Fiscal Discipline and Public Debt Sustainability

Evangelos V. Dioikitopoulos*

March 10, 2016

Abstract

In this paper, we develop an overlapping generations growth model, where the government follows a fiscal rule inspired by the empirical estimation conducted by Bohn (1998), to control the level of public debt. We study the determinants of the sustainable level of debt, the effects of different strategies to reduce the public debt and the stability of the long-run equilibrium. We show that although countries have the same structural parameter (for instance, the Southern European countries), they have to abobt different fiscal policies subject to their initial position in economic development and their historical levels of debt. Moreover, we show what has to be the dynamic responsiveness of government expenditures and taxation to the level of debt and output in order to shape a dynamic path of sustainable debt.

JEL classification: *C62; C63;C68;E37;E62;H6;H30.*

Keywords: Fiscal sustainability; Fiscal rules; Bond-financed deficits

Acknowledgments: We have benefited from comments and suggestions by C. Azariadis, H. Dellas, A. Philippopoulos, R. Wendner, S. Ghosh and participants at various conferences and seminars. The usual disclaimer applies.

---

* *Department of Management, Group of Economics, Econometrics and Finance, King's College London, London, UK, e-mail: evangelos.dioikitopoulos@kcl.ac.uk.*

1 Introduction

Over the last 5 years, particularly in Europe, we have witnessed a shift to deficit reducing policies and austerity measures to target the sustainability of public debt. The International Monetary Fund, the European Central Bank, and the European Commission in an effort to help European countries to overcome situations of exploding debt have focused on policies that firstly, target a minimum level of deficit and secondly, place some level of fiscal austerity (taxation) subject to the volume of debt of each country. However, we have witnessed that those policies are continuously re-optimized due to failure of the countries under austerity measures in achieving their targets. Such variation in the dynamic adjustment of policy instruments have resulted in an uncertain economic environment raising the need of imposing a stable dynamic fiscal policy rule for debt and taxation.[1]

In this paper, we build up a theoretical framework where productive government expenditures are financed though taxation and public debt. In particular, we extend the work of Chalk (2000), by considering a policy rule that not only depends on the need for a structural deficit to finance public expenditures but also takes care the way taxation (austerity) has to dynamically adjust in order to place the economy in a regime with sustainable debt. To this end, we introduce the empirical fiscal policy rule of Bohn (1998) and Gali et al. (2007) and we study its theoretical properties. We first find that there exists not only a limit on the level of structural deficit (as in Chalk, 2000) but also a limit on the volume of austerity in order to guarantee an sustainable equilibrium in the economy. Interestingly, we show that although countries can have the same structural parameters, they have to adobt different fiscal policies for sustainable debt subject to their level of economic development and their historical debt levels. Moreover, we show what has to be the responsiveness of government expenditures and taxation in changes to the level of debt in order to shape a dynamic path with stable income and debt. We provide numerical examples to our theoretical results.

Our paper is related to the literature on fiscal consolidation and debt sustainability. Sargent and Wallace (1981) state that there is a ceiling on government indebtedness and that permanent deficits will eventually need to be monetised. Darby (1985) has argued that if the interest rate is less than the growth rate it becomes more pleasant since the economy can simply outgrow its liabilities. However, firstly, countries that belong to a monetary union are constrained to use monetary policy. Secondly, the growth rate in economic crisis periods seems to be lower than the interest rate (also the growth rate can be negative), and, in turn, the usual transversality conditions on infinitely lived agents models can be violated. To this end, we build an overlapping generations framework that can allow the possibility of a debt bubble (Tirole, 1985) but with a consistent fiscal plan that can allow for deficits in the short-run. This happens as the policy rule allows the dynamic adjustment of taxation to the level of debt so as to place the economy to an equilibrium path of sustainable debt. In other words, our contribution relies on the two separate features of the policy rule. Interestingly, due to the presence of multiple equilibria, the procyclicality feature of the policy rule to the level of output helps a country to escape from an unstable dynamic path (creating "instability" to escape from an unstable environment). On the flip side, the "austerity" (increase in taxation) feature of the rule controls the level of debt from not entering into a "bubble" area.

[1]For instance, De Grauwe and Yuemei (2013) show that the fragility of the government bond markets to self-fulfilling liquidity crises in the Eurozone driven by negative market sentiments.

We relate our analysis to the discussion about the derivation of policy prescriptions for countries with unsustainable debt. According to Washigton Consencus (Washigton, 1989) and the Maastricht criteria for entry into EMU, usually there is a ceiling on the overall deficit in order for it to be considered acceptable and sustainable. All those rules imply the same criteria for each country under the assumption that the structural characteristics are the same. Following Azariadis and Stachuzski (2005) and recent empirical evidence by De Grauwe and Yuemei (2013), we show that multiple equilibria can arise and although countries have similar characteristics (e.g. Spain, Italy, and Greece), they may face divergent paths in debt and income, observed in reality. In particular, we show that the final position of economies depends on the initial conditions of physical capital stock and the historical levels of debt independed of the structural characteristics. As a result, initial conditions are important in determining the sustainability of a country's fiscal policy. An economy endowed with a relatively high outstanding stock of debt, inherited from a lack of fiscal discipline in the past, will simply not be able to run the same policies as a low debt economy. Indeed, highly indebted economies may have no choice but to run substantial primary surpluses. As such, large current deficits not only lead to the potential of an explosive growth in the debt to output ratio today but also substantially restrict the latitude for future administrations to engage in deficit financing in the future. In addition to Chalk (2000) we contribute by showing the joined determination of structural deficit and taxation that can place the economy in a sustainable equilibrium subject to the initial level of income and debt.

The rest of the paper is structured as follows. Section 2 sets-up the theoretical model. Section 3 derives the dynamic properties of equilibria. Section 4 derives the effects of policy instruments in the long-run and simulates the dynamics numerically. Section 5 concludes the paper.

2 The model

2.1 Supply Side

We consider an overlapping generations model with government debt and physical capital as advanced by Diamond (1965) and Tirole (1985). There are N_t consumers who each live for two periods and we assume zero population growth. They choose their consumption C_t, d_{t+1} to maximize their discounted logarithmic utility function,

$$U = \ln C_t + \beta \ln d_{t+1} \tag{1}$$

where β is the weight that agents place in their second period utility. In the first period of their life, agents inelastically supply labour and they receive a wage rate, w_t which is taxed by τ_t. Some of their income is consumed in the first period and the rest is saved for the second period. When old, the agents consume their savings and they receive a return on their savings, r_{t+1}. By solving their intertemporal problem, the savings of each individual are positively determined by the after tax wage rate and their savings propensity, $s = \frac{\beta}{1+\beta}$. Thus, savings, S, are given by,

$$S(w_t, r_{t+1}) = s(1 - \tau_t)w_t \tag{2}$$

. Since savings are not affected by the real interest rate, we have assumed without lost of generality that the government taxes only wages. We want to reflect the idea that the government can only use distortionary taxation.

2.2 Demand Side

In the demand side, there exists a continuum of firms that produces output, Y_t, using capital, k_t, labour, l_t, and public good supplied by the government g_t,

$$Y_t = Ak_t^{\alpha} l_t^{1-\alpha} g_t^{\gamma} \qquad \alpha + \gamma < 1 \tag{3}$$

The wage rate and return on capital, using labour market clearing condition, $l_t = 1$, are determined by

$$w_t = (1-\alpha)Ak_t^{\alpha} g^{\gamma} \tag{4}$$

$$R_t = \alpha Ak_t^{\alpha-1} g_t^{\gamma} \tag{5}$$

2.3 Government

Regarding the government, we assume that the supply of public capital is determined by a Samuelson Rule, which states that the marginal income generated by public capital must be equal to the marginal cost of producing public capital given by:

$$g_t = (\gamma Ak_t^{\alpha})^{\frac{1}{1-\gamma}} \tag{6}$$

From the Samuelson's rule, we can express the aggregate production function in the following way:

$$Ak^a g^{\gamma} = Ak^a (\gamma Ak^{\alpha})^{\frac{\gamma}{1-\gamma}} = \gamma^{\frac{\gamma}{1-\gamma}} A^{\frac{1}{1-\gamma}} k^{\frac{\alpha}{(1-\gamma)}} = \tilde{A}k^{\overline{\frac{\alpha}{(1-\gamma)}}} \tag{7}$$

where $\tilde{A} \equiv \gamma^{\frac{\gamma}{1-\gamma}} A^{\frac{1}{1-\gamma}}$. Since, we have assumed that $\alpha+\gamma < 1$, following standard growth literature, the marginal productivity of capital is decreasing, $\dfrac{\alpha}{(1-\gamma)} < 1$. Now, we can compute the equilibrium wages and real interest using the aggregate production function as follows:

$$w(k_t) = (1-\alpha)Ak_t^{\alpha} g_t^{\gamma} = (1-\alpha)\tilde{A}k_t^{\overline{\frac{\alpha}{(1-\gamma)}}} \tag{8}$$

$$R(k_t) = \alpha Ak_t^{\alpha-1} g_t^{\gamma} = \alpha\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-1} \tag{9}$$

Further, we assume that the government finances public expenditures not only from taxation but also by using government debt. The budget constraint of the government is given by

$$B_{t+1} = R_t B_t + g_t - \tau_t w_t \tag{10}$$

Following the fiscal rule estimated by Bohn (1998) and used by Gali et al. (2007), we assume that the primary surplus/deficit is a function of the level debt and income determined by the fiscal policy parameters, $b > 0$ and $a > 0$ given by

$$g_t - \tau_t w_t = -aB_t + by_t \tag{11}$$

Policy parameter a states what the responsiveness of the deficits to the level of debt (the higher the debt the lower the deficit so as to stabilize debt) and parameter b states the responsiveness of deficit in the level of income (higher income, more deficit to finance public spending).[2]Thus, this rule places some level of fiscal discipline, "austerity", as given by a, in the sense that under an increase of debt, taxation has to increase so as to reduce deficit and, in turn, public debt. On the flip side, as the economy develops policy parameter b allows for higher structural deficit in order to finance public spending. Interestingly, as we are going to show later on, the procyclical role of b places some level of instability in the economy which can be useful for the economy in order to escape from an unstable dynamic path. At the same time, "austerity" (parameter a) helps to ensure sustainability in public debt.

3 Equilibrium dynamics

Using the factor payments and quantities from the maximization problem of agents and firms into the dynamic equations of capital stock and debt, after some algebra (see Appendix 1), the dynamic equilibrium in the market is determined by following dynamical system of equations

$$k_{t+1} - k_t = (s(1-\alpha) + s(b-\gamma) - b)\, y(k_t) - k_t + (a(1-s) - R_t(k_t))\, B_t \tag{12}$$

$$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t) \tag{13}$$

We will first analyze the existence and uniqueness of steady-state equilibrium and, then, we will analyze the stability of equilibrium and the dynamic behavior of capital and debt. The steady-state of capital stock and debt level in the economy is that bundle, (k^*, b^*) such that $k_{t+1} - k_t = 0$ and $B_{t+1} - B_t = 0$ simultaneously.

Proposition 1 *(Existence and Uniqueness). For (i)* $(s(1-\alpha) + s(b-\gamma) - b > 0$ *and (ii)* $a < \frac{\alpha}{(1-s)(s(1-\alpha)-(1-s)b-s\gamma)} \equiv a^{\max}$ *there exist two non-trivial equilibrium steady states,* $k_{ss}^{low} > 0$ *and* $k_{ss}^{high} > 0$ *where* $k_{ss}^{low} < k_{ss}^{high}$.
Proof. *Appendix 2.* ▪

Proposition 2 *(Stability) Both steady-states are stable. The lower equilibrium,* k_{ss}^{low}*, is saddle-path stable and the higher equilibrium,* k_{ss}^{high}*, is a stable node.*
Proof. *Appendix 3.* ▪

Proposition 1 shows that if the saving propensity and the responsiveness of the structural deficit to the level income, b, are such that the investment on capital positively effects the accumulation of capital (condition (i)) and, if the response of the tax rate on the level of

[2]Note that in the case that $a = 0$ and $b = 0$ the government balances the budget public capital and is equal to $g_t = \tau_t w_t$.

debt, a, is not high enough (limits for austerity as given by condition (ii)), then there exist two strictly positive equilibrium steady-states. Proposition 2 shows that both equilibria are stable, the relatively lower one displays saddle-path stability and relatively higher one is a stable node. Given the overlapping generations framework (which relaxes the hypothesis of Ricardian equivalence and allows for the presence of bubbles), the dynamical system does not have a jump variable, thus, the only meaningful equilibrium, is the equilibrium that its stability is a node (Azariadis and Stachuzski, 2005). Following that, Proposition 1 together with Proposition 2 imply that the initial conditions, i.e. the historical levels of debt, determine the long-run position of the economy even the structural parameters of the economy can be the same. In particular, for high initial volume of debt and low initial capital stock the economy can be in a position of exploding debt leading to a debt bubble and a poverty trap. While, after a threshold level of initial capital stock and volume of debt the economy will converge to an equilibrium level of high capital stock and sustainable debt.[3] The threshold levels for the initial conditions of capital and debt above (below) them sustainability (unsustainability) for the long-run position of debt exist, depend on the strictness of the policy rule determined by a and b. In the next section, we show how government can affect the dynamics of economies with high debt and place them in a sustainable regime.

4 Policy effects and implications

In this section, we analyze the effect of the responsiveness of taxation to the level debt, a and to the level of structural deficit, b, on the long-run position of the economy, the dynamics and we discuss the policy implications.

4.1 The effects of "Austerity" in steady-state and dynamics

Proposition 3 *The policy parameter, a, negatively affects the relatively lower steady state, k_{ss}^{low}, and, positively, affects the higher steady-state equilibrium.*
Proof. *Appendix 4.* ▪

Proposition 3 states that the equilibria we derived in Proposition 1 display different properties. An increase in austerity parameter, a, negatively affects the relatively lower steady-state of the capital stock while it positively affects the relatively higher steady-state capital stock. In other words, the higher the responsiveness of the tax rate to the level of debt, the higher the gap between the two equilibria. Once the stability of the high capital stock is a stable node this gives more space for an economy with low level of capital stock and moderate level of debt to stabilize its level of debt and converge to the relatively higher level of capital stock. The interesting implication of this theoretical property is that the government by placing higher austerity can help economies with high level of initial debt and relatively low capital stock to enter to the area of sustainability. The rule can be clever in the sense that, dynamically, taxation will increase in economies with high initial level of debt and relatively low level of capital stock but once the capital stock achieves a certain threshold, taxation will dynamically reduce so as to avoid a huge crowding out effect.

[3] Those threshold levels are determined on Appendix 2 and 3.

To better illustrate this theoretical outcome we provide a numerical example. Assume two economies, country A (for instance, Italy) and country B (for instance, Greece). Assume that, both countries have the same structural characteristics such as total factor productivity, $A = 8$, the share of physical capital on the production function, $\alpha = 0.25$, same productivity in government expenditures, $\gamma = 0.15$, same time preference (savings propensity), $\beta = 0.098$ and both follow the same rule with the same weights, $a = 0.5$ and $b = 0.013$ (followed by Bohn, 1998 and similar to Gali et al., 2007). Also, both countries are developed in the sense that both belong to the area of low interest rates (see Appendix 3 for areas stability). The countries only differ in their initial level of public debt and physical capital stock. Country B has relatively lower initial capital stock and higher initial level of debt than country A. Lets assume the country's B initial capital stock is, $K_0^B = 0.5$ and the initial level of debt is $B_0^B = 0.5$ while for County A we assume that $K_0^A = 3$ and $B_0^A = 0.3$. The numerical exercise we pursue is by setting those different initial conditions but setting the same structural and policy parameters to allow both economies dynamics to proceed. Our simulations in Table 1 show that Country A will display sustainable debt and development (it will reach the high steady-state of capital stock and stable steady-state level of debt). While as it is shown in Table 2, Country B will end up in a situation of exploding debt and sharp reduction of capital stock entering to an area of a poverty trap and result to zero capital stock. Thus, as Proposition 1 and 2 imply although the countries can have the same structural characteristic and follow the same policy rule, they will display different dynamics and steady-states just by starting with different initial conditions.

Table 1
Country A: Dynamic adjustment towards the stable steady-state
with $a = 0.5$ and $b = 0.013$

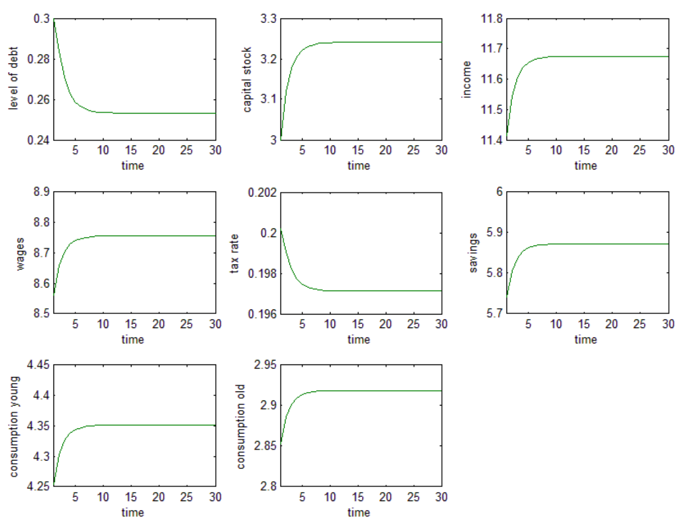
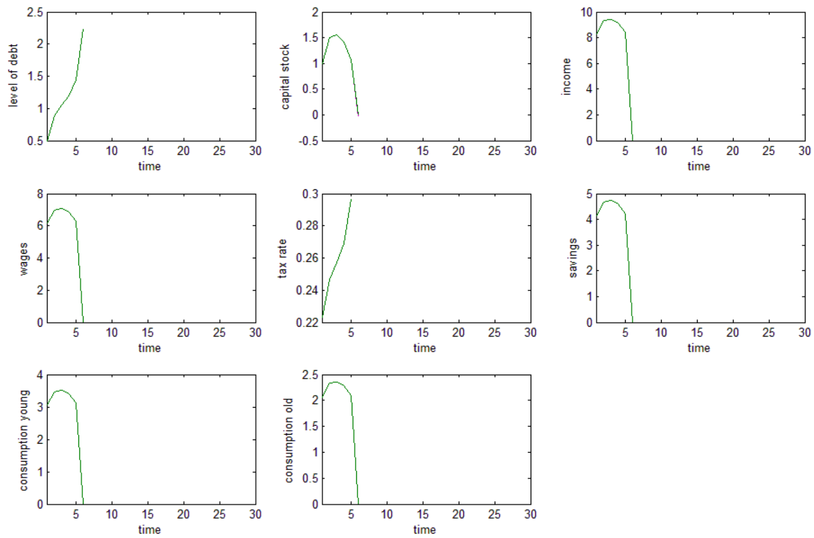
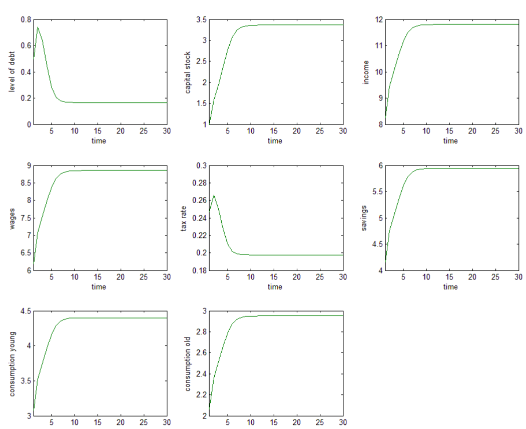


Table 2
Country B: Dynamic adjustment towards the stable steady-state
with $a = 0.5$ and $b = 0.013$



The policy implication that can be derived from this result is that the policy rule and the level of austerity has to adjust not only to the fundamentals but also to the initial state of the economy. So, in cases of exploding debt, following Proposition 3, countries have to increase the response of deficit (taxation) to the level of debt ("austerity"), a, so as to expand the area of sustainability. For this reason, in Table 3 we provide the dynamic path of Country B by changing the level of "austerity" from 0.5 to 0.8.

Table 3
Country B: Dynamic adjustment towards the stable steady-state
with $a = 0.9$ and $b = 0.013$



According to Table 3, with higher a, the policy rule can place Country B to a stable path for the capital stock and sustainable long-run level of debt. An interesting outcome of the simulation results is the non-monotonic dynamics of the tax rate. The tax rate increases at low levels of capital stock so as to decrease deficit and stabilize the level of debt. As debt falls and the economy starts to develop, then taxes fall in order to boost savings that will form a higher capital stock and a higher tax base to finance government expenditures. In other words, the two features of the rule work as follows. From one hand, we need austerity in order to put the economy in a stable equilibrium path (policy parameter a), from the other hand, and in line with the optimal Samuelson rule for the provision of public services, we need some structural deficit in order to finance productive government spending (policy parameter, b). Differently to other policy rules (which work in environments of stable dynamic paths) and state that deficits have to decrease as output increases (for consumption smoothing) this rule goes the other way and places some level of instability in the economy so as to be able to escape from the unstable path but at the same time guarantees some level of fiscal consolidation so as to avoid emergence of debt bubbles.

4.2 Structural deficits and long-run equilibrium

Last, the same work can be done (and has extensively analyzed by Chalk (2000) for the policy instrument that controls the level of structural deficit.

Proposition 4 *The structural deficit parameter, b, positively affects the lower steady state k_{ss}^{low} and negatively affects the higher, k_{ss}^{high}, steady-state equilibrium.*
Proof. *Appendix 4.* ▪

Proposition 4 states that the equilibria display different properties also for the level of structural deficit. An increase in the level of structural deficit positively affects the low steady-state while an increase in the level of structural deficit negatively affects the high steady-state. This theoretical result conforms with the result of Chalk (2000). A decline in the level of structural deficit positively affects the level of high capital stock and increases the probability that a country can escape from a poverty trap as it can increase the gap between the two equilibria and, in turn, increase the area with stable node dynamics.

5 Conclusion

In this paper we built a theoretical model that can allow the possibility of scenarios of unsustainable debt and self-fulfilling crises motivated by the recent debt crisis in Europe and empirical justification provided by De Grauwe and Yuemei (2013). To this end, we considered the theoretical properties of a policy rule that has been empirically estimated by Bohn (1998) but has not been used in a framework that can allow for the presence of debt bubbles and multiple equilibria. Then, we analyzed how we can adjust the parameters of the policy rule with the objective to find a way to place economies that face high debt, in an area of sustainable debt and economic development.

We found that such a rule has to have two distinct features. First, it needs to have a consistent fiscal plan to allow for structural deficits for the provision of government spending (even with low productivity). At the same time, it needs to place some level of austerity to restrict the economy from falling in an area of exploding debt. Our contribution relies on those two separate features of the policy rule we consider. We show that under the presence of multiple equilibria, a procyclical policy rule to the level of output helps a country to escape from an unstable dynamic path. At the same time, we show that the "austerity" (increase in taxation) feature of the rule controls the level of debt from not entering into a "bubble" area. According to our numerical simulations we higher austerity level and non-monotonic dynamic adjustment of taxation (initially increase and then decrease) endogenously, a country with high level of debt and moderate initial level of capital stock can enter to an area of sustainable debt and achieve a stable high steady-state equilibrium for the capital stock.

6 References

Bohn H., 1998. The Behaviour of U.S. Public Debt and Deficits, The Quarterly Journal of Economics, p.949-963.

Chalk, 2000. The sustainability of bond-financed deficits: An overlapping generations approach. Journal of Monetary Economics, 45, p. 293-328

Darby, M., 1985. Some pleasant monetarist arithmetic. Federal Reserve Bank of Minneapolis Quarterly Review 9 (1), p. 32-43.

De Grauwe, P. and Yuemei, J. 2013. Self-fulfilling crises in the Eurozone: An empirical test. Journal of International Money and Finance, Vol. 34, p. 15–36.

Diamond, P.A., 1965. National Debt in a Neoclassical Growth Model. The American Economic Review, Volume 55, Issue 5

Gali J., Lopez-Salido David J. and Valles J., 2007. Understanding the Effects of Government Spending on Consumption. Journal of the European Economic Association, 5(1), p. 227–270.

Sargent, T. and Wallace, N., 1981. Some unpleasant monetarist arithmetic. Federal Reserve Bank of Minneapolis Quarterly Review 5, 1-17.

Tirole J., 1985. Asset Bubbles and Overlapping Generations. Econometrica, Vol. 53, No. 6. p. 1499-1528.

Williamson, J., 1989. What Washington Means by Policy Reform, in: Williamson, John (ed.): Latin American Readjustment: How Much has Happened, Washington: Institute for International Economics 1989.

7 Appendix

Appendix 1: Derivation of the Dynamical System

The equilibrium in the market is determined by following equations:

$$B_{t+1} = R(k_t)B_t - aB_t + by_t$$

$$k_{t+1} + B_{t+1} = s(1 - \tau)w(k_t)$$

The evolution of public debt is determined by the government budget constraint and fiscal rule eqs.(10)and (11). Given the government follows the Samuleson rule to determines the public spending, the marginal needs to adjust to implement the fiscal rule,

$$g_t - \tau w = -aB_t + by_t$$

$b > 0$ and $a > 0$

$$\frac{g_t}{y_t} - \tau\frac{w_t}{y_t} = -a\frac{B_t}{y_t} + b$$

Now, the marginal tax is equal to

$$-\tau = -a\frac{B_t}{y_t} + b - \frac{g_t}{y_t}$$

From the samuelson rule

$$\frac{g_t}{y_t} = \gamma$$

then, we have that

$$-\tau(k_t, B_t)w_t = -aB_t + (b-\gamma)y_t \tag{14}$$

note that, we need to assume that $\tau(k_t, B_t) < 1$, then we obtain the following expression for ratio debt-GDP, use that w_t/y_t

$$\tau(k_t, B_t)(1-\alpha) = a\frac{B_t}{y_t} - b + \gamma < 1 \tag{15}$$

$$\tau(k_t, B_t) = a\frac{B_t}{(1-\alpha)y_t} + \frac{(\gamma - b)}{1-\alpha} < 1 \tag{16}$$

$$\frac{B_t}{y_t} < \frac{1-\alpha+b-\gamma}{a} \tag{17}$$

This expression is upper limit for debt, otherwise the government would need to force confiscatory taxation $\tau > 1$.We need to impose sΩe restriction in the parameter, in order to get a positive debt at steady state:

$$1 - \alpha + b - \gamma > 0$$

Then, we obtain the following dynamic system, given the equilibrium in the product goods market (saving must be equal to investment in real capital and government bonds) and the government budget constraint:

$$k_{t+1} = s(1-\tau(k_t, B_t))w_t(k_t) - (R_t(k_t) - a)B_t - by(k_t)$$

$$B_{t+1} = R_t(k_t)B_t - aB_t + by(k_t)$$

We simplify the expression for k_{t+1} using eq. (14),

$$
\begin{aligned}
k_{t+1} &= sw_t(k_t) - s\tau(k_t, B_t)w_t(k_t) - (R_t(k_t) - a)B_t - by(k_t) \\
k_{t+1} &= sw_t(k_t) + s\left((b-\gamma)y_t - aB_t\right) - (R_t(k_t) - a)B_t - by(k_t) \\
k_{t+1} &= sw_t(k_t) + s(b-\gamma)y_t - saB_t - R_t(k_t)B_t + aB_t - by(k_t) \\
k_{t+1} &= sw_t(k_t) + s(b-\gamma)y_t + \left(a - as - R_t(k_t)\right)B_t - by(k_t) \\
k_{t+1} &= sw_t(k_t) + s(b-\gamma)y_t + \left(a(1-s) - R_t(k_t)\right)B_t - by(k_t)
\end{aligned}
$$

.Then, evolution of capital and bond is determined by:

$$\Delta B_t = B_{t+1} - B_t = \left(R(k_t) - a - 1\right)B_t + by(k_t);$$

$$k_{t+1} = sw_t(k_t) + s(b-\gamma)y_t + \left(a(1-s) - R_t(k_t)\right)B_t - by(k_t)$$

or note that in Cobb-Doulgas case labour share $\frac{w_t(k_t)}{y(k_t)} = 1 - \alpha$

$$k_{t+1} = \left(s(1-\alpha) + s(b-\gamma) - b\right)y(k_t) + \left(a(1-s) - R_t(k_t)\right)B_t$$

In the case of Cobb-Douglas, we express the dynamic system as function of the different parameters (to review and to be moved tot he Appendix)

$$\Delta B_{t+1} = B_{t+1} - B_t = (\alpha\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-1} - a - 1)B_t + b\tilde{A}k_t^{\overline{(1-\gamma)}^{\alpha}}$$

$$k_{t+1} = \left(s(1-\alpha) + s(b-\gamma) - b\right)\tilde{A}k_t^{\frac{\alpha}{(1-\gamma)}} + \left(a(1-s) - \alpha\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-1}\right)B_t$$

Appendix 2. Existence and Uniqueness

The change of capital stock and the debt level of the economy is determined by the following dynamic system:

$$k_{t+1} - k_t = \left(s(1-\alpha) + s(b-\gamma) - b\right)y(k_t) - k_t + \left(a(1-s) - R_t(k_t)\right)B_t$$

$$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t)$$

We will first analyze the existence and uniqueness of steady-state equilibrium and then we will analyze the stability of equilibrium and the dynamic behavior of capital and debt. The steady-state of capital stock and debt level in the economy is that bundle, k, b, where $k_{t+1} - k_t = 0$ and $B_{t+1} - B_t = 0$ simultaneously.

The locus where the change of debt is zero, $B_{t+1} - B_t = 0$ is given by

$$B = \frac{by(k)}{(1+a) - R(k)} \equiv \Gamma(k)$$

The properties of $\Gamma(k)$ are the following:

1. $\lim_{k\to 0} \Gamma(k) = 0$ and $\lim_{k\to\infty} \Gamma(k) = \infty$.

2. $\Gamma(k)$ is discontinuous at $k = \check{k}$ where $\check{k} : (1+a) - R(\check{k}) = 0$ Under the Cobb-douglas function production function $\check{k} = \left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}}$

3. For $0 < k < \check{k}$ then $\Gamma(k) < 0$ and for $\check{k} < k < \infty$ then $\Gamma(k) > 0$.

Proof. Note that $y(k) > 0$ for any k and $\frac{\partial((1+a)-R(k_t))}{\partial k} = -\acute{R}(k_t) > 0$ (monotonic function). Also, $\lim_{k\to 0}(1+a) - R(k_t) = -\infty$ and $\lim_{k\to\infty}(1+a) - R(k_t) = (1+a) > 0$. This means that for $0 < k < \check{k}$ then $(1+a) - R(k_t) < 0$ and for $\check{k} < k < \infty$ then $(1+a) - R(k_t) > 0$ For $R(k) = \alpha\tilde{A}k^{\frac{\alpha}{1-\gamma}-1}$ that is $(1+a) - \alpha\tilde{A}k^{\frac{\alpha}{1-\gamma}-1} > 0 \Rightarrow \hat{k} < \left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}}$

4. The limit behavior of $\Gamma(k)$ from the left and the right of discontinuity is given by: $\lim_{k\to\check{k}^-} \Gamma(k) = -\infty$ and $\lim_{k\to\check{k}^+} \Gamma(k) = \infty$.

5. The first order derivative $\Gamma(k)$ is given by: $\frac{\partial\Gamma(k)}{\partial k} = b\frac{y^{'}(k_t)((1+a) - R(k_t)) + (R'(k_t))y(k_t)}{((1+a) - R(k_t))^2}$ which after simplification (see footnote[4])

$$\frac{\partial\Gamma(k)}{\partial k} = b\frac{\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right)R(k)}{((1+a) - R(k_t))^2}$$

For $0 < k < \check{k}$ then $\frac{\partial\Gamma(k)}{\partial k} < 0$.

[4] """""""Simplification of the numerator of first order derivative

$$\frac{\alpha}{(1-\gamma)}\frac{y(k)}{k}((1+a) - R(k)) + \alpha(\tfrac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k^2}y(k) \Rightarrow$$
$$\left(\frac{1}{(1-\gamma)}((1+a) - R(k)) + (\tfrac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k}\right)\frac{\alpha y(k)}{k}$$
$$\left((\tfrac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{(1-\gamma)}) + (\tfrac{\alpha}{1-\gamma} - 1)\tfrac{R(k)}{\alpha}\right)R(k)$$
$$\left(\tfrac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{(1-\gamma)} + \tfrac{\alpha}{1-\gamma}\tfrac{R(k)}{\alpha} - \tfrac{R(k)}{\alpha}\right)R(k)$$
$$\left(\tfrac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{(1-\gamma)} + \tfrac{R(k)}{1-\gamma} - \tfrac{R(k)}{\alpha}\right)R(k)$$
$$\left(\tfrac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{\alpha}\right)R(k)$$

$$\frac{\partial\Gamma(k)}{\partial k} = b\frac{\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right)R(k)}{((1+a) - R(k_t))^2}$$ """"""""""

This happens because $0 < k < \check{k}$, $y^{'}(k_t)((1+a) - R(k_t)) < 0$ and $(R'(k_t))y(k_t) < 0$ given that $y^{'}(k_t) > 0$, $(1+a) - R(k_t) < 0$ and $R'(k_t) < 0$ and $y(k_t) > 0$.

Definition 1 *Define* $k_{\min} \equiv \left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$

For $\check{k} < k < \infty$ then,
(i) $\frac{\partial \Gamma(k)}{\partial k} < 0$ for $\check{k} < k < k_{\min}$
(ii) $\frac{\partial \Gamma(k)}{\partial k} > 0$ for $k_{\min} < k < \infty$.

Proof. $\frac{\partial \Gamma(k)}{\partial k} < 0$ if $y^{'}(k)((1+a) - R(k)) + (R'(k))y(k) < 0$ which following $R(k) = \alpha \tilde{A} k_t^{\frac{\alpha}{1-\gamma}-1} = \alpha \frac{y(k)}{k}$ and $R_k = \alpha(\frac{\alpha}{1-\gamma} - 1)\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-2} = \alpha(\frac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k^2}$, $y^{'}(k_t) = \dfrac{\alpha}{(1-\gamma)} \tilde{A} k_t^{\dfrac{\alpha}{(1-\gamma)}-1} = \dfrac{\alpha}{(1-\gamma)} \frac{y(k)}{k}$ we have

$$\frac{\alpha}{(1-\gamma)}\frac{y(k)}{k}((1+a) - R(k)) + \alpha(\tfrac{\alpha}{1-\gamma} - 1)\tfrac{y(k)}{k^2}y(k) < 0 \Rightarrow$$
$$\frac{\alpha}{(1-\gamma)}((1+a) - R(k)) + \alpha(\tfrac{\alpha}{1-\gamma} - 1)\tfrac{y(k)}{k} < 0$$
$$\frac{\alpha}{(1-\gamma)}((1+a) - \alpha\tfrac{y(k)}{k}) + \alpha(\tfrac{\alpha}{1-\gamma} - 1)\tfrac{y(k)}{k} < 0$$
$$(1+a)\frac{\alpha}{(1-\gamma)} - \alpha\frac{\alpha}{(1-\gamma)}\tfrac{y(k)}{k} + \alpha\tfrac{\alpha}{1-\gamma}\tfrac{y(k)}{k} - \alpha\tfrac{y(k)}{k} < 0$$
$$(1+a)\frac{\alpha}{(1-\gamma)} - \alpha\tfrac{y(k)}{k} < 0$$
$$\frac{(1+a)}{(1-\gamma)} - \tfrac{y(k)}{k} < 0$$
$$\frac{(1+a)}{(1-\gamma)} - \tfrac{R(k)}{\alpha} < 0 \Rightarrow (1+a)\alpha - (1-\gamma)R(k) < 0 \text{ that is}$$

$(1+a)\alpha - (1-\gamma)\alpha\tilde{A}k^{\frac{\alpha}{1-\gamma}-1} < 0 \Rightarrow k^{\frac{\alpha}{1-\gamma}-1} > \frac{(1+a)}{(1-\gamma)\tilde{A}} \Rightarrow k < \left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1}{\frac{\alpha}{1-\gamma}-1}} \Rightarrow k < \left(\frac{(1+a)}{(1-\gamma)\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \equiv k_{\min}$. The opposite otherwise.

$$b\frac{\left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right)R(k)}{((1+a) - R(k_t))^2}$$

Second order derivative:

$$\frac{\partial^2 \Gamma(k)}{\partial k^2} = b\frac{\left(\frac{(1+a)\acute{R}}{(1-\gamma)} - \frac{2R\acute{R}}{\alpha}\right)(1+a-R)^2 + \left(\frac{(1+a)}{(1-\gamma)} - \frac{R}{\alpha}\right)R2\,(1+a-R)\,\acute{R}}{(1+a-R)^4}$$

taking common factor $\acute{R}$ and eliminating $(1+a-R)$

$$b\acute{R}\frac{\left(\frac{(1+a)}{(1-\gamma)} - \frac{2R}{\alpha}\right)((1+a) - R) + \left(\frac{(1+a)}{(1-\gamma)} - \frac{R}{\alpha}\right)R2}{(1+a-R)^3} =$$

$$b\acute{R}\frac{\left(\frac{(1+a)}{(1-\gamma)}-\frac{2R}{\alpha}\right)((1+a)-R)+\left(\frac{(1+a)2R}{(1-\gamma)}-\frac{2R^2}{\alpha}\right)}{(1+a-R)^3} = b\acute{R}\frac{\left(\frac{(1+a)}{(1-\gamma)}(1+a)-\frac{2R}{\alpha}(1+a)\right)-\left(\frac{(1+a)R}{(1-\gamma)}-\frac{2R^2}{\alpha}\right)+\Big(}{(1+a-R)^3}$$

$$\frac{b\acute{R}(1+a)}{(1-\gamma)\alpha}\frac{\alpha(1+a)-(1-\gamma)2R+\alpha R}{(1+a-R)^3} = \frac{b\acute{R}(1+a)}{(1-\gamma)\alpha}\frac{\alpha(1+a)-(1-\gamma)2R+\alpha R}{(1+a-R)^3} =$$

$$\frac{\partial^2\Gamma(k)}{\partial k^2} = \frac{b\acute{R}}{(1-\gamma)\alpha}\frac{R(\alpha-2(1-\gamma))+\alpha(1+a)}{(1+a-R)^3}$$

Analysis of $\frac{\partial^2\Gamma(k)}{\partial k^2}$. We analyze the case after the discontinuity, that is, for $\check{k} < k < \infty$

Then, for $\check{k} < k < \infty$ then $1+a-R > 0$ then $\frac{\partial^2\Gamma(k)}{\partial k^2} > 0$ if $R(\alpha-2(1-\gamma))+\alpha(1+a) < 0$ $\Rightarrow R(2(1-\gamma)-\alpha) > \alpha(1+a) \Rightarrow R > \frac{\alpha(1+a)}{2(1-\gamma)-\alpha} \Rightarrow k < \left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \equiv \tilde{k}$. So, for $\check{k} < k < \tilde{k}$, $\frac{\partial^2\Gamma(k)}{\partial k^2} > 0$

We also, want to show that $\tilde{k}$ is indeed above the discontinuity $\check{k}$

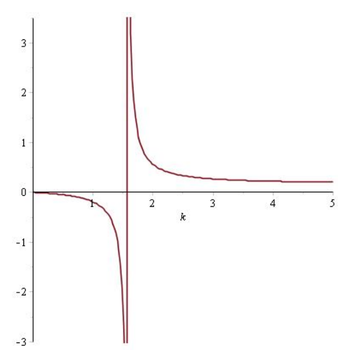
First, we compare $\tilde{k}$ with $\check{k}$, we need $\tilde{k} > \check{k} \Rightarrow \left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} > \left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}} \Rightarrow$ $\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)} < \frac{(1+a)}{\alpha\tilde{A}} \Rightarrow$

$\alpha < (2(1-\gamma)-\alpha) \Rightarrow 2\alpha < 2(1-\gamma) \Rightarrow a-(1-\gamma) < 0$ which holds.

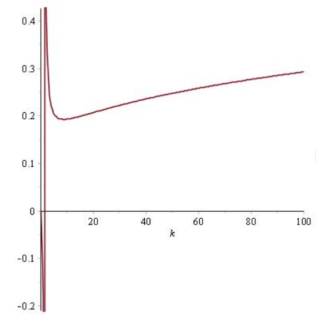
Thus the function is convex for $\check{k} < k < \tilde{k}$, $\frac{\partial^2\Gamma(k)}{\partial k^2} > 0$ and concave for $\tilde{k} < k < \infty$, $\frac{\partial^2\Gamma(k)}{\partial k^2} < 0$. Last, $\lim_{k\to\infty}\frac{\partial^2\Gamma(k)}{\partial k^2} = 0$

From the properties above the graph of the debt locus is given by:

For $0 < k < 5$ (includes the discontinuity)



For $5 < k < \infty$ (limiting behavior and the minimum. Better of the second part)



The locus where the change of capital stock is zero, $K_{t+1} - K_t = 0$ is given by

$$\Theta(k) = \frac{(s(1-\alpha) - (1-s)b - s\gamma)\, y(k) - k}{(R(k) - a(1-s))}$$

where $y(k) = \tilde{A}k^{\frac{\alpha}{(1-\gamma)}}$ and $y_k = \tilde{A}\frac{\alpha}{(1-\gamma)}k^{\frac{\alpha}{(1-\gamma)}-1} = \frac{\alpha}{(1-\gamma)}\frac{y(k)}{k}$ and the limit behavior is: $\lim_{t\to 0} y(k) = 0, \lim_{t\to\infty} y(k) = \infty$ $\lim_{t\to 0} y_k = \infty$ and $\lim_{t\to\infty} y_k = 0$
and
$R(k) = \alpha\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-1} = \alpha\frac{y(k)}{k}$ and $R_k = \alpha(\frac{\alpha}{1-\gamma} - 1)\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-2} = \alpha(\frac{\alpha}{1-\gamma} - 1)\frac{y(k)}{k^2} = (\frac{\alpha}{1-\gamma} - 1)\frac{R(k)}{k}$

1. $\lim_{k\to 0}\Theta(k) = \lim_{k\to 0}\Theta(k) = \frac{\Omega y(0) - 0}{(R(0) - a(1-s))} = 0$ and $\lim_{k\to\infty}\Theta(k) = \frac{\frac{\partial((s(1-\alpha)-(1-s)b-s\gamma)y(k)-k)}{\partial k}}{\frac{\partial((R(k)-a(1-s)))}{\partial k}} = \lim_{k\to\infty}\frac{(s(1-\alpha)-(1-s)b-s\gamma)\acute{y}(k)-1}{\acute{R}(k)} = \lim_{k\to\infty}\frac{(s(1-\alpha)-(1-s)b-s\gamma)y''(k)}{R''(k)} = \infty$

2. $\Theta(k)$ is discontinuous at $k = \hat{k}$ where $\hat{k} : R(\hat{k}) - a(1-s) = 0$. Under a Cobb-douglas production function $\hat{k} = \left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$

Remark 1 *We show that the discontinuity of the debt locus to be below the discontinuity of the k locus. That is* $\left(\frac{(1+a)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-1-\gamma}} < \left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \Rightarrow (1+a) > a(1-s) \Rightarrow (1+a) > a(1-s) \Rightarrow 1 > -as$ *where for* $a > 0$ *and* $\alpha \in (0,1)$ *this always holds.*

Assumption 1 *We assume a positive effect of income (investment) on the accumulation of capital stock which happens under the following condition* $(s(1-\alpha) + s(b-\gamma) - b > 0$

3. Define k_{AUT}: $(s(1-\alpha) - (1-s)b - s\gamma)\, y(k_{AUT}) - k_{AUT} = 0$ (in other words $B = 0$) which in the Cobb-douglas case is given by: $(s(1-\alpha) - (1-s)b - s\gamma)\,\tilde{A}k^{\frac{\alpha-(1-\gamma)}{(1-\gamma)}} - 1 = 0 \Rightarrow k_{AUT} = \left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}}$.

Assumption 2 *Parametric condition such that:* $\hat{k} > k_{AUT}$ *is:* $\left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} > \left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}} =$ $\frac{a(1-s)}{\alpha\tilde{A}} < \frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)} \Rightarrow a(1-s)((s(1-\alpha)-(1-s)b-s\gamma)) < \alpha$ *which imposes limits on austerity* $a < \frac{\alpha}{(1-s)(s(1-\alpha)-(1-s)b-s\gamma)} \equiv a^{\max}$.

Later on I will show that with a more restrictive assumption we can guarantee concavity.

Then, because of concavity of $y(k)$ it is easy to show that the value of $\Theta(k)$ is given by the following remark.

Remark 2 *(i) for* $0 < k < k_{AUT}$ *then* $\Theta(k) > 0$ *and* $R(\hat{k}) - a(1-s) > 0$
(ii) for $k_{AUT} < k < \hat{k}$ *then* $\Theta(k) < 0$ *and* $R(\hat{k}) - a(1-s) > 0$
(iii) for $\hat{k} < k < \infty$ *then* $\Theta(k) > 0$ *and* $R(\hat{k}) - a(1-s) < 0$

4. The limit behavior of $\Theta(k)$ at the discontinuity is given by:

$\lim_{k\to\hat{k}^-} \Theta(k) = -\infty$ and $\lim_{k\to\hat{k}^+} \Theta(k) = \infty$.

5. The first order derivative of $\Theta(k)$.

Define $\Omega \equiv (s(1-\alpha)-(1-s)b-s\gamma)$

$\frac{\partial\Theta(k)}{\partial k} = \frac{(\Omega y_k - 1)(R(k)-a(1-s))-(\Omega y(k)-k)R_k}{(R(k)-a(1-s))^2} =$

We then use the following equations

$R(k) = \alpha\frac{y(k)}{k}$, and $R_k = \alpha(\frac{\alpha}{1-\gamma}-1)\tilde{A}k_t^{\frac{\alpha}{1-\gamma}-2} = \alpha(\frac{\alpha}{1-\gamma}-1)\frac{y(k)}{k^2} = (\frac{\alpha}{1-\gamma}-1)\frac{R(k)}{k}, y_k = \frac{\alpha}{(1-\gamma)}\frac{y(k)}{k} = \frac{1}{(1-\gamma)}R(k)$

Then, the derivative gets: (we express everything in $R(k)$)

$\frac{\partial\Theta(k)}{\partial k} = \frac{(\Omega\frac{1}{(1-\gamma)}R(k)-1)(R(k)-a(1-s))-(\Omega\frac{R(k)k}{\alpha}-k)(\frac{\alpha}{1-\gamma}-1)\frac{R(k)}{k}}{(R(k)-a(1-s))^2} =$
$= \frac{(\Omega\frac{1}{(1-\gamma)}R(k)-1)(R(k)-a(1-s))-(\Omega\frac{R(k)k}{\alpha}-k)(\frac{\alpha}{1-\gamma}-1)\frac{R(k)}{k}}{(R(k)-a(1-s))^2} =$

$\frac{(\Omega\frac{R(k)}{(1-\gamma)}(R(k)-C)-(R(k)-C)-(\Omega\frac{R(k)k}{\alpha}(\frac{\alpha}{1-\gamma}-1)\frac{R(k)}{k}-k(\frac{\alpha}{1-\gamma}-1)\frac{R(k)}{k})}{(R(k)-C)^2} = \frac{\Omega\frac{R^2}{(1-\gamma)}-C\Omega\frac{R}{(1-\gamma)}-R+C-(\Omega\frac{R^2}{\alpha}(\frac{\alpha}{1-\gamma}-1)-(\frac{\alpha}{1-\gamma}-1)R)}{(R(k)-C)^2} =$
$\frac{\Omega\frac{R^2}{(1-\gamma)}-C\Omega\frac{R}{(1-\gamma)}-R+C-\frac{\Omega R^2}{1-\gamma}+\Omega\frac{R^2}{\alpha}+\frac{\alpha}{1-\gamma}R-R}{(R(k)-C)^2} = \frac{\frac{\Omega}{\alpha}R^2-(\frac{\Omega a(1-s)-\alpha}{(1-\gamma)}+2)R+a(1-s)}{(R-a(1-s))^2}$

$$\frac{\partial\Theta(k)}{\partial k} = \frac{\frac{\Omega}{\alpha}R^2-(\frac{\Omega a(1-s)-\alpha}{(1-\gamma)}+2)R+a(1-s)}{(R-a(1-s))^2}$$

Define $A = \frac{\Omega}{\alpha}$, $C = a(1-s)$ and $\Xi = (\frac{\Omega C-\alpha}{(1-\gamma)}+2) = (\frac{a(AC-1)}{(1-\gamma)}+2)$

$$\frac{\partial\Theta(k)}{\partial k} = \frac{AR^2-\Xi R+C}{(R(k)-C)^2}$$

which is a quadratic equation with **at most two roots**.

5.1 (Limiting behavior) By applying the de hospital rule

$$\lim_{k\to 0}\frac{\partial\Theta(k)}{\partial k}=\frac{\Omega}{\alpha}>0 \text{ and } \lim_{k\to\infty}\frac{\partial\Theta(k)}{\partial k}=\frac{1}{(a(1-s))}>0$$

5.2 $\frac{\partial\Theta(k)}{\partial k}>0$ if $AR^2-\Xi R+C>0$ and $\frac{\partial\Theta(k)}{\partial k}<0$ for $AR^2-\Xi R+C<0$ which depends on the number of roots.

Discriminant: $\Xi^2-4AC=(\frac{a(AC-1)}{(1-\gamma)}+2)^2-4AC=\frac{a^2(AC^2-2AC+1)}{(1-\gamma)^2}+\frac{4a(AC-1)}{(1-\gamma)}+4-4AC=\frac{a^2(AC^2-2AC+1)}{(1-\gamma)^2}+\frac{4a(AC-1)}{(1-\gamma)}+4(1-AC)=$

6.(Second order derivatives)

The first order derivative is given by:

$\frac{\partial\Theta(k)}{\partial k}=\frac{AR^2-\Xi R+C}{(R(k)-C)^2}$

Taking the second order derivative we obtain that:

$\frac{\partial^2\Theta(k)}{\partial k^2}=\frac{(A2R\acute{R}-\Xi\acute{R})(R-C)^2-(AR^2-\Xi R+C)(R(k)-C)2\acute{R}}{(R-C)^4}=$

$\acute{R}\frac{(A2R-\Xi)(R-C)-(AR^2-\Xi R+C)2}{(R-C)^3}=$

$\acute{R}\frac{A2R(R-C)-\Xi(R-C)-2AR^2+2\Xi R-2C)}{(R-C)^3}=$

$\acute{R}\frac{(A2RR-A2RC)-\Xi R+\Xi C-2AR^2+2\Xi R-2C)}{(R-C)^3}=$

$\acute{R}\frac{A2R^2-A2RC-\Xi R+\Xi C-2AR^2+2\Xi R-2C}{(R-C)^3}=$

$\acute{R}\frac{-A2RC+\Xi C+\Xi R-2C}{(R-C)^3}=$

$\acute{R}\frac{R(\Xi-2AC)+C(\Xi-2)}{(R-C)^3}=\acute{R}\frac{R(\Xi-2AC)+C((\frac{\Omega C-\alpha}{(1-\gamma)})}{(R-C)^3}\Rightarrow\acute{R}\frac{R(\Xi-2AC)+C(\alpha(\frac{\frac{\Omega}{\alpha}C-1}{(1-\gamma)})}{(R-C)^3}=$

$$\frac{\partial^2\Theta(k)}{\partial k^2}=\acute{R}\frac{R(\Xi-2AC)+C(\alpha\frac{AC-1}{(1-\gamma)})}{(R-C)^3}$$

The derivative is negative until the discontinuity $0<k<\hat{k}$ $(R-C>0)$ of the kk locus if: $R(\Xi-2AC)+C(\alpha(\frac{AC-1}{(1-\gamma)})>0$ because $\acute{R}<0$. Thus, we need that,

$$R>\frac{-C(\alpha(\frac{AC-1}{(1-\gamma)}))}{(\Xi-2AC)}$$

$$\alpha\tilde{A}k_t^{\frac{\alpha-(1-\gamma)}{1-\gamma}}>\frac{-C(\alpha(\frac{AC-1}{(1-\gamma)}))}{(\Xi-2AC)}$$

$$k<\left(\frac{-C(\alpha(\frac{AC-1}{(1-\gamma)}))}{\alpha\tilde{A}(\Xi-2AC)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}\equiv\tilde{k}$$

this is a necessary and sufficient condition for concavity. We now want to show if this is true for $0 < \tilde{k} < \hat{k}$ ($\tilde{k}$ below the discontinuity $\hat{k}$).

$$\left(\frac{-C(\alpha(\frac{AC-1}{(1-\gamma)}))}{(\Xi-2AC)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} < \left(\frac{C}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$$
$$\left(\frac{-C(\alpha(\frac{AC-1}{(1-\gamma)}))}{(\Xi-2AC)}\right) > \left(\frac{C}{\alpha\tilde{A}}\right)$$
$$\left(\frac{-(\alpha(\frac{AC-1}{(1-\gamma)}))}{(\Xi-2AC)}\right) > 1$$
$$-(\alpha(\frac{AC-1}{(1-\gamma)})) > (\Xi - 2AC)$$
$$(\alpha(\frac{AC-1}{(1-\gamma)})) < (\Xi - 2AC)$$

Note that $\Xi = (\frac{\Omega C-\alpha}{(1-\gamma)} + 2) = (\frac{\frac{\Omega}{\alpha}C-1}{(1-\gamma)} + 2) = (\frac{\alpha(\frac{\Omega}{\alpha}C-1)}{(1-\gamma)} + 2) = (\frac{\alpha(AC-1)}{(1-\gamma)} + 2)$

Substituting to the inequality $(\alpha(\frac{AC-1}{(1-\gamma)})) < \frac{\alpha(AC-1)}{(1-\gamma)} + 2 - 2AC \Rightarrow 0 < +2 - 2AC \Rightarrow 2AC < 2 \Rightarrow AC < 1.$ Which holds from the assumption that limits austerity (see Remark) as we have

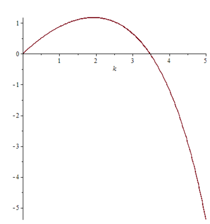
$\frac{a(1-s)}{\alpha} < \frac{1}{(s(1-\alpha)-(1-s)b-s\gamma)} \Rightarrow \frac{C}{\alpha} < \frac{1}{\Omega} \Rightarrow \frac{\Omega}{\alpha} < \frac{1}{C} \Rightarrow A < \frac{1}{C} \Rightarrow AC < 1.$

Lemma 1 *Under Remark 1, then* $\Theta(k)$ *is concave and inverse U-shaped for* $0 < k < \hat{k}$ *and convex (U-shaped) for* $\hat{k} < k < \infty$*.*

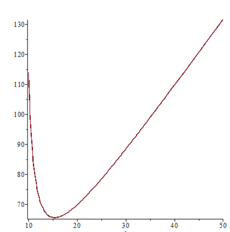
Properties of first and second derivative of $\Theta(k)$ and Remark 1.

The graph of the kk locus

Before the discontinuity



After the discontinuity



The steady states are defined by the following expression

$$F(k) = \Theta(k) - \Gamma(k)$$
$$F(k) = \frac{(\Omega)\, y(k) - k}{(R(k) - C)} - \frac{by(k)}{(1+a) - R(k)}$$

i. $F(0) = 0$

For $0 < k < \check{k}$, $\Theta(k) > 0$ and $\Gamma(k) < 0$ thus, $F(k) > 0$. Also, $\lim\limits_{k \to \check{k}^-} F(k) = +\infty$

Then, $\lim\limits_{k \to \check{k}^-} F(k) = -\infty$ and $\lim\limits_{k \to \check{k}^+} \acute{F}(k) > 0$. So, just after the discontinuity of the debt locus the $F(k)$ function is increasing.

Also, $\lim\limits_{k \to \hat{k}^-} F(k) = -\infty$, $\lim\limits_{k \to \hat{k}^-} \acute{F}(k) < 0$.

So, $F(k)$ is increasing from the discontinuity of the debt locus and it is decreasing at the discontinuity of the capital stock locus.

Since, $\check{k} < k < \hat{k}$ the derivative changes sign, we are going to explore if the maximum of the function is positive.

$\acute{F}(k) = \frac{\partial \Theta(k)}{\partial k} - \frac{\partial \Gamma(k)}{\partial k}$, $\frac{\partial \Theta(k_{\max})}{\partial k} - \frac{\partial \Gamma(k_{\max})}{\partial k} = 0 \Rightarrow \frac{AR^2 - \Xi R + C}{(R(k)-C)^2} - \left(\frac{(1+a)}{(1-\gamma)} - \frac{R}{\alpha}\right) R = 0$
$AR^2 - \Xi R + C - \left(\frac{(1+a)}{(1-\gamma)} - \frac{R(k)}{\alpha}\right) R(k)\,(R(k) - C)^2 = 0$
$AR^2 - \Xi R + C - \left(\frac{(1+a)}{(1-\gamma)} - \frac{R}{\alpha}\right) R\,(R^2 - 2RC + C^2) = 0$

$AR^2 - \Xi R + C - \left(\frac{(1+a)-(1-\gamma)R}{(1-\gamma)\alpha}\right)(R^3 - 2R^2C + RC^2) = 0$
$AR^2 - \Xi R + C - \frac{R^3(1+a)}{(1-\gamma)\alpha} + \frac{2R^2C(1+a)}{(1-\gamma)\alpha} - \frac{RC^2(1+a)}{(1-\gamma)\alpha} + \frac{R^3(1-\gamma)R}{(1-\gamma)\alpha} - \frac{2R^2C(1-\gamma)R}{(1-\gamma)\alpha} + \frac{RC^2(1-\gamma)R}{(1-\gamma)\alpha} = 0$

$$F^{''}(k) = \acute{R}\frac{R(\Xi - 2AC) + C(\alpha\frac{AC-1}{(1-\gamma)})}{(R-C)^3} - \frac{b\acute{R}}{(1-\gamma)\alpha}\frac{R(\alpha - 2(1-\gamma)) + \alpha(1+a)}{(1+a-R)^3}$$

.

Because we proved that $\acute{R}\frac{R(\Xi-2AC)+C(\alpha\frac{AC-1}{(1-\gamma)})}{(R-C)^3} < 0$ after the discontinuity of the debt locus and between the k austerity, then, for concavity of $F(k)$ we need $\frac{b\acute{R}}{(1-\gamma)\alpha}\frac{R(\alpha - 2(1-\gamma)) + \alpha(1+a)}{(1+a-R)^3} > 0$ which from the analysis of the debt locus after the discontinuity hold for $R(\alpha - 2(1-\gamma)) + \alpha(1+a) < 0 \Rightarrow k < \left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \equiv \tilde{k}$. Thus, for $k < \tilde{k}$ then $F^{''}(k) < 0$. Thus, if that $\tilde{k}$ is below the discontinuity of the k-locus $\hat{k} = \left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}}$.

Thus, a sufficient parametric condition for concavity of $F(k)$ in the area between the discontinuities, $\check{k} < k < \hat{k}$, is that $\tilde{k} < \hat{k}$ that is

$\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} < \left(\frac{a(1-s)}{\alpha\tilde{A}}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} \Rightarrow \frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)} > \frac{a(1-s)}{\alpha\tilde{A}} \Rightarrow (1+a)\alpha > a(1-s)(2(1-\gamma) - \alpha)$.

Lemma 2 *If* $(1+a)\alpha > a(1-s)(2(1-\gamma)-\alpha)$ *then in the area between the discontinuities* $\check{k} < k < \hat{k}$, $F''(k) < 0$.

This proposition means that if an equilibrium exists will be multiple (except the knife edge tangency condition). Furthermore,the debt locus will be convex at the tangency and k locus concave.

Another possible a sufficient parametric condition for concavity of $F(k)$ is to look at the are between the discontinuities of debt locus and the k_{AUT} (because in the area between k_{AUT} and the discontinuity of k-locus the debt is negative and no equilibrium can exist). So, in this case a sufficient condition is $\tilde{k} < k_{AUT}$.

$\left(\frac{(1+a)}{\tilde{A}(2(1-\gamma)-\alpha)}\right)^{\frac{1-\gamma}{\alpha-(1-\gamma)}} < \left(\frac{1}{\tilde{A}(s(1-\alpha)-(1-s)b-s\gamma)}\right)^{\frac{(1-\gamma)}{\alpha-(1-\gamma)}} \Rightarrow \frac{(1+a)}{(2(1-\gamma)-\alpha)} > \frac{1}{(s(1-\alpha)-(1-s)b-s\gamma)} \Rightarrow (1+a)\,(s(1-\alpha)-(1-s)b-s\gamma) > (2(1-\gamma)-\alpha)$.

If $(1+a)\,(s(1-\alpha)-(1-s)b-s\gamma) > (2(1-\gamma)-\alpha)$ then in the area between the $\check{k} < k < k_{AUT}$, $F''(k) < 0$.

Appendix 3. Stability

In this section, we are going to analyze the stability properties and the type of each equilibrium. We are going to construct the phase diagram and analyze the arrows of motion.

The dynamic equation for debt is given by

$B_{t+1} - B_t = (R(k_t) - a - 1)B_t + by(k_t)$

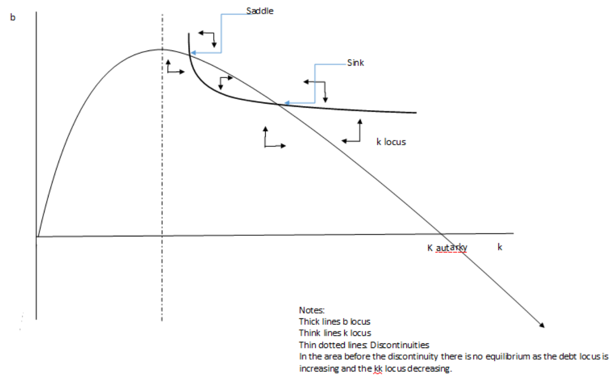
Remind that, for $\check{k} < k < \infty$ then $(1+a) - R(k) > 0$. Then, for $B_{t+1} - B_t > 0$, $R(k_t) - a - 1)B_t + by(k_t) > 0$ that is $B_t < \frac{by(k_t)}{(1+a)-R(k)}$. Thus, for any B_t lower then the $\Gamma(k)$ locus and because $\Gamma(k)$ is convex, the debt is decreasing (increasing under the $\Gamma(k)$ locus).

The dynamic equation for the capital stock is given by

$k_{t+1} - k_t = (s(1-\alpha) + s(b-\gamma) - b)\,y(k_t) - k_t + (a(1-s) - R_t(k_t))\,B_t$

For $k_{t+1} - k_t > 0$ if $(s(1-\alpha) + s(b-\gamma) - b)\,y(k_t) - k_t + (a(1-s) - R_t(k_t))\,B_t > 0$. Remind that, for $\check{k} < k < k_{AUT}$ then $\Theta(k) > 0$ and $R(\hat{k}) - a(1-s) > 0$. Dividing the inequality by $R(\hat{k}) - a(1-s) > 0$ we get $\frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} - B_t > 0 \Rightarrow B_t < \frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} \Rightarrow B_t < \frac{(s(1-\alpha)+s(b-\gamma)-b)y(k_t)-k_t}{R(\hat{k})-a(1-s)} \Rightarrow B_t < \Theta(k)$. Because $\Theta(k)$ is a concave function, for every B below the $\Theta(k)$ locus the capital stock is increasing and below the $\Theta(k)$ locus, it is decreasing.

According to this analysis, the phase diagram and the arrows of motion are given by:



From the above diagram we can deduct that there are two stable equilibria. The lower equilibrium is saddle-path stable and the second equilibrium is stable node.

Appendix 4. Steady-State Effects of Policy Parameters

The equilibrium steady-state of capital is given by:

$$F(k) = \frac{(\Omega)\, y(k) - k}{(R(k) - C)} - \frac{by(k)}{(1+a) - R(k)}$$

where $\Omega(b) \equiv (s(1-\alpha) - (1-s)b - s\gamma)$, $C(a) \equiv a(1-s)$

We first want to examine the effect of austerity parameter on steady-state capital stock. From the implicit function theorem we have:

$\frac{\partial k}{\partial a} = -\frac{\frac{\partial F(k)}{\partial k}}{\frac{\partial F(k)}{\partial a}}$

$\frac{\partial F(k)}{\partial a} = \frac{(\Omega y(k) - k)}{(R(k)-C)^2} + \dfrac{by(k)}{((1+a) - R(k))^2} > 0$ from $0 < k < k_{AUT}$.
$\frac{\partial F(k)}{\partial k} > 0$ from $0 < k < k_{\max}$ and $\frac{\partial F(k)}{\partial k} < 0$ from $k_{\max} < k < k_{AUT}$.

Given that the one equilibrium, k_{ss}^{low} is below $k_{\max}$ and the other, k_{ss}^{high}, above $k_{\max}$ display different properties resulting to Proposition 3.

Secondly, we examine the effect of structural deficit parameter on steady-state capital stock. From the implicit function theorem we have:

$\frac{\partial k}{\partial b} = -\frac{\frac{\partial F(k)}{\partial k}}{\frac{\partial F(k)}{\partial b}}$
$(s(1-\alpha) - (1-s)b - s\gamma)$

$\frac{\partial F(k)}{\partial b} = \frac{-(1-s)}{(R(k)-C)} - \dfrac{y(k)}{(1+a) - R(k)}$

We know that for $0 < k < k_{AUT}$, $R(\hat{k}) - C > 0$ and for $\check{k} < k < \infty$, $(1+a) - R(k) > 0$. Thus, in the area we are interested $\check{k} < k < k_{AUT}$ we have:

$\frac{\partial F(k)}{\partial b} < 0$, for $\check{k} < k < k_{AUT}$, thus, resulting to Proposition 4.