(i, x')$ is IR of $p(i, x)$.

As the proof is logically similar to the proof of Corollary 2 we omit it here. We include the following calculus result for the sake of completeness (we omit its obvious proof).

Corollary B.2 *Suppose that p is C^2 and belongs to class II, let X be an open interval and $x' > x$. If $\frac{\partial^2}{\partial i \partial x} p(i, x) < 0$, $(i, x) \in Int(A)$, then $p(i, x)$ satisfies (9) on A. If $\frac{\partial^2}{\partial i \partial x} Ln\left(p(i, x)\right) < 0$, $(i, x) \in Int(A)$, then $p(i, x)$ satisfies (10) on A.*

Relation between difference and ratio test While the ratio test and the difference test can both be used to establish an OR or IR of $p(\cdot)$, they are not equivalent, and we explore their relation in this section.

Proposition B.2 *Suppose that the premise of proposition 1 is satisfied. If* (10) *is satisfied, then* (9) *holds for all* i *such that* $p(i,x') \geq p(i,x)$*. If* (9) *is satisfied, then* (10) *holds for all* $j > i$ *such that* $j \notin G(i)$ *and* $p(i,x') \leq p(i,x)$*.*

Proof: If (10) is satisfied, then equivalently $\frac{p(i,x')-p(i,x)}{p(i,x)} > \frac{p(j,x')-p(j,x)}{p(j,x)}$ whenever $j > i$, and $j \notin G(i)$. Suppose that $p(i,x') \geq p(i,x)$, and take $j > i$, $j \notin G(i)$. Hence $p(i,x) > p(j,x)$, and the first claim follows from $\frac{p(i,x')-p(i,x)}{p(j,x)} > \frac{p(j,x')-p(j,x)}{p(j,x)}$. If (9) is satisfied, then:

$$\frac{p(i,x')}{p(i,x)} > \frac{p(j,x')-p(j,x)}{p(i,x)} + 1 \qquad j > i,\, j \notin G(i) \tag{48}$$

Now, because of (48) condition (10) is satisfied if $\frac{p(j,x')-p(j,x)}{p(i,x)} + 1 \geq \frac{p(j,x')}{p(j,x)}$, or equivalently, if $\frac{p(j,x')-p(j,x)}{p(i,x)} \geq \frac{p(j,x')-p(j,x)}{p(j,x)}$. But this inequality must hold, because as $p(i,x') \leq p(i,x)$ (48) implies that $p(j,x') - p(j,x) < 0$. ∎

If p is the (stage I) success chance, then proposition B.2 says that if p satisfies setwise decreasing differences, the setwise decreasing competitive balance property is satisfied at least for the "loosing" range, where $p(i,x') \leq p(i,x)$. Conversely, if the decreasing competitive balance property is satisfied, then the "winning" range, where $p(i,x') \geq p(i,x)$, satisfies setwise decreasing differences.

To place the last result in the relevant theoretical context, note that $p(i,x)$ is setwise strictly submodular on $A \equiv [0,1] \times X$ if and only if $p(i,x)$ has setwise strictly decreasing differences on its domain, i.e. (9) is satisfied on A. Similarly, $p(i,x)$ has setwise strictly decreasing ratios on A if and only if $p(i,x)$ is setwise strictly log-submodular on A, i.e. if and only if $Ln(p(i,x))$ is setwise strictly submodular on A. It is a known result that if $p(\cdot,\cdot)$ were monotonic on A, then log-supermodularity would imply supermodularity, and submodularity would imply log-submodularity (see Topkis (1998), p. 65). However, because in our context $p(i,x)$ generically cannot be montonic in both arguments,[52] the result does not apply to our setting, but proposition B.2 can be viewed as an extension of the result to the case of a partially monotonic function.

[52] E.g. in the continuous case the integral condition implies the existence of at least one intersection of $p(i,x)$ and $p(i,x')$.

B.2 Additional results

Logit-demand with an outside option In the Logit model of section 4.1.2 we have assumed that the consumer always purchases one alternative, but what if there is an outside option? A simple way of incorporating an outside option into the Logit (see, e.g., Anderson et al., 1992) is to write expected demand of firm i as

$$q_i = \frac{e^{-\lambda P_i}}{\int e^{-\lambda P_s} + x}, \qquad x = e^{\lambda V_0}.$$

If $x = 0$ this corresponds to the previous model, with a uniform quality level but heterogeneous costs $c(i)$. We have $\int q_i di < 1$ iff $x > 0$, because with $x > 0$ there is a non-zero probability that the consumer does not purchase at all. Define $p(i)$ as the market share of i *relative to total supply*, $p(i) \equiv \frac{q_i}{\int q_i} di$. Then $q_i = p(i)\frac{T}{T+x}$, where $T = \int e^{-\lambda P_s}$, and profits are

$$\Pi_i = P_i q_i - c(i) q_i = \frac{T}{T+x}\left(\left(-\frac{Ln(p(i)T)}{\lambda}\right)p(i) - c(i)p(i)\right)$$

which is essentially (25) scaled by a positive number. Hence (26) - (28) remain valid for $x > 0$ (with the reinterpretation of $p(i)$), and the main distributional effects of λ are preserved. In particular, an increase in λ leads to an OR of $p(\cdot)$ and increases relative profits as well as relative prices $\frac{P_i}{P_j}$ (because $c(i) < c(j)$, $j \rhd i$). Moreover, a change in the value of the outside option ($dx \neq 0$) has no equilibrium effects on market shares nor on relative profits nor on *relative* (expected) quantities supplied. By (26) prices P_i are invariant to x, and by (27) $T'(x) = 0$, showing that the only effect of (a change in) the outside option is to change individual and aggregate supply, and to scale all equilibrium profits by a factor, while preserving market shares and market prices.[53]

[53]Note however that quantity differences, $q_i - q_j > 0$, and profit differences $\Pi(i) - \Pi(j) > 0$ decrease in x.