Dynamic Adjustment of Fiscal Policy under a Debt Crisis

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Abstract

In an overlapping generations framework that allows for the presence of a debt crisis senario (debt bubbles), we introduce productive government expenditures and a dynamic fiscal rule that combines fiscal stimulus and fiscal consolidation. We formally argue that for the pursuit of escaping from a situation of exploding debt and low economic activity a fiscal rule has to be procyclical to increases in output and, at the same time, has to control for the level of debt. Then, when the economy enters the sustainability area, the same rule, has to endogenously adapt to the actual level of debt and income in order to stimulate private investment through lower taxes. We provide a numerical example to our theorerical results and we show that the tax rate has to adjust non-monotonically during the process of recovery reflecting the two counter-balancing properties of the examined fiscal policy rule.

JEL classification: *E6;H6;H30.*

Keywords: Fiscal sustainability; Fiscal rules; Bond-financed deficits

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1 Introduction

Over the last years, particularly in Europe, we have witnessed a shift to austerity measures and deficit reducing policies to target the sustainability of public debt. The International Monetary Fund (IMF), the European Central Bank (ECB), and the European Commission in an effort to help European countries to overcome situations of exploding debt have focused on policies that place some level of fiscal austerity (increase in taxation and spending cuts) to control the volume of debt of each country. However, first, we have witnessed that those policies, due to their discretionary nature, are continuously re-optimized given the failure of some countries to achieve their targets. Second, using almost the same fiscal policy measures in similar countries (e.g. Portugal and Greece) we observe divergent results in economic outcomes (for a detailed review see Brendon and Corsetti, 2016). Such variation in the dynamic adjustment of policy instruments and divergence in the expected economic outcomes have resulted in an uncertain economic environment raising the need of imposing a stable dynamic fiscal policy rule subject to the state of the economy.

The aim of this paper is to examine the properties of a fiscal policy rule for debt sustainability in a framework that allows for the presence of debt bubbles. From one hand, increasing productive government spending stimulates an economy with low private investment and, in turn, output, on the other hand, without considering a consistent finance plan about the level of debt, an expansionary policy can generate a debt bubble. According to our model the effectiveness of fiscal stimulus and consolidation for debt sustainability is determined by the initial conditions on the level of debt and capital stock. To this end, we provide a fiscal rule that can endogenously adjust to the need for stimulus and consolidation as the economy develops. In particular, we show that a fiscal rule has to be procyclical in output increases (contrary to perceived notions) but at a high initial level of debt, taxation has to increase (endogenously) in order to finance the deficits.[1] After a threshold level of income, taxation negatively adjusts to output increases and government expenditures are financed through a higher tax base.

Our paper is related to the literature on fiscal consolidation and debt sustainability and contributes in several manners. Earlier work by Sargent and Wallace (1981) states that there is a ceiling on government indebtedness and that permanent deficits will eventually need to be monetized. However, some countries either belong to a monetary union or monetary policy is constrained by the zero lower bound. Based upon that, among others, Eggertsson (2011), Christiano, Eichenbaum and Rebello (2011) and Coehen et al. (2012) highlight the role of fiscal stimulus and they show that the government spending multipliers are potentially larger when the zero bound is binding. However, their modelling approach does not allow for the presence of debt bubbles that can be triggered by fiscal stimulus and the fact that fiscal multipliers depend on the state of the cycle and the level of debt as empirical evidence by Auerbach and Gorodnichenko (2012, 2013) and Ramey and Zubairy (2016) indicates. Furthermore, Corsetti et al. (2013) highlight that the benefits to fiscal expansion could easily be undone if the fiscal solvency of the government comes to be questioned – an issue that is of obvious relevance to Southern European countries at present.

[1]In the optimal neoclassical growth model of infinitely lived agents debt bubbles are ruled out optimally and a procyclical fiscal rule crowds out private investment strongly and generates instability. However, under the existence of debt bubbles and unstable debt dynamics that can occur in an OLG framework, a procyclical policy in output can place the economy in the sustainability area (through increases in productivity) as we will show later on.

To this end, by advancing the seminal work of Tirole (1985) and that of Chalk (2000), we provide a theoretical framework, considering the aforementioned empirical evidence, where, first, we do not allow for monetary policy, and second, we allow for the presence of debt bubbles to take into account the unfavourable consequences of fiscal stimulus on debt. Third, though a policy rule inspired by Bohn (1998), we consider state depended fiscal stimulus (through productive government expenditures) to remedy a recession but at the same time we control for the level of debt.

Regarding policy implications, we argue that fiscal asymetries that may not rely solely on fundamentals but on selfull-filling pessimism derived from initial conditions as recent empirical evidence indicates (De Grauwe and Yuemei, 2013, Ramey and Zubairy, 2016). In particular, we show that multiple equilibria a la Azariadis and Stachurski (2005) can arise and although countries have similar characteristics (e.g. Spain, Italy, Portugal and Greece) and follow similar policies, they may face divergent paths in debt and income, as observed in the data (Brendon and Corsetti, 2016). Our result is in line to Favero and Giavazzi (2007) where the absence of a debt feedback effect on taxes and government spending can result in incorrect estimates of the dynamic effects of fiscal shocks. However, we advance their study by theoretically showing that the feedback effect of debt on taxes may not be monotonic subject to the initial conditions (in line with Ramey and Zubairy, 2016). Once those non-linearities (phase of business cycle along with the level of debt) are taken into account, empirical studies may come in more precise results regarding fiscal multipliers.

Section 2 sets up the model and Section 3 examines the equilibrium properties, existence, uniqueness and stability. Section 4 investigates the effect of the policy parameters on steady-state output and we provide a simple numerical example about the short-run dynamics. Section 5 concludes the paper.

2 The model

2.1 Demand Side

We consider an overlapping generations model advanced by Diamond (1965) and Tirole (1985). There are N_t consumers who each live for two periods. They choose their consumption today, C_t, and tomorrow, d_{t+1}, to maximize intertemporal utility as given by the following utility function,

$$U = \ln C_t + \beta \ln d_{t+1} \tag{1}$$

where β is the weight that agents place in their second period utility. In the first period of their life, agents inelastically supply labour and they receive a wage rate, w_t which is taxed by τ_t. When old, the agents consume their savings and they receive a return on their savings, r_{t+1}. By solving their intertemporal problem, the savings, S, of each individual are positively determined by the after tax wage rate and their savings propensity, $s = \frac{\beta}{1+\beta}$,

$$S(w_t) = s(1 - \tau_t)w_t \tag{2}$$

2.2 Supply Side

On the supply side, there exists a continuum of firms that produces output, Y_t, using capital, k_t, labour, l_t, and a public good supplied by the government g_t,

$$Y_t = Ak_t^{\alpha} l_t^{1-\alpha} g_t^{\gamma} \qquad \alpha + \gamma < 1 \tag{3}$$

The wage rate and return on capital, using the labour market clearing condition, $l_t = 1$, are determined by

$$w_t = (1-\alpha)Ak_t^{\alpha} g_t^{\gamma} \tag{4}$$

$$R_t = \alpha Ak_t^{\alpha-1} g_t^{\gamma} \tag{5}$$

2.3 Government

We assume that the supply of the public good is determined by a Samuelson Rule, which states that the marginal benefit generated by the public good (expenditures) must be equal to the marginal cost of its production given by:

$$g_t = (\gamma A k_t^{\alpha})^{\frac{1}{1-\gamma}} \tag{6}$$

Using (6), we can compute the equilibrium wage and real interest as follows:

$$w(k_t) = (1-\alpha)Ak_t^{\alpha} g_t^{\gamma} = (1-\alpha)\tilde{A}k_t^{\frac{\alpha}{1-\gamma}} \tag{7}$$

$$R(k_t) = \alpha Ak_t^{\alpha-1} g_t^{\gamma} = \alpha \tilde{A} k_t^{\frac{\alpha}{1-\gamma}-1} \tag{8}$$

where $\tilde{A} \equiv \gamma^{\frac{\gamma}{1-\gamma}} A^{\frac{1}{1-\gamma}}$. Further, we assume that the government finances public expenditures not only from taxation but also by issuing government debt. The budget constraint of the government is given by

$$B_{t+1} = R_t B_t + g_t - \tau_t w_t \tag{9}$$

Following the fiscal rule estimated by Bohn (1998) and similar to Gali et al. (2007), we assume that the primary surplus/deficit is a function of the level of debt and income determined by the fiscal policy parameters, $a > 0$ and $b > 0$ given by

$$g_t - \tau_t w_t = -aB_t + by_t \tag{10}$$

Policy parameter a states what is the responsiveness of the deficits to the level of debt ("debt control" channel) and parameter b states the responsiveness of deficit in the level of income ("fiscal stimulus" channel). Thus, this rule places some level of fiscal discipline, "austerity", as given by a, in the sense that under an increase of debt, taxation has to increase so as to reduce deficit and, in turn, public debt. On the flip side, as the economy develops policy parameter b allows for higher structural deficit in order to finance public spending.[2]

[2] Expessing the primary deficit as a ratio of y_t we obtain $\frac{g_t - \tau_t w_t}{y_t} = -a\frac{B_t}{y_t} + b$. Thus, $b > 0$ is the part of deficit to income ratio that is structural. Interestingly, we show later on that even in the presence of a structural deficit (which is the case in many countries) our rule is able to place the economy in a sustainable path.

3 Equilibrium dynamics

Given that in equilibrium saving must be equal to investment in real capital and government bonds, after some algrebra (see Appendix 1), the dynamic equilibrium is determined by the following dynamical system of equations

$$k_{t+1} - k_t = (s(1-\alpha) + s(b-\gamma) - b)\, y(k_t) - k_t + (a(1-s) - R_t(k_t))\, B_t \tag{11}$$

$$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t) \tag{12}$$

The steady-state of capital stock and debt level in the economy is the bundle, (k^*, b^*) such that $k_{t+1} - k_t = 0$ and $B_{t+1} - B_t = 0$ hold simultaneously.

Proposition 1 *(Existence and Uniqueness). For (i) $b < \frac{s(1-\alpha-\gamma)}{(1-s)} \equiv b^{\max}$ and (ii) $a < \frac{\alpha}{(1-s)(s(1-\alpha-\gamma)-(1-s)b)} \equiv a^{\max}$ there exist two non-trivial equilibrium steady states, $k_{ss}^{low} > 0$ and $k_{ss}^{high} > 0$ where $k_{ss}^{low} < k_{ss}^{high}$.*
Proof. *Appendix 2.* ▪

Proposition 2 *(Stability) Both steady-states are stable. The lower equilibrium, k_{ss}^{low}, is saddle-path stable and the higher equilibrium, k_{ss}^{high}, is a stable node.*
Proof. *Appendix 3.* ▪

Proposition 1 shows that if the level of structural deficit, b, is not high enough to crowd-out investment in capital and, if the response of the tax rate to the level of debt, a, is not high enough (limits for "austerity", condition (ii)), then there exist two strictly positive non-trivial equilibrium steady-states.[3] Proposition 2 shows that both equilibria are stable, the relatively lower one displays saddle-path stability and relatively higher one is a stable node. Proposition 1 together with Proposition 2 imply that the initial conditions, the initial level of debt and physical capital, determine the long-run position of the economy even the structural parameters of the economy can be the same (see the phase diagram in Figure 1). In particular, other things equal, for relatively high initial volume of debt and relatively low initial capital stock the economy can be in a position of exploding debt leading to a debt bubble (point B, Figure 1). While, after a threshold level of initial capital stock and volume of debt the economy will converge to an equilibrium level of high capital stock and sustainable debt (point A, Figure 1). We illustrate this point using a numerical example in the following section.

4 Policy Effects and Implications

In this section we study the properties of the policy parameters of the fiscal policy rule and provide the associated policy implications.

Proposition 3 *The policy parameter, a, negatively affects the relatively lower steady state, k_{ss}^{low}, and, positively, affects the higher steady-state equilibrium.*
Proof. *Appendix 4.* ▪

[3]For an alternative exposition, as can be seen in Appendix 1, the fiscal rule can be expressed in terms of the tax rate as $\tau(k_t, B_t) = (aB_t - (b-\gamma)y(k_t))/w_t(k_t)$.

Proposition 3 states that the equilibria we derived in Proposition 1 display different properties. An increase in the "austerity" parameter, a, decreases the relatively lower steady-state of the capital stock while it increases the relatively higher steady-state capital stock. In other words, the higher the responsiveness of the tax rate to the level of debt, the higher the gap between the two equilibria. The interesting implication of this theoretical property is that the government by implementing higher austerity can help economies with high level of initial debt and relatively low capital stock to enter to the area of sustainability. At the same time, the rule changes endogenously once the capital stock achieves a certain threshold, and taxation dynamically reduces so as to avoid a huge crowding out effect of the private sector.

To better illustrate our analytical results, we provide a simple numerical example using standard parameter values from the growth literature. Assume two countries, A (e.g. Spain) and B (e.g. Greece). Both countries have the same structural characteristics such as total factor productivity, $A = 8$, share of physical capital on the production function, $\alpha = 0.25$, productivity of the public good, $\gamma = 0.15$, time preference, $\beta = 0.098$ and both follow the same rule with weights, $a = 0.5$ and $b = 0.013$ (using the estimates by Bohn, 1998). Also, both countries are developed in the sense that both belong to the area of low interest rates (right hand side of the discontinuity in Figure 1). They only differ in their initial level of debt and capital stock. Country B has relatively lower initial capital stock, $K_0^B = 0.5$, and higher initial level of debt, $B_0^B = 0.5$, than country A, $K_0^A = 3$ and $B_0^A = 0.3$ correspondigly. Then, our numerical exercise is to examine the way that the dynamics of each country evolve by setting those different initial conditions for each country. Table 1 shows that Country A reaches the high steady-state of capital stock with sustainable steady-state debt, while in Table 2, Country B ends up in a situation of exploding debt. Thus, as Proposition 1 and 2 imply although those countries have the same structural characteristics and follow the same policy rule, they will display different dynamics and steady-states just by starting with different level of inherited debt and capital stock.

Table 1
Country A: Dynamic adjustment towards the stable steady-state
with $a = 0.5$ and $b = 0.013$

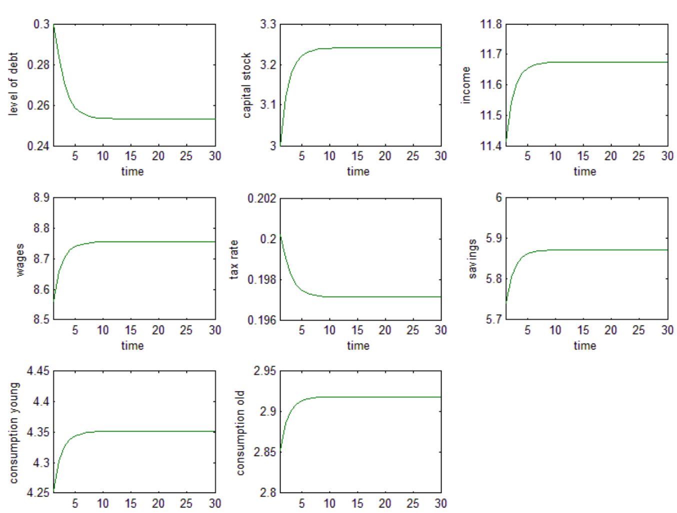
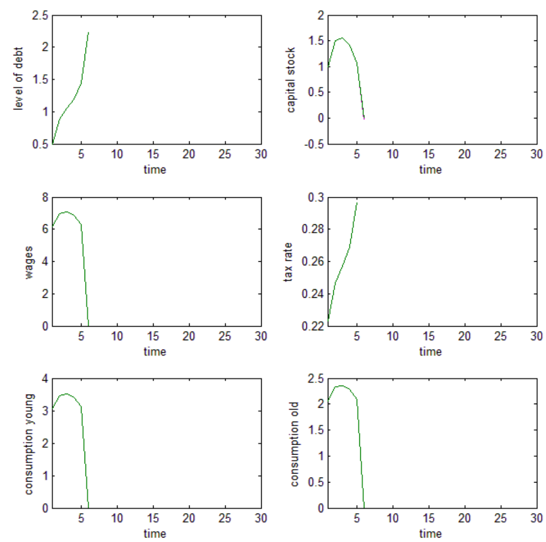


Table 2
Country B: Dynamic adjustment towards exploding debt
with $a = 0.5$ and $b = 0.013$

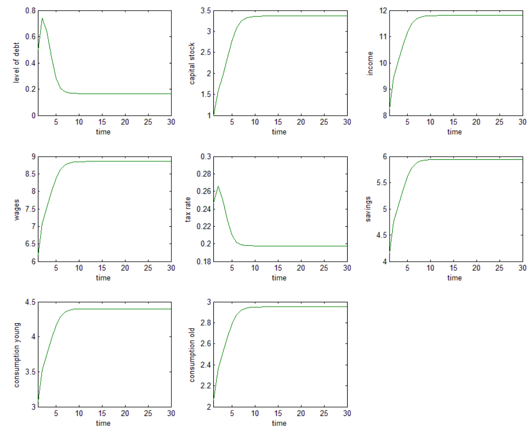


The policy implication that can be derived from this result is that, in the design of a policy rule, the choise of the level of austerity has to depend not only on the fundamentals but also on the initial state of the inherited debt and income. So, in cases of exploding debt, following Proposition 3, countries have to increase the response of taxation to the level of debt, a, so as to expand the area of sustainability. In Table 3 we provide the dynamic path of Country B by only increasing the level of a from 0.5 to 0.8.

Table 3

Country B: Dynamic adjustment towards the stable steady-state

with $a = 0.8$ and $b = 0.013$



According to Table 3, with higher a, the policy rule can place Country B to a stable path for the capital stock associated with sustainable long-run level of debt. An interesting outcome is the endogenous non-monotonic dynamics of the tax rate. The tax rate increases at low levels of capital stock so as to decrease deficit and stabilize the level of debt. As debt falls and income increases, taxes fall in order to boost savings that will form a higher capital stock and a higher tax base to finance government expenditures. In other words, the two features of the rule work as follows. On the one hand, higher "austerity" is enabled to put the economy in a stable equilibrium path. On the other hand, in line with the optimal Samuelson rule for the provision of public services, deficit increases in order to finance productive government spending but through higher tax base and lower taxes. Differently to other policy rules (which work in environments of stable dynamic paths) and state that deficits have to decrease as output increases (for consumption smoothing) this rule guarantees some level of fiscal consolidation so as to avoid the emergence of debt bubbles but also provides fiscal stimulus so as the economy to be able to escape from the unstable path.

Under the theoretical foundations of this short paper, we believe that, first, the empirical investigation of policy rules with the aforementioned properties and, second, a detailed calibration of different countries under a debt crisis open interesting research directions with subsequent implications for the design of fiscal policy rules for debt sustainability.

Last, we do the same work for the policy instrument that controls the level of structural deficit (and has extensively analyzed by Chalk, 2000).

Proposition 4 *The structural deficit parameter, b, positively affects the lower steady state k_{ss}^{low} and negatively affects the higher, k_{ss}^{high}, steady-state equilibrium.*
Proof. *Appendix 4.* ▪

Proposition 4 states that the equilibria display different properties also for the level of structural deficit. An increase in the level of structural deficit positively affects the low steady-state while an increase in the level of structural deficit negatively affects the high steady-state.

This theoretical result conforms with the result of Chalk (2000). A decline in the level of structural deficit positively affects the level of high capital stock and increases the probability that a country can escape from a poverty trap as it can increase the gap between the two equilibria and, in turn, increase the area with stable node dynamics. However, as has already been shown by Chalk (2000) and condition (i) of Proposition 1 there are limit in the use of structural deficits.

5 Conclusion

Motivated by the recent debt crisis experience in the Eurozone we examined the theoretical properties of an empirical rule inspired by Bohn (1998) in an overlapping generations framework with productive public expenditures. Departing from discretionary policy making followed nowadays by the European Commision and the IMF, we suggest a policy rule that can endogenously adapt to the actual level of debt and income in the economy. We show that even in the presence of structural deficits a rule that is clever enough to place stability/instability when necessary (subject to the initial conditions) can build a sustainable path for debt and output. In particular, the rule has to provide fiscal consolidation to guarantee a stable path for investors but at the same time it has to provide fiscal stimulus (productive government expenditures) to stimulate the production of an economy with low private investment.

We beleive that our theoretical results brings forward interesting testable predictions for empirical research.

6 Appendix

Appendix 1: Derivation of the Dynamical System

Given that in equilibrium saving must be equal to investment in real capital and government bonds, the dynamic equilibrium is given by the following dynamical system

$$B_{t+1} = R(k_t)B_t - aB_t + by_t$$

$$k_{t+1} + B_{t+1} = s(1 - \tau_t)w(k_t)$$

and equation (10). Given that the government follows the Samuleson rule to determine the public spending, the marginal tax needs to adjust to implement the fiscal rule,

$$g_t - \tau w = -aB_t + by_t$$

where deviding by y_t

$$\frac{g_t}{y_t} - \tau\frac{w_t}{y_t} = -a\frac{B_t}{y_t} + b$$

and solving for τ_t the marginal tax is equal to

$$-\tau_t = -a\frac{B_t}{y_t} + b - \frac{g_t}{y_t}$$

. Also, from the Samuelson rule and the production function we get

$$\frac{g_t}{y_t} = \gamma$$

. Then, we have that

$$-\tau(k_t, B_t)w_t = -aB_t + (b-\gamma)y_t \tag{13}$$

. We simplify the expression for k_{t+1} using eq. (13) and $\frac{w_t}{y_t} = 1 - a$,

$$k_{t+1} - k_t = \left(s(1-\alpha) + s(b-\gamma) - b\right) y(k_t) - k_t + \left(a(1-s) - R_t(k_t)\right) B_t$$

which is equation (12).

Appendix 2. Existence and Uniqueness

From (11) and (12) the change of capital stock and the debt level of the economy is determined by the following dynamic system:

$$k_{t+1} - k_t = \left(s(1-\alpha) + s(b-\gamma) - b\right) y(k_t) - k_t + \left(a(1-s) - R_t(k_t)\right) B_t$$

$$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t)$$

We will first analyze the existence and uniqueness of steady-state equilibrium and, then, we will analyze the stability of equilibrium and the dynamic behavior of capital and debt. The steady-state of capital stock and debt level in the economy is that bundle, k, b, where $k_{t+1} - k_t = 0$ and $B_{t+1} - B_t = 0$ simultaneously.

The locus where the change of debt is zero, $B_{t+1} - B_t = 0$ is given by

$$B = \frac{by(k)}{(1+a) - R(k)} \equiv \Gamma(k)$$

The properties of $\Gamma(k)$ are the following:

1. $\lim_{k\to 0} \Gamma(k) = 0$ and $\lim_{k\to\infty} \Gamma(k) = \infty$.

2. $\Gamma(k)$ is discontinuous at $k = \check{k}$ where $\check{k} : (1+a) - R(\check{k}) = 0$ Under the Cobb-douglas function production function $\check{k} = \left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}}$

3. For $0 < k < \check{k}$ then $\Gamma(k) < 0$ and for $\check{k} < k < \infty$ then $\Gamma(k) > 0$.

Proof. Note that $y(k) > 0$ for any k and $\frac{\partial((1+a)-R(k_t))}{\partial k} = -\acute{R}(k_t) > 0$ (monotonic function). Also, $\lim_{k\to 0}(1+a) - R(k_t) = -\infty$ and $\lim_{k\to\infty}(1+a) - R(k_t) = (1+a) > 0$. This means that for $0 < k < \check{k}$ then $(1+a) - R(k_t) < 0$ and for $\check{k} < k < \infty$ then $(1+a) - R(k_t) > 0$ For

$R(k) = \alpha \tilde{A} k^{\frac{\alpha}{1-\gamma}-1}$ that is $(1+a) - \alpha \tilde{A} k^{\frac{\alpha}{1-\gamma}-1} > 0 \Rightarrow \hat{k} < \left(\frac{(1+a)}{\alpha \tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}}$

4. The limit behavior of $\Gamma(k)$ from the left and the right of discontinuity is given by: $\lim_{k \to \check{k}^-} \Gamma(k) = -\infty$ and $\lim_{k \to \check{k}^+} \Gamma(k) = \infty$.

5. The first order derivative $\Gamma(k)$ is given by:
$\frac{\partial \Gamma(k)}{\partial k} = b \frac{y'(k_t)((1+a) - R(k_t)) + (R'(k_t))y(k_t)}{((1+a) - R(k_t))^2}$ which after simplification (see footnote)[4]

$$\frac{\partial \Gamma(k)}{\partial k} = b \frac{\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right) R(k)}{((1+a) - R(k_t))^2}$$

For $0 < k < \check{k}$ then $\frac{\partial \Gamma(k)}{\partial k} < 0$.

This happens because $0 < k < \check{k}$, $y'(k_t)((1+a) - R(k_t)) < 0$ and $(R'(k_t))y(k_t) < 0$ given that $y'(k_t) > 0$, $(1+a) - R(k_t) < 0$ and $R'(k_t) < 0$ and $y(k_t) > 0$.

Definition 1 *Define* $k_{\min} \equiv \left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$

For $\check{k} < k < \infty$ then,
(i) $\frac{\partial \Gamma(k)}{\partial k} < 0$ for $\check{k} < k < k_{\min}$
(ii) $\frac{\partial \Gamma(k)}{\partial k} > 0$ for $k_{\min} < k < \infty$.

Proof. $\frac{\partial \Gamma(k)}{\partial k} < 0$ if $y'(k)((1+a) - R(k)) + (R'(k))y(k) < 0$ which following $R(k) = \alpha \tilde{A} k_t^{\frac{\alpha}{1-\gamma}-1} = \alpha \frac{y(k)}{k}$ and $R_k = \alpha(\frac{\alpha}{1-\gamma} - 1)\tilde{A} k_t^{\frac{\alpha}{1-\gamma}-2} = \alpha(\frac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k^2}$, $y'(k_t) = \frac{\alpha}{(1-\gamma)} \tilde{A} k_t^{\frac{\alpha}{(1-\gamma)}-1} = \frac{\alpha}{(1-\gamma)} \frac{y(k)}{k}$ we have

[4]"""""""Simplification of the numerator of first order derivative

$$\frac{\alpha}{(1-\gamma)} \frac{y(k)}{k}((1+a) - R(k)) + \alpha(\frac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k^2} y(k) \Rightarrow$$
$$\left(\frac{1}{(1-\gamma)}((1+a) - R(k)) + (\frac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k}\right) \frac{\alpha y(k)}{k}$$
$$\left((\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{(1-\gamma)}) + (\frac{\alpha}{1-\gamma} - 1)\frac{R(k)}{\alpha}\right) R(k)$$
$$\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{(1-\gamma)} + \frac{\alpha}{1-\gamma}\frac{R(k)}{\alpha} - \frac{R(k)}{\alpha}\right) R(k)$$
$$\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{(1-\gamma)} + \frac{R(k)}{1-\gamma} - \frac{R(k)}{\alpha}\right) R(k)$$
$$\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right) R(k)$$

$$\frac{\partial \Gamma(k)}{\partial k} = b \frac{\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right) R(k)}{((1+a) - R(k_t))^2}$$ """""""""""

$$\frac{\alpha}{(1-\gamma)}\tfrac{y(k)}{k}((1+a)-R(k))+\alpha(\tfrac{\alpha}{1-\gamma}-1)\tfrac{y(k)}{k^2}y(k)<0\Rightarrow$$
$$\frac{\alpha}{(1-\gamma)}((1+a)-R(k))+\alpha(\tfrac{\alpha}{1-\gamma}-1)\tfrac{y(k)}{k}<0$$
$$\frac{\alpha}{(1-\gamma)}((1+a)-\alpha\tfrac{y(k)}{k})+\alpha(\tfrac{\alpha}{1-\gamma}-1)\tfrac{y(k)}{k}<0$$
$$(1+a)\frac{\alpha}{(1-\gamma)}-\alpha\frac{\alpha}{(1-\gamma)}\tfrac{y(k)}{k}+\alpha\tfrac{\alpha}{1-\gamma}\tfrac{y(k)}{k}-\alpha\tfrac{y(k)}{k}<0$$
$$(1+a)\frac{\alpha}{(1-\gamma)}-\alpha\tfrac{y(k)}{k}<0$$
$$\frac{(1+a)}{(1-\gamma)}-\tfrac{y(k)}{k}<0$$
$$\frac{(1+a)}{(1-\gamma)}-\tfrac{R(k)}{\alpha}<0\Rightarrow(1+a)\alpha-(1-\gamma)R(k)<0 \text{ that is}$$

$(1+a)\alpha-(1-\gamma)\alpha\tilde{A}k^{\frac{\alpha}{1-\gamma}-1}<0\Rightarrow k^{\frac{\alpha}{1-\gamma}-1}>\frac{(1+a)}{(1-\gamma)\tilde{A}}\Rightarrow k<\left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1}{\frac{\alpha}{1-\gamma}-1}}\Rightarrow k<\left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}\equiv k_{\min}$. The opposite otherwise.

Second order derivative:

$$\frac{\partial^2\Gamma(k)}{\partial k^2}=b\frac{\left(\frac{(1+a)\acute{R}}{(1-\gamma)}-\frac{2R\acute{R}}{\alpha}\right)(1+a-R)^2+\left(\frac{(1+a)}{(1-\gamma)}-\frac{R}{\alpha}\right)R2\,(1+a-R)\,\acute{R}}{(1+a-R)^4}$$

taking common factor $\acute{R}$ and eliminating $(1+a-R)$

$$b\acute{R}\frac{\left(\frac{(1+a)}{(1-\gamma)}-\frac{2R}{\alpha}\right)((1+a)-R)+\left(\frac{(1+a)}{(1-\gamma)}-\frac{R}{\alpha}\right)R2}{(1+a-R)^3}=$$

$$b\acute{R}\frac{\left(\frac{(1+a)}{(1-\gamma)}-\frac{2R}{\alpha}\right)((1+a)-R)+\left(\frac{(1+a)2R}{(1-\gamma)}-\frac{2R^2}{\alpha}\right)}{(1+a-R)^3}=b\acute{R}\frac{\left(\frac{(1+a)}{(1-\gamma)}(1+a)-\frac{2R}{\alpha}(1+a)\right)-\left(\frac{(1+a)R}{(1-\gamma)}-\frac{2R^2}{\alpha}\right)+\big(}{(1+a-R)^3}$$

$$\frac{b\acute{R}(1+a)}{(1-\gamma)\alpha}\frac{\alpha(1+a)-(1-\gamma)2R+\alpha R}{(1+a-R)^3}=\frac{b\acute{R}(1+a)}{(1-\gamma)\alpha}\frac{\alpha(1+a)-(1-\gamma)2R+\alpha R}{(1+a-R)^3}=$$

$$\frac{\partial^2\Gamma(k)}{\partial k^2}=\frac{b\acute{R}}{(1-\gamma)\alpha}\frac{R(\alpha-2(1-\gamma))+\alpha(1+a)}{(1+a-R)^3}$$

Analysis of $\frac{\partial^2\Gamma(k)}{\partial k^2}$. We analyze the case after the discontinuity, that is, for $\check{k}<k<\infty$

Then, for $\check{k}<k<\infty$ then $1+a-R>0$ then $\frac{\partial^2\Gamma(k)}{\partial k^2}>0$ if $R(\alpha-2(1-\gamma))+\alpha(1+a)<0$ $\Rightarrow R(2(1-\gamma)-\alpha)>\alpha(1+a)\Rightarrow R>\frac{\alpha(1+a)}{2(1-\gamma)-\alpha}\Rightarrow k<\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}\equiv\tilde{k}$. So, for $\check{k}<k<\tilde{k}$, $\frac{\partial^2\Gamma(k)}{\partial k^2}>0$

We also, want to show that $\tilde{k}$ is indeed above the discontinuity $\check{k}$

First, we compare $\tilde{k}$ with $\check{k}$, we need $\tilde{k}>\check{k}\Rightarrow\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}>\left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}}\Rightarrow$ $\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}<\frac{(1+a)}{\alpha\tilde{A}}\Rightarrow$

$\alpha<(2(1-\gamma)-\alpha)\Rightarrow 2\alpha<2(1-\gamma)\Rightarrow a-(1-\gamma)<0$ which holds.

Thus the function is convex for $\check{k} < k < \tilde{k}$, $\frac{\partial^2 \Gamma(k)}{\partial k^2} > 0$ and concave for $\tilde{k} < k < \infty$, $\frac{\partial^2 \Gamma(k)}{\partial k^2} < 0$. Last, $\lim_{k \to \infty} \frac{\partial^2 \Gamma(k)}{\partial k^2} = 0$.

Now we are going to analyze the locus where the change of capital stock is zero, $K_{t+1} - K_t = 0$ is given by

$$\Theta(k) = \frac{(s(1-\alpha) - (1-s)b - s\gamma)\, y(k) - k}{(R(k) - a(1-s))}$$

where $y(k) = \tilde{A} k^{\overline{(1-\gamma)}^{\alpha}}$ and $y_k = \tilde{A} \frac{\alpha}{(1-\gamma)} k^{\overline{(1-\gamma)}^{\alpha} - 1} = \frac{\alpha}{(1-\gamma)} \frac{y(k)}{k}$ and the limit behavior is: $\lim_{t \to 0} y(k) = 0, \lim_{t \to \infty} y(k) = \infty$ $\lim_{t \to 0} y_k = \infty$ and $\lim_{t \to \infty} y_k = 0$
and
$R(k) = \alpha \tilde{A} k_t^{\frac{\alpha}{1-\gamma} - 1} = \alpha \frac{y(k)}{k}$ and $R_k = \alpha(\frac{\alpha}{1-\gamma} - 1) \tilde{A} k_t^{\frac{\alpha}{1-\gamma} - 2} = \alpha(\frac{\alpha}{1-\gamma} - 1) \frac{y(k)}{k^2} = (\frac{\alpha}{1-\gamma} - 1) \frac{R(k)}{k}$

1. $\lim_{k \to 0} \Theta(k) = \lim_{k \to 0} \Theta(k) = \frac{\Omega y(0) - 0}{(R(0) - a(1-s))} = 0$ and $\lim_{k \to \infty} \Theta(k) = \frac{\frac{\partial((s(1-\alpha)-(1-s)b-s\gamma)y(k)-k)}{\partial k}}{\frac{\partial((R(k)-a(1-s)))}{\partial k}} = \lim_{k \to \infty} \frac{(s(1-\alpha)-(1-s)b-s\gamma)\acute{y}(k)-1}{\acute{R}(k)} = \lim_{k \to \infty} \frac{(s(1-\alpha)-(1-s)b-s\gamma)y''(k)}{R''(k)} = \infty$

2. $\Theta(k)$ is discontinuous at $k = \hat{k}$ where $\hat{k} : R(\hat{k}) - a(1-s) = 0$. Under a Cobb-douglas production function $\hat{k} = \left(\frac{a(1-s)}{\alpha \tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$

Remark 1 *We show that the discontinuity of the debt locus to be below the discontinuity of the k locus. That is* $\left(\frac{(1+a)}{\alpha \tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}} < \left(\frac{a(1-s)}{\alpha \tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \Rightarrow (1+a) > a(1-s) \Rightarrow (1+a) > a(1-s) \Rightarrow 1 > -as$ *where for* $a > 0$ *and* $\alpha \in (0,1)$ *this always holds.*

Assumption 1 *We assume a positive effect of income (investment) on the accumulation of capital stock which happens under the following condition* $(s(1-\alpha) + s(b-\gamma) - b > 0$ *or* $b < \frac{s(1-\alpha-\gamma)}{(1-s)} \equiv b^{\max}$

3. Define k_{AUT}: $(s(1-\alpha) - (1-s)b - s\gamma)\, y(k_{AUT}) - k_{AUT} = 0$ (in other words $B = 0$) which in the Cobb-douglas case is given by: $(s(1-\alpha) - (1-s)b - s\gamma)\, \tilde{A} k^{\overline{(1-\gamma)}^{\alpha-(1-\gamma)}} - 1 = 0 \Rightarrow k_{AUT} = \left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}}$.

Assumption 2 *Parametric condition such that:* $\hat{k} > k_{AUT}$ *is:* $\left(\frac{a(1-s)}{\alpha \tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} > \left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}} =$
$\frac{a(1-s)}{\alpha \tilde{A}} < \frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)} \Rightarrow a(1-s)((s(1-\alpha) - (1-s)b - s\gamma)) < \alpha$ *which imposes limits on austerity* $a < \frac{\alpha}{(1-s)(s(1-\alpha)-(1-s)b-s\gamma)} \equiv a^{\max}$.

Then, because of concavity of $y(k)$ it is easy to show that the value of $\Theta(k)$ is given by the following remark.

Remark 2 *(i) for* $0 < k < k_{AUT}$ *then* $\Theta(k) > 0$ *and* $R(\hat{k}) - a(1-s) > 0$
(ii) for $k_{AUT} < k < \hat{k}$ *then* $\Theta(k) < 0$ *and* $R(\hat{k}) - a(1-s) > 0$
(iii) for $\hat{k} < k < \infty$ *then* $\Theta(k) > 0$ *and* $R(\hat{k}) - a(1-s) < 0$

4. The limit behavior of $\Theta(k)$ at the discontinuity is given by:

$\lim_{k \to \hat{k}^-} \Theta(k) = -\infty$ and $\lim_{k \to \hat{k}^+} \Theta(k) = \infty$.

5. The first order derivative of $\Theta(k)$.

Define $\Omega \equiv (s(1-\alpha) - (1-s)b - s\gamma)$

$$\frac{\partial \Theta(k)}{\partial k} = \frac{(\Omega y_k - 1)(R(k) - a(1-s)) - (\Omega y(k) - k)R_k}{(R(k)-a(1-s))^2} =$$

We then use the following equations
$R(k) = \alpha \frac{y(k)}{k}$, and $R_k = \alpha(\frac{\alpha}{1-\gamma} - 1)\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-2} = \alpha(\frac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k^2} = (\frac{\alpha}{1-\gamma} - 1)\frac{R(k)}{k}, y_k = \frac{\alpha}{(1-\gamma)} \frac{y(k)}{k} = \frac{1}{(1-\gamma)} R(k)$
Then, the derivative gets: (we express everything in $R(k)$)

$$\frac{\partial \Theta(k)}{\partial k} = \frac{(\Omega \frac{1}{(1-\gamma)} R(k) - 1)(R(k) - a(1-s)) - (\Omega \frac{R(k)k}{\alpha} - k)(\frac{\alpha}{1-\gamma} - 1)\frac{R(k)}{k}}{(R(k)-a(1-s))^2} =$$
$$= \frac{(\Omega \frac{1}{(1-\gamma)} R(k) - 1)(R(k) - a(1-s)) - (\Omega \frac{R(k)k}{\alpha} - k)(\frac{\alpha}{1-\gamma} - 1)\frac{R(k)}{k}}{(R(k)-a(1-s))^2} =$$

$$\frac{(\Omega \frac{R(k)}{(1-\gamma)}(R(k)-C) - (R(k)-C) - (\Omega \frac{R(k)k}{\alpha}(\frac{\alpha}{1-\gamma}-1)\frac{R(k)}{k} - k(\frac{\alpha}{1-\gamma}-1)\frac{R(k)}{k})}{(R(k)-C)^2} = \frac{\Omega \frac{R^2}{(1-\gamma)} - C\Omega \frac{R}{(1-\gamma)} - R + C - (\Omega \frac{R^2}{\alpha}(\frac{\alpha}{1-\gamma}-1) - (\frac{\alpha}{1-\gamma}-1)R)}{(R(k)-C)^2} =$$
$$\frac{\Omega \frac{R^2}{(1-\gamma)} - C\Omega \frac{R}{(1-\gamma)} - R + C - \frac{\Omega R^2}{1-\gamma} + \Omega \frac{R^2}{\alpha} + \frac{\alpha}{1-\gamma}R - R}{(R(k)-C)^2} = \frac{\frac{\Omega}{\alpha}R^2 - (\frac{\Omega a(1-s)-\alpha}{(1-\gamma)} + 2)R + a(1-s)}{(R-a(1-s))^2}$$

$$\frac{\partial \Theta(k)}{\partial k} = \frac{\frac{\Omega}{\alpha}R^2 - (\frac{\Omega a(1-s)-\alpha}{(1-\gamma)} + 2)R + a(1-s)}{(R-a(1-s))^2}$$

Define $Z = \frac{\Omega}{\alpha}$, $C = a(1-s)$ and $\Xi = (\frac{\Omega C - \alpha}{(1-\gamma)} + 2) = (\frac{a(AC-1)}{(1-\gamma)} + 2)$

$$\frac{\partial \Theta(k)}{\partial k} = \frac{ZR^2 - \Xi R + C}{(R(k)-C)^2}$$

which is a quadratic equation with **at most two roots**.

5.1 (Limiting behavior) By applying the de hospital rule

$$\lim_{k \to 0} \frac{\partial \Theta(k)}{\partial k} = \frac{\Omega}{\alpha} > 0 \text{ and } \lim_{k \to \infty} \frac{\partial \Theta(k)}{\partial k} = \frac{1}{(a(1-s))} > 0$$

5.2 $\frac{\partial\Theta(k)}{\partial k} > 0$ if $ZR^2 - \Xi R + C > 0$ and $\frac{\partial\Theta(k)}{\partial k} < 0$ for $ZR^2 - \Xi R + C < 0$ which depends on the number of roots.

Discriminant: $\Xi^2 - 4ZC = (\frac{a(AC-1)}{(1-\gamma)} + 2)^2 - 4ZC = \frac{a^2(ZC^2-2ZC+1)}{(1-\gamma)^2} + \frac{4a(ZC-1)}{(1-\gamma)} + 4 - 4ZC = \frac{a^2(ZC^2-2ZC+1)}{(1-\gamma)^2} + \frac{4a(ZC-1)}{(1-\gamma)} + 4(1-ZC)$

6.(Second order derivatives)

The first order derivative is given by:

$\frac{\partial\Theta(k)}{\partial k} = \frac{ZR^2-\Xi R+C}{(R(k)-C)^2}$

Taking the second order derivative we obtain that:

$\frac{\partial^2\Theta(k)}{\partial k^2} = \frac{(Z2R\acute{R}-\Xi\acute{R})(R-C)^2-(ZR^2-\Xi R+C)(R(k)-C)2\acute{R}}{(R-C)^4} =$

$\acute{R}\frac{(Z2R-\Xi)(R-C)-(ZR^2-\Xi R+C)2}{(R-C)^3} =$

$\acute{R}\frac{Z2R(R-C)-\Xi(R-C)-2ZR^2+2\Xi R-2C)}{(R-C)^3} =$

$\acute{R}\frac{(Z2RR-Z2RC)-\Xi R+\Xi C-2ZR^2+2\Xi R-2C)}{(R-C)^3} =$

$\acute{R}\frac{Z2R^2-Z2RC-\Xi R+\Xi C-2ZR^2+2\Xi R-2C}{(R-C)^3} =$

$\acute{R}\frac{-Z2RC+\Xi C+\Xi R-2C}{(R-C)^3} =$

$\acute{R}\frac{R(\Xi-2ZC)+C(\Xi-2)}{(R-C)^3} = \acute{R}\frac{R(\Xi-2ZC)+C((\frac{\Omega C-\alpha}{(1-\gamma)})}{(R-C)^3} \Rightarrow \acute{R}\frac{R(\Xi-2ZC)+C(\alpha(\frac{\frac{\Omega}{\alpha}C-1}{(1-\gamma)})}{(R-C)^3} =$

$$\frac{\partial^2\Theta(k)}{\partial k^2} = \acute{R}\frac{R(\Xi-2ZC)+C(\alpha\frac{ZC-1}{(1-\gamma)})}{(R-C)^3}$$

The derivative is negative until the discontinuity $0 < k < \hat{k}$ $(R - C > 0)$ of the kk locus if: $R(\Xi - 2ZC) + C(\alpha(\frac{ZC-1}{(1-\gamma)}) > 0$ because $\acute{R} < 0$. Thus, we need that,

$$R > \frac{-C(\alpha(\frac{AC-1}{(1-\gamma)}))}{(\Xi-2ZC)}$$

$$\alpha\tilde{A}k_t^{\frac{\alpha-(1-\gamma)}{1-\gamma}} > \frac{-C(\alpha(\frac{ZC-1}{(1-\gamma)}))}{(\Xi-2ZC)}$$

$$k < \left(\frac{-C(\alpha(\frac{ZC-1}{(1-\gamma)}))}{\alpha\tilde{A}(\Xi-2ZC)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \equiv \tilde{k}$$

this is a necessary and sufficient condition for concavity. We now want to show if this is true for $0 < \tilde{k} < \hat{k}$ ($\tilde{k}$ below the discontinuity $\hat{k}$).

$\left(\frac{-C(\alpha(\frac{ZC-1}{(1-\gamma)}))}{(\Xi-2ZC)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} < \left(\frac{C}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$

$\left(\frac{-C(\alpha(\frac{ZC-1}{(1-\gamma)}))}{(\Xi-2ZC)}\right) > \left(\frac{C}{\alpha\tilde{A}}\right)$

$\left(\frac{-(\alpha(\frac{ZC-1}{(1-\gamma)}))}{(\Xi-2ZC)}\right) > 1$
$-(\alpha(\frac{ZC-1}{(1-\gamma)})) > (\Xi - 2ZC)$
$(\alpha(\frac{AC-1}{(1-\gamma)})) < (\Xi - 2ZC)$

Note that $\Xi = (\frac{\Omega C-\alpha}{(1-\gamma)} + 2) = (\frac{\frac{\Omega}{\alpha}C-1}{(1-\gamma)} + 2) = (\frac{\alpha(\frac{\Omega}{\alpha}C-1)}{(1-\gamma)} + 2) = (\frac{\alpha(ZC-1)}{(1-\gamma)} + 2)$

Substituting to the inequality $(\alpha(\frac{ZC-1}{(1-\gamma)})) < \frac{\alpha(ZC-1)}{(1-\gamma)} + 2 - 2ZC \Rightarrow 0 < +2 - 2ZC \Rightarrow 2ZC < 2 \Rightarrow ZC < 1$. Which holds from the assumption that limits austerity (see Remark) where we have $\frac{a(1-s)}{\alpha} < \frac{1}{(s(1-\alpha)-(1-s)b-s\gamma)} \Rightarrow \frac{C}{\alpha} < \frac{1}{\Omega} \Rightarrow \frac{\Omega}{\alpha} < \frac{1}{C} \Rightarrow Z < \frac{1}{C} \Rightarrow ZC < 1$.

Lemma 1 *Under Remark 1, then $\Theta(k)$ is concave and inverse U-shaped for $0 < k < \hat{k}$ and convex (U-shaped) for $\hat{k} < k < \infty$.*

To illustrate this Lemma with diagrams we get:

The steady states are defined by the following expression

$$F(k) = \Theta(k) - \Gamma(k)$$
$$F(k) = \frac{(\Omega)\,y(k) - k}{(R(k) - C)} - \frac{by(k)}{(1+a) - R(k)}$$

i. $F(0) = 0$

For $0 < k < \check{k}$, $\Theta(k) > 0$ and $\Gamma(k) < 0$ thus, $F(k) > 0$. Also, $\lim_{k\to\check{k}^-} F(k) = +\infty$

Then, $\lim_{k\to\check{k}^-} F(k) = -\infty$ and $\lim_{k\to\check{k}^+} \acute{F}(k) > 0$. So, just after the discontinuity of the debt locus the $F(k)$ function is increasing.

Also, $\lim_{k\to\hat{k}^-} F(k) = -\infty$, $\lim_{k\to\hat{k}^-} \acute{F}(k) < 0$.

So, $F(k)$ is increasing from the discontinuity of the debt locus and it is decreasing at the discontinuity of the capital stock locus.

Since, $\check{k} < k < \hat{k}$ the derivative changes sign, we are going to explore if the maximum of the function is positive.

$\acute{F}(k) = \frac{\partial\Theta(k)}{\partial k} - \frac{\partial\Gamma(k)}{\partial k}, \frac{\partial\Theta(k_{\max})}{\partial k} - \frac{\partial\Gamma(k_{\max})}{\partial k} = 0 \Rightarrow \frac{ZR^2-\Xi R+C}{(R(k)-C)^2} - \left(\frac{(1+a)}{(1-\gamma)} - \frac{R}{\alpha}\right) R = 0$
$ZR^2 - \Xi R + C - \left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right) R(k)\,(R(k) - C)^2 = 0$
$ZR^2 - \Xi R + C - \left(\frac{(1+a)}{(1-\gamma)} - \frac{R}{\alpha}\right) R\,(R^2 - 2RC + C^2) = 0$

$ZR^2 - \Xi R + C - \left(\frac{(1+a)-(1-\gamma)R}{(1-\gamma)\alpha}\right)(R^3 - 2R^2C + RC^2) = 0$
$ZR^2 - \Xi R + C - \frac{R^3(1+a)}{(1-\gamma)\alpha} + \frac{2R^2C(1+a)}{(1-\gamma)\alpha} - \frac{RC^2(1+a)}{(1-\gamma)\alpha} + \frac{R^3(1-\gamma)R}{(1-\gamma)\alpha} - \frac{2R^2C(1-\gamma)R}{(1-\gamma)\alpha} + \frac{RC^2(1-\gamma)R}{(1-\gamma)\alpha} = 0$

$$F^{''}(k) = \acute{R}\frac{R(\Xi - 2ZC) + C(\alpha\frac{ZC-1}{(1-\gamma)})}{(R-C)^3} - \frac{b\acute{R}}{(1-\gamma)\alpha}\frac{R(\alpha - 2(1-\gamma)) + \alpha(1+a)}{(1+a-R)^3}$$

.

Because we proved that $\acute{R}\frac{R(\Xi-2ZC)+C(\alpha\frac{ZC-1}{(1-\gamma)})}{(R-C)^3} < 0$ after the discontinuity of the debt locus and between the k austerity, then, for concavity of $F(k)$ we need $\frac{b\acute{R}}{(1-\gamma)\alpha}\dfrac{R(\alpha-2(1-\gamma))+\alpha(1+a)}{(1+a-R)^3} > 0$ which from the analysis of the debt locus after the discontinuity hold for $R(\alpha - 2(1-\gamma)) + \alpha(1+a) < 0 \Rightarrow k < \left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \equiv \tilde{k}$. Thus, for $k < \tilde{k}$ then $F^{''}(k) < 0$. Thus, if that $\tilde{k}$ is below the discontinuity of the k-locus $\hat{k} = \left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$.

Thus, a sufficient parametric condition for concavity of $F(k)$ in the area between the discontinuities, $\check{k} < k < \hat{k}$, is that $\tilde{k} < \hat{k}$ that is

$\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} < \left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \Rightarrow \frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)} > \frac{a(1-s)}{\alpha\tilde{A}} \Rightarrow (1+a)\alpha > a(1-s)(2(1-\gamma)-\alpha)$.

Lemma 2 *If* $(1+a)\alpha > a(1-s)(2(1-\gamma)-\alpha)$ *then in the area between the discontinuities* $\check{k} < k < \hat{k}$, $F^{''}(k) < 0$.

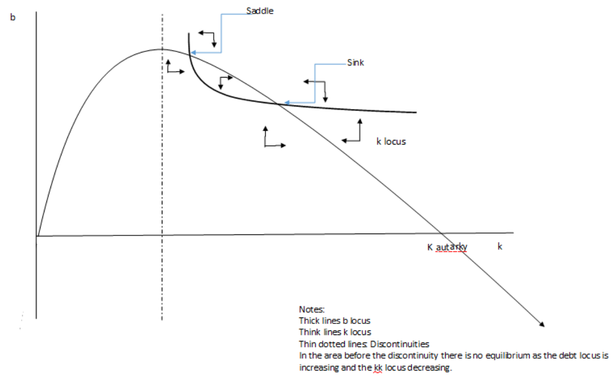
This Lemma means that if an equilibrium exists will be multiple. Furthermore,the debt locus will be convex at the tangency and the k locus concave.

A sufficient parametric condition for concavity of $F(k)$ is to investigate between the discontinuities of debt locus and the k_{AUT} (because in the area between k_{AUT} and the discontinuity of k-locus the debt is negative and no equilibrium can exist). So, in this case a sufficient condition is $\tilde{k} < k_{AUT}$.

$\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} < \left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}} \Rightarrow \frac{(1+a)}{(2(1-\gamma)-\alpha)} > \frac{1}{(s(1-\alpha)-(1-s)b-s\gamma)} \Rightarrow (1+a)\left(s(1-\alpha)-(1-s)b-s\gamma\right) > (2(1-\gamma)-\alpha)$.

If $(1+a)\left(s(1-\alpha)-(1-s)b-s\gamma\right) > (2(1-\gamma)-\alpha)$ then in the area between the $\check{k} < k < k_{AUT}$, $F^{''}(k) < 0$.

Under Lemma 1 and Lemma 2 and conditions (i) and (ii) of Proposition 1 we show that the shape of the k locus is inversed U-Shaped and the shape of the b-locus decreasing after the area of the discontinuity. So two equilibrium steady-state exist. The diagram following diagram graphically illustrates our theoretical result (note the before the discontinuity there cannot exist an equilibrium with $k > 0$ because the debt-locus has negative values)



Appendix 3. Stability

In this section, we are going to analyze the stability properties and the type of each equilibrium. We are going to construct the phase diagram and analyze the arrows of motion.

The dynamic equation for debt is given by

$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t)$

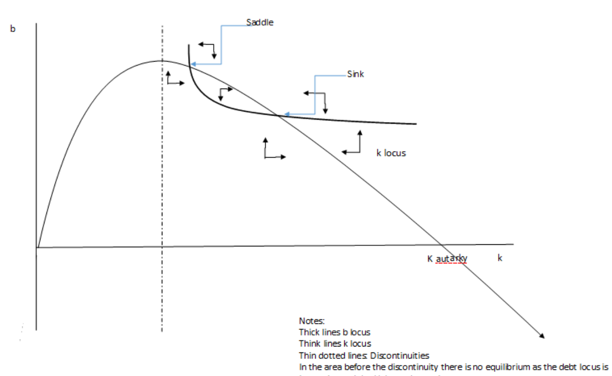
Remind that, for $\check{k} < k < \infty$ then $(1 + a) - R(k) > 0$. Then, for $B_{t+1} - B_t > 0$, $R(k_t) - a - 1)B_t + by(k_t) > 0$ that is $B_t < \frac{by(k_t)}{(1+a)-R(k)}$. Thus, for any B_t lower then the $\Gamma(k)$ locus and because $\Gamma(k)$ is convex, the debt is decreasing (increasing under the $\Gamma(k)$ locus).

The dynamic equation for the capital stock is given by

$k_{t+1} - k_t = (s(1 - \alpha) + s(b - \gamma) - b)\, y(k_t) - k_t + (a(1 - s) - R_t(k_t))\, B_t$

For $k_{t+1} - k_t > 0$ if $(s(1 - \alpha) + s(b - \gamma) - b)\, y(k_t) - k_t + (a(1 - s) - R_t(k_t))\, B_t > 0$. Remind that, for $\check{k} < k < k_{AUT}$ then $\Theta(k) > 0$ and $R(\hat{k}) - a(1 - s) > 0$. Dividing the inequality by $R(\hat{k}) - a(1 - s) > 0$ we get $\frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} - B_t > 0 \Rightarrow B_t < \frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} \Rightarrow$ $B_t < \frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} \Rightarrow B_t < \Theta(k)$. Because $\Theta(k)$ is a concave function, for every B below the $\Theta(k)$ locus the capital stock is increasing and below the $\Theta(k)$ locus, it is decreasing.

According to this analysis, the phase diagram and the arrows of motion are given by:



we can deduct that there are two stable equilibria. The lower equilibrium is saddle-path stable and the second equilibrium is stable node.

Appendix 4. Steady-State Effects of Policy Parameters

The equilibrium steady-state of capital is given by:

$$F(k) = \frac{(\Omega)\, y(k) - k}{(R(k) - C)} - \frac{by(k)}{(1+a) - R(k)}$$

where $\Omega(b) \equiv (s(1-\alpha) - (1-s)b - s\gamma)$, $C(a) \equiv a(1-s)$

Firstly, we examine the effect of austerity parameter on steady-state capital stock. From the implicit function theorem we have:

$\frac{\partial k}{\partial a} = -\frac{\frac{\partial F(k)}{\partial k}}{\frac{\partial F(k)}{\partial a}}$

$\frac{\partial F(k)}{\partial a} = \frac{(\Omega y(k) - k)}{(R(k)-C)^2} + \dfrac{by(k)}{((1+a) - R(k))^2} > 0$ from $0 < k < k_{AUT}$.

$\frac{\partial F(k)}{\partial k} > 0$ from $0 < k < k_{\max}$ and $\frac{\partial F(k)}{\partial k} < 0$ from $k_{\max} < k < k_{AUT}$.

Given that the one equilibrium, k_{ss}^{low} is below $k_{\max}$ and the other, k_{ss}^{high}, above $k_{\max}$ those to equilibria display different properties resulting to Proposition 3.

Secondly, we examine the effect of structural deficit parameter on steady-state capital stock. From the implicit function theorem we have:

$\frac{\partial k}{\partial b} = -\frac{\frac{\partial F(k)}{\partial k}}{\frac{\partial F(k)}{\partial b}}$

$(s(1-\alpha) - (1-s)b - s\gamma)$

$\frac{\partial F(k)}{\partial b} = \frac{-(1-s)}{(R(k)-C)} - \dfrac{y(k)}{(1+a) - R(k)}$

We know that for $0 < k < k_{AUT}$, $R(\hat{k}) - C > 0$ and for $\check{k} < k < \infty$, $(1+a) - R(k) > 0$. Thus, in the area we are interested $\check{k} < k < k_{AUT}$ we have:

$\frac{\partial F(k)}{\partial b} < 0$, for $\check{k} < k < k_{AUT}$, thus, resulting to Proposition 4.

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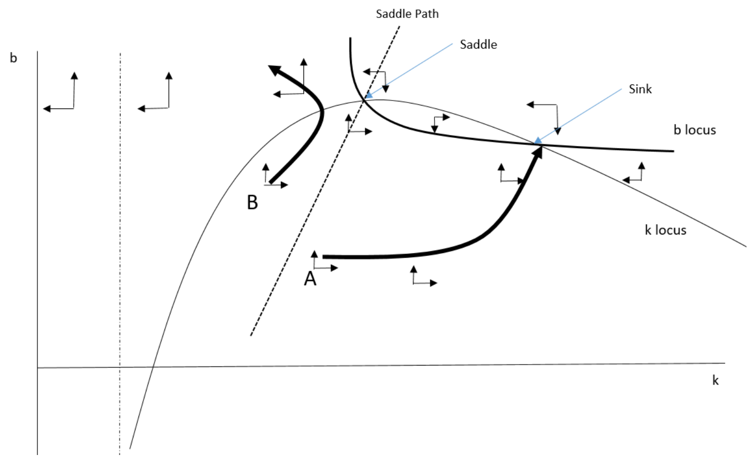
Figure 1: Phase Diagram


Saddle Path
Saddle
Sink
b
b locus
k locus
B
A
k