Optimal monetary policy in a New Keynesian model with heterogeneous expectations*

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Abstract

In a world where expectations are heterogeneous, how should we design the optimal policy? Are canonical policies robust if expectations heterogeneity is considered or they would be associated with large welfare losses? We aim to answer these questions in a stylized simple model of New Keynesian kind, where agents' beliefs are not homogeneous. Assuming that a fraction of agents can form their expectation by some adaptive or extrapolative schemes, we focus on optimal monetary policy by second-order approximation of the policy objective from the consumers' utility functions. We find that the introduction of bounded

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rationality in the New Keynesian framework matters. The presence of heterogeneous agents adds a new dimension to the central bank's optimization problem: consumption inequality. Optimal policies need to be designed to stabilize the cross-variability of heterogeneous expectations; in fact, as long as different individual consumption plans depend on different expectations paths, a central bank aimed to reduce consumption inequality should minimize the cross-sectional variability of expectations. Moreover, the traditional trade-off between the price dispersion and aggregate consumption variability is also quantitatively affected by heterogeneity.

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1 Introduction

The New Keynesian approach has become the workhouse for academic and practical discussions of monetary policy. There, optimal monetary policies are designed on the rational expectations paradigm; however, heterogeneity in the expectations formation mechanism is well documented in both survey data and laboratory experiments. In a world where expectations are heterogeneous, how should we design the optimal policy? Are canonical policies robust if expectations heterogeneity is considered or they would be associated with large welfare losses? We aim to answer these questions in a stylized simple model where agents' beliefs are not homogeneous.

The empirical evidence shows that inflation and output forecasts are not always fully rational.[1] Several approaches have been proposed to build a framework that could rationalize this fact. For instance, Krusell and Smith (1996), Mankiw (2000), Amato and Laubach (2003) and Galí *et al.* (2004, 2007) introduce in macroeconomic models a fraction of agents who are not completely rational by taking their decisions according to some "rule of thumb," i.e. consuming all their income.[2] Mankiw and Reis (2001, 2007) propose an approach based on sticky information. Evans and Honkapohja (2001, 2003) focus on learning models. Other approaches borrow some ideas from behavioral economics or develop some concepts of near-rationality.[3] By deviating somehow

[1] See, e.g., Thomas (1999), Akerlof *et al.* (2000), Mankiw *et al.* (2001), Branch (2004), Bovi (2013) and reference therein. Recent evidence based on laboratory experiments is documeted by Hommes (2011).

[2] An alternative interpretation of the assumption is to see it a short cut to model limited asset market participation (see e.g. Bilbiie, 2008).

[3] Among others see, e.g., Roberts (1997), Akerlof *et al.* (2000), Gaffeo *et al.*, (2010), De Grauwe (2010), De Grauwe and Kaltwasser (2012).

from the full rational paradigm, all these studies have challenged important economic puzzles such as inflation intrinsic persistence, data-consistent disinflation path, fiscal and monetary policy effects.

Along the above lines, a notable number of studies explicitly considers non-homogeneous expectations and takes the issue of how this heterogeneity potentially affects aggregate economic dynamics. These studies include Brock and Hommes (1997), Preston (2005), Branch and Evans (2006), Branch and McGough (2009) and Massaro (2013). In particular, by using two alternative approaches to model expectations heterogeneity, Branch and McGough (2009) and Massaro (2013) develop parsimonious micro-founded sticky price models that are consistent with boundedly rational individuals.

In Branch and McGough (2009) and Massaro (2013), expectations operators may differ across groups of agents, who are of two kinds (a fraction of them is rational and the remainder are not), and agents' optimal choices are modeled to be consistent to their specific forecasts. By aggregating individual decision rules, Branch and McGough (2009) and Massaro (2013) derive aggregate demand and supply equations of New Keynesian kind embedding bounded rationality. The resulting reduced form model is analytically tractable and encompasses the representative rational agent canonical benchmark as a special case.

Formally, as Preston (2005), Massaro (2013) focuses on long horizon forecasts, assuming that agents with subjective expectations choose optimal plans looking at forecasts of macroeconomic conditions over an infinite horizon. By contrast, Branch and McGough (2009) assume that individuals with subjective beliefs choose optimal plans that satisfy their individual Euler equations. By an axiomatic approach to heterogeneous expectations, they provide restrictions on the admissible forms of non-fully rational beliefs sufficient to ensure the laws of motion for the aggregate variables are analytically tractable and easily comparable to those obtained under complete rationality.[4] As a result, in Massaro (2013) the predicted aggregate dynamics hinges on long horizon forecasts, while in Branch and McGough (2009) it depends on one-period ahead subjective heterogenous forecasts.

Both Branch and McGough (2009) and Massaro (2013) show that heterogenous expectations can undermine some standard results about the determinacy of the rational expectation equilibrium (REE). However, they do not consider optimal policies, but assume that the central bank sets the interest rate according to a Taylor-kind rule. In other words, they investigate the effects

[4] In particular, by their restrictions, they are able to decouple the aggregate dynamics from the dynamics of the wealth distribution.

of policies based on simple interest rules, showing that specifications that are determinate under rationality may exhibit explosive or multiple equilibria in the case of bounded rationality.

Our paper extends the Branch and McGough's (2009) model to compute optimal monetary policy. We use the LQ approach developed and refined by Rotemberg and Woodford (1997), Woodford (2003: Chapter 6), Benigno and Woodford (2012). We derive a second-order approximation of the policy objective from the consumers, who are assumed to have the same specification for the utility functions, as different agents only differ for the way they form their beliefs. Then, we use our consistent welfare measure to investigate the effects of bounded rationality on optimal monetary policy. We consider the optimal response of the endogenous variables of the model assuming that the central bank optimizes either under commitment or under discretion. In the first case the monetary authority by a commitment to a policy plan is able to affect private sector's expectations. In the second case the central bank, taking private agent's forecasts as given, minimizes welfare losses in each period.

We find that the introduction of bounded rationality in the New Keynesian framework matters. The existence of a group of non-rational agents, who form their forecasts by adaptive or extrapolative mechanisms, implies that optimal policies need to be designed to stabilize the cross-variability of heterogenous expectations. The rationale of our result can be found in the fact that heterogenous agents add a new dimension to the central bank's optimization problem: consumption inequality. As long as different individual consumption plans depend on different expectations paths, a central bank aimed to reduce the variability in individual consumption should minimize the cross-sectional variability of expectations.

The central bank problem has now three dimensions: *i)* minimize the variability of aggregate consumption, *ii)* minimize the cost associated to price dispersion and *iii)* minimize the cross-sectional variance of consumption. As said, the latter requires to stabilize expectation variability. However, the traditional trade-off between the formers is also affected by heterogeneity. In our context, the cost of price dispersion increases with the size of the group of bounded agents. As long as they grow, the emphasis of stabilizing inflation versus output declines. This occurs because the dynamics of price dispersion is more complex than in the standard case (where it only relies on inflation). Here, price dispersion also depends on output stabilization.

The rest of the paper is organized as follows. Section 2 presents the Branch and McGough's (2009) axioms that generalizes the New Keynesian model to include non-homogeneous expectations and presents the resulting analytically tractable reduced form model that encompasses the representative rational agent benchmark as a special case. Section 3 derives the welfare criterion as a second-order approximation of the policy objective, assuming that the steady state is not distorted. Section 4 illustrates the properties of optimal policies under bounded rationality. It compares these to the canonical policies and Taylor rules and discusses the implications of agents' heterogeneity for commitment and discretion. Section 5 concludes.

2 The economy

We consider a simple yeoman-farmer economy. The economy is populated by a continuum of mass one of infinitely-lived households who produce and consume. A fraction α of them (rational households) has rational expectations. The remaining fraction (non-rational households) forms expectations according to a mechanism of bounded rationality. The two kinds of households are indexed by $\mathcal{R}$ and $\mathcal{B}$, which refer to rational and boundedly households, respectively. Each household produces a differentiated good by using its own labor and consumes a "bundle," a composite good composed by all the products. We assume price stickiness by assuming that in every period each yeoman-farmer faces an exogenous constant probability of being able to reset its price. The model is borrowed from Branch and McGough (2009) and described in the following sub-sections. The next one describes the axiomatic approach used to model heterogeneous expectations. The remaining sub-sections describe the model equations in details, the private sector's first-order conditions, and the log-linearized economy, respectively.[5]

2.1 The axiomatic approach to heterogenous expectations

Heterogeneous expectations are introduced by following the axiomatic approach developed by Branch and McGough (2009). Formally, denoting by $\mathcal{E}^i$ a generic (subjective) expectations operator (i.e., $\mathcal{E}^i_t x_{t+1}$ is the time t expectation on the value assumed by variable x at $t+1$ formed by an agent of type i), we impose the following assumptions: *i)* each expectation operator, $\mathcal{E}^i$, fixs observables; *ii)* all agents' beliefs coincide in the steady state; *iii)* $\mathcal{E}^i$ is a linear operator;[6] *iv)* $\mathcal{E}^i$ satisfies the law of iterated expectations; *v)* if x is a variable forecasted by agents at time t and time $t+k$ then $\mathcal{E}^i_t \mathcal{E}^j_{t+k} x_{t+k} = \mathcal{E}^i_t x_{t+k}$, $i \neq j$; *vi)* all agents have common expectations on expected differences in limiting wealth.

As discussed by Branch and McGough (2009), assumptions *i)*-*iv)* are consistent with reason-

[5] We refer to Branch and McGough (2009) for details.

[6] It is worth noticing that we only need to assume that if x, y, αx, $x+y$ are forecasted, $\mathcal{E}^i(\alpha x) = \alpha \mathcal{E}^i x$ and $\mathcal{E}^i(x+y) = \mathcal{E}^i x + \mathcal{E}^i y$. Moreover, if for all $k \geqslant 0$, x_{t+k} and $\sum_k \beta^{t+k} x_{t+k}$ are forecasted by agents then $\mathcal{E}^i_t \sum_k \beta^{t+k} x_{t+k} = \sum_k \beta^{t+k} \mathcal{E}^i_t x_{t+k}$.

able specifications of agent behavior. They implies that the forecast of a known quantity should be exactly the known quantity, some continuity in beliefs, agents incorporate some economic structure into their forecasting model, agents' beliefs satisfy the law of iterated expectations at an individual level.

Assumptions *v)-vi)* are necessary for aggregation. The former implies that agents' forecasts satisfy law of iterated expectations at an aggregate level. It requires that an individual of type i's belief of the future expectations of agents of type j is i's expectation. In other words, we are imposing a particular structure on higher-order beliefs. The axiom that agents agree on limiting wealth distributions avoids to abstract from the wealth distribution dynamics that otherwise affect the formulation of forecasts by expectations type causing a problem for aggregation. This allows us to remain close in form to the homogeneous case.[7]

By using the above assumptions, we can define the aggregate expectations as a weighted average of groups' expectations: $\mathcal{E}_t x_t = (1-\alpha)\,\mathcal{E}_t^{\mathcal{B}} x_t + \alpha \mathcal{E}_t^{\mathcal{R}} x_t$, where $\mathcal{R}$ and $\mathcal{B}$ identify the expectation operator for rational and boundedly rational households, respectively.

Our economy is populated by rational and adaptive agents; we assume that $\mathcal{E}_t^{\mathcal{R}} x_{t+1} = E_t x_{t+1}$, i.e. rational agents have one-step ahead perfect foresight on economic variables. However, it is worth noticing that they are not fully rational since they are not able to correctly understand the forecasts of bounded agents (they have wrong second-order beliefs—see assumption *iv.*) Regarding non-rational individuals, in line with the literature, we assume that they form their beliefs on the basis of a simple linear perceived law of motion (i.e., $x_t = \theta x_{t-1}$.) Therefore, $\mathcal{E}_t^{\mathcal{B}} x_t = \theta x_{t-1}$, i.e., the operator $\mathcal{E}^{\mathcal{B}}$ is a form of adaptive ($\theta < 1$) or extrapolative ($\theta > 1$) expectations, where θ is defined as the "adaption operator". We refer to $\theta = 1$, as the case of naive expectations. Applying the law of iterated expectations, we obtain $\mathcal{E}_t^{\mathcal{B}} x_{t+1} = \theta^2 x_{t-1}$.[8]

2.2 The model

Each household i produces as monopolist its own differentiated product and directly purchases a composite good seeking to maximize the expected value of the following utility function:

$$\mathcal{E}_0^i \sum_{t=0}^{\infty} \beta^t \left[u\left(C_t^i\right) - \nu(Y_t(i)) \right] \tag{1}$$

[7] For further details, see Branch and McGough (2009).

[8] See Branch and McGough (2009) for details. See also Pesaran (1987), Brock and Hommes (1997, 1998), Branch and McGough (2005) for empirical support and further considerations.

where $\beta \in (0,1)$ is the discount factor; $\mathcal{E}^i \in \left\{\mathcal{E}^{\mathcal{R}}, \mathcal{E}^{\mathcal{B}}\right\}$ indicates a generic (subjective) expectations operator of agent i, who can belong either to the rational or boundedly rational subset. The terms $u\left(C_t^i\right)$ and $\nu(Y_t(i))$ indicate the utility from consuming the composite good (C_t^i) and the disutility from producing the differentiated product $(Y_t(i))$, respectively.[9]

In each period, a number of randomly selected agents are allowed to reoptimize their prices. Each firm may reset its price only with probability $1-\xi_p$ in any given period, independently of the time elapsed since the last adjustment occurred. Thus, each period a proportion $1-\xi_p$ of producers reset their prices, while a fraction ξ_p keep their prices unchanged. As a result, the average duration of a price is given by $(1-\xi_p)^{-1}$.

Optimal prices will depend on the expectations about future marginal costs; thus, they will differ between the two different type of agents. Moreover, the Calvo lottery also implies a heterogeneity within each type of agents since only a fraction of them are able to reset their prices. Acting as price setters, individual agents face the risk associated with this Calvo lottery. Following a common procedure in heterogeneous agent models, to somehow limit heterogeneity between types, it is assumed that agents are engaged in a form of risk sharing mechanism to ensure themselves from the risk of the Calvo price setting. A benevolent financial regulator collects all income and then redistributes to each type of household the average income of that agent's type.[10]

It follows that individuals are fully insured against the risk associated to the possibility that they will be not able to adjust prices. Since agents have different expectations, the insurance contracts are designed in order to guarantee different amount of expected real income: $\Omega_t^{\mathcal{B}} = \int_\alpha^1 \frac{P_t(i)Y_t(i)}{(1-\alpha)P_t}di$ and $\Omega_t^{\mathcal{R}} = \int_0^\alpha \frac{P_t(i)Y_t(i)}{\alpha P_t}di$. Aggregating each agent type expected real income we obtain the real output of the economy:

$$Y_t = \alpha\Omega_t^{\mathcal{R}} + (1-\alpha)\Omega_t^{\mathcal{B}} \tag{2}$$

Moreover, because of the insurance mechanism, according to the agent's type $\tau \in \{\mathcal{R}, \mathcal{B}\}$, the

[9] There exists a continuum of goods represented by an interval $[0,1]$, $C_t^i \equiv \left(\int_0^1 C_t^i(j)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$ is then a Dixit-Stiglitz consumption basket and $C_t^i(j)$ is the quantity of good j consumed by the household i in period t. The consumer price index is defined as $P_t \equiv \left[\int_0^1 P_t(i)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}$.

[10] Among others, the same risk-sharing mechanism is used by Kocherlakota (1996), Shi (1999) and Mankiw and Reis (2007). It is entirely standard in models with heterogeneous agents. Alternatives are also discussed by Branch and McGough (2009).

real budget constraint of household i is:

$$C_t^i + B_t^i = \frac{1+i_{t-1}}{1+\pi_t} B_{t-1}^i + \Omega_t^\tau \tag{3}$$

where $B_t^i P_t$ is quantity of one-period, nominal riskless discount bonds purchased in period t, and maturing in $t+1$, held by agent i (each bond pays one unit of money at maturity and its price is $Q_t = (1+i_t)^{-1}$); $1+i_t$ indicates the gross nominal interest rate on a riskless one period bond purchase in period t; $1+\pi_t$ defines the gross inflation rate.

Finally, the existence of fully enforceable contracts requires the agents to behave as if they will receive their full marginal revenue from producing more, by assuming that agents should choose price and output as if they faced their perceived trade-off. In fact, each agent's income is independent of his effort because of the presence of the insurance company. Thus, without enforceable contracts, any agent would choose an effort equal to zero due to a free-riding behavior.

2.3 First-order conditions

Each household has to decide the optimal consumption (saving) plan, that is how to allocate its consumption expenditures among the different goods, and choosing the optimal price (i.e., his effort) if selected in the Calvo lottery.

The optimal consumption plan of household i is obtained by the maximization of (1) subject to (3) and a solvency constraint. Because the insurance mechanism, the optimal plan is the same within agents belonging to the same type. It can be obtained by a simple variational argument. As a result, agents of each type $\tau \in \{\mathcal{R}, \mathcal{B}\}$ make choices on consumption respecting the intertemporal Euler equation:

$$\frac{1}{1+i_t} = \beta \mathcal{E}_t^\tau \left[\frac{P_t}{P_{t+1}} \frac{u_c\left(C_t^\tau\right)}{u_c\left(C_{t+1}^\tau\right)} \right] \qquad \tau \in \{\mathcal{R}, \mathcal{B}\} \tag{4}$$

Optimal allocation of household's consumption expenditures among the different goods, $C_t\left(j\right)$, requires that the consumption index C_t is maximized for any level of expenditure $\int_0^1 P_t\left(i\right) C_t\left(j\right) dj$. Solving the intratemporal goods allocation problem, the set of demand equations for each type $\tau \in \{\mathcal{R}, \mathcal{B}\}$ of household is:

$$Y_t^\tau(j) = \left(\frac{P_t^j}{P_t}\right)^{-\varepsilon} \left(\Omega_t^\tau + \left(1+i_{t-1}\right) \frac{B_{t-1}^\tau}{P_t} - \frac{B_t^\tau}{P_t}\right) \tag{5}$$

In aggregate terms and given the bond market clearing condition, we derive the standard demand for the good j:

$$Y_t(j) = \alpha \left(\frac{P_t^j}{P_t} \right)^{-\varepsilon} \Omega_t^{\mathcal{R}} + (1-\alpha) \left(\frac{P_t^j}{P_t} \right)^{-\varepsilon} \Omega_t^{\mathcal{B}} = \left(\frac{P_t^j}{P_t} \right)^{-\varepsilon} Y_t \tag{6}$$

Each producer j belonging to type $\tau \in \{\mathcal{R}, \mathcal{B}\}$ chooses the price (P_t^j) solving:

$$\max_{P_t^j} \mathcal{E}_0^\tau \sum_{i=0}^{\infty} \left(\beta \xi_p \right)^i \left[(1-T) \lambda_{t+i} (P_t^j) Y_{t+i}(j) - \nu \left(Y_{t+i}(j) \right) \right] \tag{7}$$

subject to (6). The first term of the sum (7) is the marginal utility of additional nominal income, which can be interpreted as the contribution to utility derived by sales revenues; the second one is the production cost in terms of effort. A production subsidy T is introduced to eliminate distortions in the steady state.

By substitution of the demand function into the household's objective function and deriving with respect to P_t^j , we get

$$\mathcal{E}_0^\tau \sum_{i=0}^{\infty} \left(\beta \xi_p \right)^i \left[\frac{\partial u \left(C_{t+i}^\tau \right)}{\partial C_{t+i}^\tau} \left(\frac{P_t^j}{P_{t+i}} \right)^{-\varepsilon} Y_{t+i} - \left(\frac{P_t^j}{P_{t+i}} \right)^{-\varepsilon - 1} Y_{t+i} \frac{\partial \nu \left(Y_{t+i}(j) \right)}{\partial Y_{t+i}(j)} \right] = 0 \tag{8}$$

where we used the fact that $T = -\frac{1}{\varepsilon - 1}$ to implement the optimal steady state and $\lambda_t^j P_t = \partial u(C_t^j)/\partial C_t^j$.

2.4 The log-linear economy

By log-linearization of (4), after some substitutions, we obtain:

$$c_t^\tau = \mathcal{E}_t^\tau c_{t+1}^\tau - \sigma(i_t - \mathcal{E}_t^\tau \pi_{t+1}) \qquad \tau \in \{\mathcal{R}, \mathcal{B}\} \tag{9}$$

where $\sigma^{-1} \equiv -\bar{C} \ u_{CC}/u_C > 0$ is the inverse of the intertemporal elasticity of substitution of consumption (i.e., the coefficient of relative risk aversion.) We define the log deviations of the variables from their steady state values with lower case letters; i.e., $c_t^\tau = \log(C_t^\tau / \bar{C})$.

By using the budget constraint, $c_t^\tau = \omega_t^\tau \equiv \left(\beta^{-1} \log(B_{t-1}^\tau) - \log(B_t^\tau) \right) / \bar{Y} + \log(\Omega_t^\tau / \bar{\Omega})$,[11] the

[11] The realtionship is obtained by exploiting the fact that $B^{\mathcal{R}} = B^{\mathcal{B}} = 0$. See Branch and McGough (2009: 1040).

log-linearized consumer's Euler equation (9) can be then rewritten in terms of wealth as:

$$\omega_t^\tau = \mathcal{E}_t^\tau \omega_{t+1}^\tau - \sigma(i_t - \mathcal{E}_t^\tau \pi_{t+1}) \qquad \tau \in \{\mathcal{R}, \mathcal{B}\} \tag{10}$$

The above condition holds for both type of agents, thus it is possible to derive the usual IS relation by aggregating (10) provided that the axioms *i)-vii)* are satisfied by the expectational operator $\mathcal{E}_t^\tau$.

Specifically, iterating (10) forward we obtain

$$\omega_t^\tau = \omega_\infty - \sigma \mathcal{E}_t^\tau \sum_{k \geq 0} (i_{t+k} - \pi_{t+k+1}) \qquad \tau \in \{\mathcal{R}, \mathcal{B}\} \tag{11}$$

where ω_∞ is the limiting wealth common between types.

By using the bond market clearing condition $\alpha \log(B_t^{\mathcal{R}}) + (1-\alpha)\log(B_t^{\mathcal{B}}) = 0$ and aggregating the budget constraints, we achieve:

$$y_t = \alpha \omega_t^{\mathcal{R}} + (1-\alpha)\omega_t^{\mathcal{B}} \tag{12}$$

Finally, replacing (11) into (12), as long as $\omega_\infty^{\mathcal{R}} = \omega_\infty^{\mathcal{B}} = \omega_\infty$, we obtain an IS curve that is similar to the relation derived in the standard New Keynesian framework with the exception of the conditional expectation operator, which is substituted by a convex combination of the heterogeneous expectations operators of the two types of agents. Formally,

$$y_t = \mathcal{E}_t y_{t+1} - \sigma \left(i_t - \mathcal{E}_t \pi_{t+1} \right) \tag{13}$$

where $\mathcal{E}_t y_{t+1} = \alpha \mathcal{E}_t^{\mathcal{R}} y_{t+1} + (1-\alpha)\mathcal{E}_t^{\mathcal{B}} y_{t+1}$ and $\mathcal{E}_t \pi_{t+1} = \alpha \mathcal{E}_t^{\mathcal{R}} \pi_{t+1} + (1-\alpha)\mathcal{E}_t^{\mathcal{B}} \pi_{t+1}$.

Regarding the supply side, solving forward (8) and making use of assumptions *iii)-v)* the log-linear version of the optimal price equation of agent j belonging to type $\tau \in \{\mathcal{R}, \mathcal{B}\}$ becomes:

$$\mathcal{E}_0^\tau \sum_{i=0}^{\infty} \left(\beta \xi_p\right)^i \left[\log\left(\frac{P_t^j}{P_{t+i}}\right) - \frac{\sigma^{-1}}{1+\eta\varepsilon} c_{t+i} - \frac{\eta}{1+\eta\varepsilon} y_{t+i} \right] = 0 \tag{14}$$

where $\eta = \nu_{YY}\bar{Y}/\nu_Y$. Therefore,

$$\log\left(\frac{P_t^j}{P_t}\right) = \xi_p \beta \mathcal{E}_t^\tau \pi_{t+1} + \left(1 - \beta\xi_p\right) \left[\frac{\sigma^{-1}}{1+\eta\varepsilon} \omega_t^\tau + \frac{\eta}{1+\eta\varepsilon} y_t \right] + \beta \xi_p \mathcal{E}_t^\tau \log\left(\frac{P_{t+1}^j}{P_{t+1}}\right) \tag{15}$$

Aggregating (15) for different types of agents and combining with the definition of the aggregate price dynamics yields to an AS curve similar to the relation derived in the standard New Keynesian framework with the same expectation operator of used that for (13):

$$\pi_t = \beta \mathcal{E}_t \pi_{t+1} + \frac{\left(1-\xi_p\right)\left(1-\beta\xi_p\right)\left(\eta+\sigma^{-1}\right)}{\xi_p\left(1+\varepsilon\eta\right)} y_t \tag{16}$$

i.e.

$$\pi_t = \alpha\beta E_t \pi_{t+1} + (1-\alpha)\beta\theta^2 \pi_{t-1} + \kappa y_t + e_t \tag{17}$$

where $\kappa = \frac{\left(1-\xi_p\right)\left(1-\beta\xi_p\right)\left(\eta+\sigma^{-1}\right)}{\xi_p(1+\varepsilon\eta)}$. Note that (17) has been augmented by a supply disturbance.

The Phillips curve (17) shows that the inclusion of boundedly rational agents in our New Keynesian framework implies inflation persistence. As long as the proportion of non-rational agents increases, the importance of the backward component relative of the forward one in the Phillips curve also increases. The effect of a change in the share of boundedly rational individuals on the backward term of (17) is magnified by the extrapolative expectations (i.e., $\partial\pi_{t-1}/\partial\alpha\theta < 0$).[12]

3 Welfare criterion

We now proceed to compute a quadratic Taylor series approximation to utility of the household i. The first term of (1) can be approximated as

$$\tilde{u}\left(C_t^i\right) = \bar{C} u_C \left(c_t(i) + \frac{1-\sigma^{-1}}{2} c_t^2(i) \right) + t.i.p. + \mathcal{O}\left(\left\|\xi^3\right\|\right) \tag{18}$$

Note that, in the steady state, $\bar{C} = \bar{C}^{\mathcal{R}} = \bar{C}^{\mathcal{B}}$ by assumption; the term $\mathcal{O}\left(\left\|\xi^3\right\|\right)$ indicates the terms of order greater than two; $t.i.p.$ collects the terms independent of policy. Integrating (18) over i, after cumbersome algebra, we obtain

$$\int_0^1 \tilde{u}\left(C_t^i\right) di = \bar{C} u_C \left\{ c_t + \frac{1-\sigma^{-1}}{2} \left[c_t^2 - var_i(c_t(i))\right] \right\} + t.i.p. + \mathcal{O}\left(\left\|\xi^3\right\|\right) \tag{19}$$

where we used the relation $var_i(c_t(i)) = E_i\left(c_t^2\right) - \left(E_i c_t\right)^2$. Note also that the cross-sectional variance of the consumption is equal to $var_i(c_t(i)) = \alpha\left(1-\alpha\right)\left(c_t^{\mathcal{R}} - c_t^{\mathcal{B}}\right)^2$.[13]

Regarding effort, it is worth noticing that each agents potentially supplies a different quantity

[12] Recall that the portion of non-rational agents is decreasing in α.

[13] It follows from $E_i\left(c_t^2\right) = \alpha\left(c_t^{\mathcal{R}}\right)^2 + (1-\alpha)\left(c_t^{\mathcal{B}}\right)^2$ and $\left(E_i c_t\right)^2 = c_t^2$.

of output (this depends on the agent's type and the fact if he is or not extracted in the Calvo lottery). Second order approximation of the second term of (1) leads to

$$\tilde{\nu}(Y_t(i))di = u_N \bar{N}\left(y_t(i) + \frac{1+\eta}{2}y_t^2(i)\right) + t.i.p. + \mathcal{O}\left(\left\|\xi^3\right\|\right) \tag{20}$$

and, after integration, we obtain

$$\int_0^1 \tilde{\nu}(Y_t(i))di = u_N \bar{N}\left(E_i y_t(i) + \frac{1+\eta}{2}\left(\left[E_i y_t(i)\right]^2 + var_i(y_t(i))\right)\right) + t.i.p. + \mathcal{O}\left(\left\|\xi^3\right\|\right) \tag{21}$$

where $\int_\alpha^1 \bar{N}(i) = \bar{N}$ since price dispersion is zero in the steady state. Then, by considering $Y(i) = \left(P_t(i)/P_t\right)^{-\varepsilon} Y_t$, we get

$$\int_0^1 \tilde{\nu}(Y_t(i))di = u_N \bar{N}\left(y_t + \frac{1+\eta}{2}y_t^2 + \frac{\varepsilon^{-1}+\eta}{2}var_i(y_t(i))\right) + t.i.p. + \mathcal{O}\left(\left\|\xi^3\right\|\right) \tag{22}$$

and $var_i(y_t(i)) = \varepsilon^2 var_i \log\left(p_t(i)\right)$. Thus, in the non-distorted steady state, where holds the equality $\bar{C}u_C/u_N = \bar{N}$, the above expression can be rewritten as

$$\int_0^1 \tilde{\nu}(Y_t(i))di = \bar{C}u_C\left(y_t + \frac{\varepsilon\left(1+\varepsilon\eta\right)}{2}var_i \log(p_t(i)) + \frac{1+\eta}{2}y_t^2\right) + t.i.p. + \mathcal{O}\left(\left\|\xi^3\right\|\right) \tag{23}$$

Combining (19) and (23), the approximated intertemporal utility can be expressed as

$$\textstyle\sum_{t=0}^{\infty} \beta^t \frac{\mathcal{U}_t}{\bar{C}u_c} = -\sum_{t=0}^{\infty} \beta^t \left[L_t + t.i.p. + \mathcal{O}\left(\left\|\xi^3\right\|\right)\right] \tag{24}$$

where the instantaneous loss is

$$L_t = \frac{\eta+\sigma^{-1}}{2}y_t^2 + \frac{\varepsilon\left(1+\varepsilon\eta\right)}{2}var_i \log(p_t(i)) + \frac{1-\sigma^{-1}}{2}var_i(c_t(i)) \tag{25}$$

and[14]

$$\begin{aligned} var_i(\log p_t(i)) &= \frac{\xi_p}{\left(1-\beta\xi_p\right)\left(1-\xi_p\right)}\pi_t^2 + \frac{\left(1-\alpha\right)\xi_p^2\left[\pi_t - \beta\theta^2\pi_{t-1} - \kappa\left(\frac{c_t^{\mathcal{B}}+\eta\sigma y_t}{1+\eta\sigma}\right)\right]^2}{\alpha\left(1-\beta\xi_p\right)\left(1-\xi_p\right)} & (26) \\ var_i(c_t(i)) &= \alpha\left(1-\alpha\right)\left(c_t^{\mathcal{R}} - c_t^{\mathcal{B}}\right)^2 & (27) \end{aligned}$$

Unlike the traditional case, welfare (25) is composed by three components when heterogenous expectations are introduced in a New Keynesian framework by assuming that a fraction of agents

[14] Price and consumption dispersions derivation are provided in Appendix.

form their expectations according to a mechanism of bounded rationality.

As in the the standard case, the first two components of (25) relate costs associated with consumption variability (y_t^2) and price dispersion ($var_i \log(p_t(i))$. However, here the dispersion of prices (26) is not linked only to current inflation, but it has a more complex structure. Note also that the dispersion of prices positively depends on the proportion of boundedly rational agents. Everything equal, it increases in the fraction of agents who form their expectations in a non-rational way.

An additional third term captures the cost linked to inequality in the consumption of the two different types of agents ($var_i(c_t(i))$. This cost is not linear in the degree of bounded rationality. It has a peak when $\alpha = 0.5$, since in this case the distribution of agents exhibits the highest dispersion (see (27)). It is worth noticing that inequality is increasing in the variability of expectations across types as the different beliefs drive different choices.

4 Optimal policies under bounded rationality

4.1 Calibration

We calibrate the model to the U.S. economy. The time unit is meant to be a quarter. The calibration of the structural parameters are chosen to equal those estimated or calibrated by Rotemberg and Woodford (1997) by using structural vector auto-regression (SVAR) methodology and microeconomic evidence, respectively. We assume that the subjective discount rate β is 0.99 so that $(\beta^{-1} - 1)$ equals the long-run average real interest rate. In the goods market, the intratemporal elasticity of substitution between the differentiated goods (price elasticity of demand) is set equal to 7.84, this implies a markup of 15%. The parameter ξ_p, which represents the frequency of price adjustment, is set at 0.66, so that prices are fixed on average for three quarters. Finally, η, which is the elasticity of the marginal disutility of producing output with respect to an increase in output, is set at on the on the basis of data regarding labor costs.[15] Calibration is summarized in the following table.

Table 1 – Baseline calibration

$\beta = 0.99$	$\sigma^{-1} = 0.16$	$\varepsilon = 7.84$	$\eta = 0.47$	$\xi_p = 0.66$

Regarding the parameters characterizing heterogeneity (α, θ), we consider different calibrations. We assume a baseline value for α equal to 0.7, implying that 30% of households form

[15] See Rotemberg and Woodford (1997) for details.

their expectations using a mechanism of bounded rationality, whereas 70% are rational. We explore the effects of α for a range $\alpha \in [0.5, 1]$.[16] The model for $\alpha = 1$ clearly represents the homogeneous-rational agents standard New Keynesian model.

The baseline value for the adaption parameter θ is set equal to one. It implies that boundedly rational households recognize with one-period lag changes in inflation and the output gap (naive expectations). Again, we consider different specifications for it to test the robustness of our results. We report numerical simulation for a range between 0.8 and 1.2. It is worth noticing that different values of θ imply different "rule of thumb" for non-rational households in setting their expectations. Lower values of θ involve mean-reverting strategies according to which deviations from the average are expected to revert it—e.g. if inflation is below the average, increases in inflation are expected. Higher values of θ entail trend-following behaviors: deviations from the average are expected to be confirmed—e.g. when the current inflation trend is upward (current inflation is above the average), the expectation is that the inflation will continue to follow the trend.

4.2 Optimal monetary policy, expectations stabilization and determinacy

We begin our analysis by investigating the behavior of a model characterized by bounded rationality when the economy is perturbed by a stochastic disturbance. In Figure 1 we plot the impulse response functions for the inflation following a cost-push shock when a timeless perspective commitment is implemented. In particular, in the figure below is illustrated the dynamic response of the aggregate inflation joint with the expected inflaton for both the category of agents

[16] However, we explore the model properties and results robustness for the whole existence field of α. Results are available upon request.

(bounded and rational).

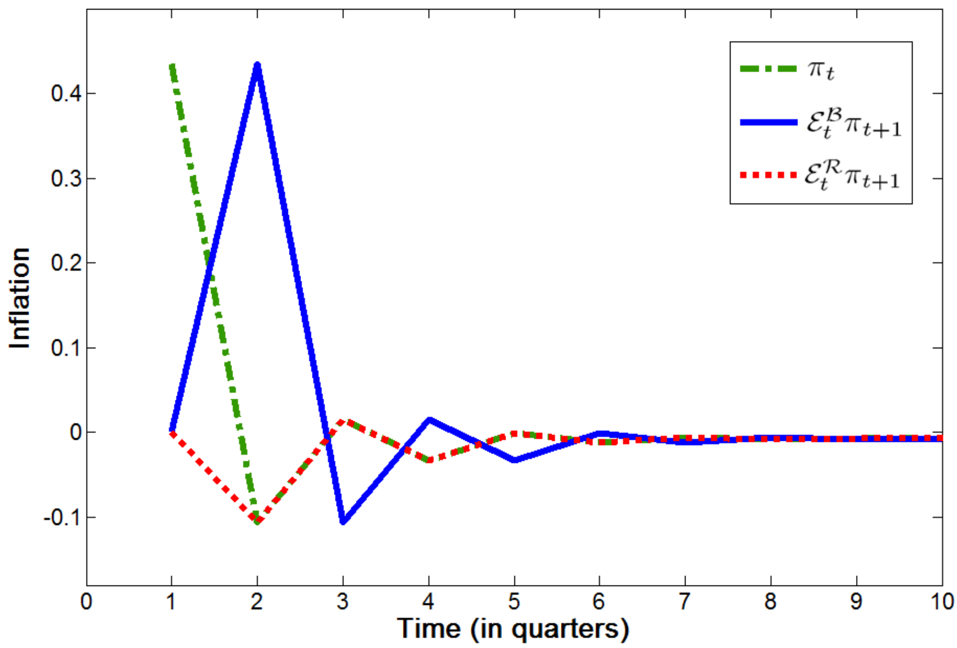

Figure 1 - Inflation IRF to a cost-push shock (commitment regime).

Differently from the standard case, where all the agents are perfectly rational (i.e. $\alpha = 1$), when a certain degree of heterogeneity is introduced, the optimal stabilization path of the inflation exhibits some fluctuations before returning to its steady state. To grasp the intuition about this dynamics, we consider the role of the expectations in a framework characterized by bounded rationality. Let assume that at time $t = 1$ a shock hits the economy: the policy maker rises the interest rate in order to compress the inflationary pressure, creating a small deflation. The fraction of boundedly rational agents form their expectations assuming that inflation is equal to its previous value: their belief is biased and, as a consequence, they are overestimating inflation, as it was brought down by the contractionary monetary policy pursued by the central banker. In the next period aggregate inflation exhibits a slight recover, but, again, inflation forecast of the non-rational agents is wrong, as they now are underestimating the inflation level. Thus, the policy maker needs to adjust the nominal interest rate up and down in order to stabilize inflation expectations. Obviously, except the first period, the inflation forecast of the rational agents is always correct. This is the rationale that underpins the dynamics depicted in Figure 1.

Figure 2 illustrates the impulse response functions for inflation and output gap both in commitment and in discretion. As before, we assume the economy is hit by a cost-push shock given by an i.i.d. process; IRFs are obtained under the model calibration reported in Table 1. We also plot the dynamic response of π_t and y_t for the standard model, where all the agents are rational, i.e. $\alpha = 1$ (dashed lines).

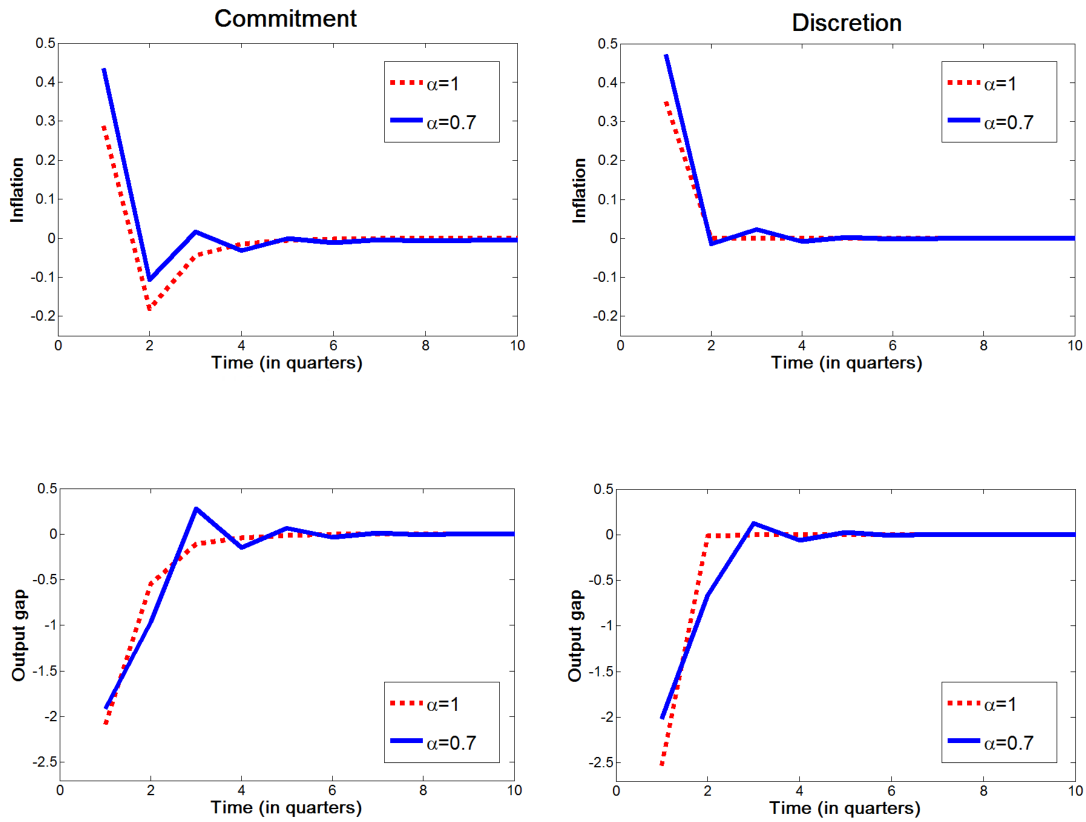

Figure 2 - Optimal stabilization policy for both regime (cost-push shock).

Because of the mechanism described previously, inflation and output gap exhibit a fluctuating dynamics in both regime, when some degree of bounded rationality is allowed. Moreover, for $\alpha = 0.7$, the output gap is more stabilized with respect to the standard fully rational case: this is due to the fact that price dispersion is affected by y_t (see (26)). Thus, reducing the variability of the output gap, the policy maker is able to exploit a better inflation-output gap trade-off.

The above results are robust for different calibrations about the adapation parameter and the share of boundedly rational agents. In a reasonable range, we obtain IRFs that are qualitatively similar to the ones represented in Figure 2.[17] Our additional simulations are also used to test the REE determinacy.

In our context we find that, when the monetary authority behaves following an optimal rule (both under commitment and discretion policy regime), the model is always determined, regardless the fraction of bounded agents. This result is in accordance with the findings of Clarida *et. al* (1999), who showed that implementing optimal policy is sufficient to guarantee the determinacy of the model.

It is worth noticing that in the same setup, Branch and McGough (2009) found that, under a standard Taylor rule, the determinacy region hinges both on the share of bounded agents and the adaption parameter.

4.3 Gains from commmitment and bounded rationality

Along this section we study how heterogeneous beliefs affect the additional losses, expressed in terms of welfare worsening, induced from a discretionary policy over a commitment. In Table 2 we report the welfare loss, both for discretion and commitment, and the percentage loss of the former regime over the latter.

Table 2 - Welfare loss and commitment gains

		α		
θ	Policy regime	0.5	0.7	0.9
1.2	commitment	211.255	131.051	80.729
	discretion	217.199	135.578	86.178
	% loss	2.814	3.455	6.750
1.0	commitment	184.459	115.936	74.796
	discretion	197.222	125.566	84.228
	% loss	6.919	8.306	12.611
0.8	commitment	140.832	92.200	65.719
	discretion	190.717	122.409	83.984
	% loss	35.421	32.765	27.792

[17] Further results are available upon request.

First, we can observe that, other things equal, a larger share of non-rational agents always involves higher welfare losses. As α decreases, we have the following effects: the economy becomes more persistent (see equation (17)) and, moreover, price dispersion goes up. As a consequence, inflation stabilization costs more.

Therefore, also the value of θ has effects on the welfare loss. A higher θ (i.e. extrapolative expectations) increases the degree of inflation persistence, magnifying the welfare losses.

As in the standard case, acting following a commitment rule always lead to stabilization welfare gains over a discretion. The proportion of boundedly rational individuals plays a key role in influencing the marginal gains of the commitment. As explained above, a larger share of non-rational agents makes the economy more persistent and induces more price dispersion. These two types of distortion have opposite effects on commitment marginal gains. On one hand, as illustrated by Steinsson (2003), more inertia entails more backward lookingness of the Phillips curve, reducing the efficacy of the commitment. Gains from a commitment are strictly related to the ability of a policy maker to manipulate private agents' expectations: this benefit tends to vanish as the backward looking component becomes predominant with respect to the forward one. On the other hand, commitment performs better relative to discretion as the price dispersion rises. For $\theta \geq 1$ the former effect dominates the latter and commitment gains reduces as the portion of bounded agents increases. Contrariwise, when forecast are formed according to an adaptive rule, i.e. $\theta < 1$, dispersion effect prevails over persistence effect and we find commitment gains as α decreases. This is due to the fact that under $\theta \geq 1$ inflation inertia is magnified, whereas when $\theta < 1$ the value attached to the lag component progressively drops.

Finally, we consider what is the effect of θ, for a given α, on the marginal gains from commitment. In this case gains are linear, as the smaller is θ the higher the gains are. The explanation is that a smaller θ reduces the degree of backward lookingness, allowing the policy maker to exloit a better manipulation of the expectations.

4.4 Optimal policies vs. Taylor rules

This section investigates the welfare gains deriving by the implementation of an optimal policy with respect to a standard Taylor rule. Specifically, in the canonical model, a simple Taylor rule, responding only to inflation according to the Taylor principle, leads to similar dynamics compared to optimal discretionary policies. We want to check the robusteness of this result by looking at how the gains from discretion on the Taylor rule are affected by the degree of bounded rationality.

Formally, under the calibration reported in Table 1, we consider two model specifications which differ only in how the interest rate is set: in one case the policy maker acts under discretion, whereas in the other one, the central bank adjusts the nominal interest rate according to the following Taylor rule:

$$i_t = \phi_\pi \pi_t \tag{28}$$

where $\phi_\pi = 1.5$. In the table below we provide the results of our numerical simulations, for several values of α and θ.

Table 3 - Optimal policy vs. Taylor rule

		$1-\alpha$			
θ	Policy regime	0	0.1	0.3	0.5
1.2	Discretion		86.178	135.578	217.199
	Taylor rule		142.249	302.436	475.872
	% gain		65.065	123.071	119.095
1.0	Discretion	70.085	84.228	125.566	197.222
	Taylor rule	73.428	115.712	217.444	343.414
	% gain	4.769	37.379	73.112	74.025
0.8	Discretion		83.984	122.409	190.717
	Taylor rule		100.139	170.647	268.804
	% gain		19.236	39.407	40.944

Table 3 shows that, in a standard framework where all the agents are rational, i.e. $1-\alpha = 0$, the Taylor rule is suboptimal, but, at the same time, discretion gains are small (about 4.5%). Note that these gain are independent of θ. As we introduce heterogeneous beliefs, we observe that the welfare losses associated with a Taylor rule dramatically grow up and the marginal gains from a discretion are very large. They reach higher costs between 40% and 120% when $\alpha \leq 0.7$, depending on θ, and between 20% and 65% when α is about 0.9.

Our results indicate that, although the costs of a Taylor rule compared to optimal inflation targeting are small when the standard case is considered, they become huge once that some degrees of bounded rationality are allowed. It is worth noting that the increasing costs of the Taylor rules are due to the fact that they cannot stabilize the individual beliefs variability, so they are associated to increasing costs in terms of inequality. Moreover, considering the canonical trade-off between price and aggregate consumption stabilization, as explained, the weight in the Taylor rule should decrease in α to mimic optimal policies.

In Figure 3, we plot the marginal gains, expressed in percent terms, from a discretion over a Taylor rule. According to Table 3, they are very close to zero when all the agents are rational; as the share of non-rational agents increases, acting following a simple Taylor rule induces very high marginal losses. Clearly, these losses are much more pronounced when a commitment is considered.

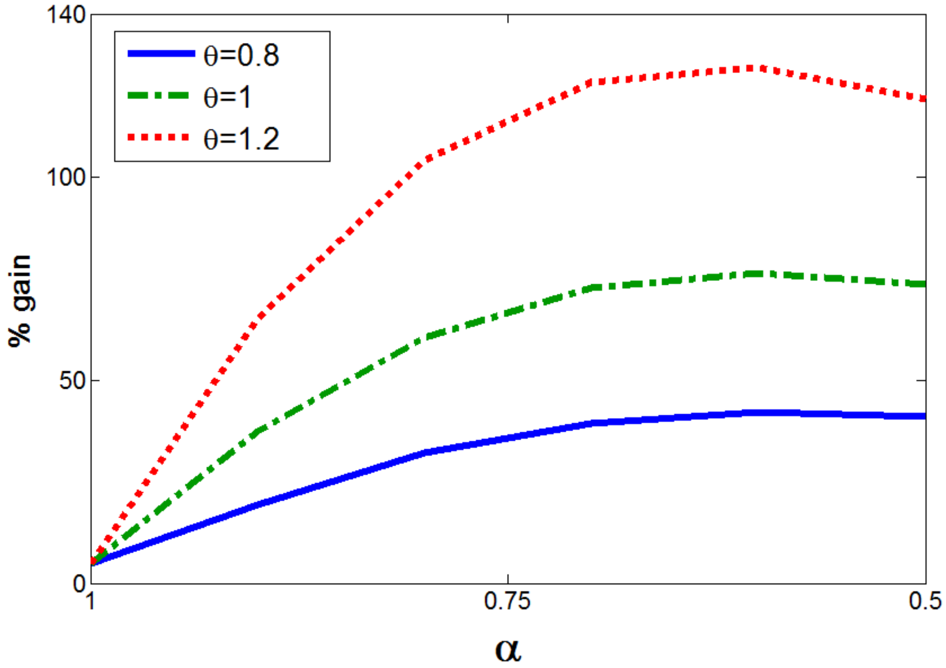

Figure 3 - Discretion gains over a Taylor rule.

5 Conclusions

Starting from the observation that it is well documented that people form their expectations according to different mechanisms, this paper studied the impact of heterogeneous expectations on optimal monetary policies in a New Keynesian framework. The point of departure of our work is Branch and McGough (2009), who introduced bounded rationality in a small-scale New Keynesian DSGE model and provide determinacy analysis. We extended their framework by deriving a welfare-based criterion consistent with the hypothesis of heterogeneous beliefs. Then we use it to investigate what are the implications of heterogeneity of optimal monetary policy design. We also tested the robustness of canonical inflation targeting policies under both discretion and commitment regimes and the REE determinacy when some agents exhibit bounded rationality.

Our simulations clearly showed as welfare losses depend on the share of bounded agents: as their share increases welfare deteriorates, as a result of two combined effects. On the one side, the economy becomes more persistent; on the other side, price dispersion increases, inducing further distortions.

Regarding policy implications, the assumption that some agents form their expectations using an adaptive or extrapolative mechanisms implies that the central bank faces a three-dimension problem: minimize the variability of aggregate consumption, minimize the cost associated to price dispersion and minimize the economy inequality (cross-sectional variance of consumption). The latter requires to stabilize expectation variability. As long as different individual consumption plans depend on different expectation paths, in fact, the variability in individual consumption is reduced by minimizing the cross-sectional variability of expectations. Moreover, the traditional trade-off between the stabilization of aggregate consumption and price dispersion is also affected by heterogeneity: the cost of price dispersion increases with the size of the group of boundedly rational agents, but the emphasis of stabilizing inflation versus output declines as long as heterogeneity increases, due to the more complex dynamics of price dispersion that here also depends on output stabilization.

Differently from recent literature based on Taylor rules in the same context, we found that optimal policies are always associated to REE determinacy under both discretion and commitment. Comparing the regime-welfare performances, we showed, as expected, that commitment always guarantees lower welfare losses than discretion. Following Steinsson (2003), we also investigate the relative gains of commitment over discretion by considering different degrees of heterogeneity. The relative gains can be associated to two different effects: *i)* Commitment gains are higher when the forward-looking component of the Phillips curve progressively enhances compared to the backward one (persistence effect); *ii)* Commitment gains are higher when price dispersion is more costly (dispersion effect). In our framework, an increasing in the share of boundedly rational subjects has an ambiguous effect on the relative gains of commitment since, on the one hand, it reduces the lead component of the Phillips curve, but, on the other hand, it raises the price dispersion. Commitment gains are more likely to be observed when agents form their expectations using an adaptive mechanisms rather than extrapolative. Note that extrapolative expectations magnify the persistence effect.

Finally, we highlighted the importance of pursuing an optimal stabilization policy rather than following simple interest rule. We showed that in a world characterized by a fraction of non-rational agents, the costs of neglecting an optimal policy rule hardly rise, leading to significative welfare losses. Here a Taylor rule is not able to mimic the optimal policy design, because it over-react to inflation and does not stabilize individual expectations. The former involves suboptimal choices in the trade-off between aggregate consumption and price dispersion stabilization; the latter leads to costs in terms of consumption inequality across individuals.

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Appendix – Price dispersion derivation

This appendix provides the derivation of the price dispersion for Branch and McGough's (2009) model. The discounted sum of the price dispersion evolves as follows:

$$\begin{aligned}\textstyle\sum_{t=0}^{\infty}\beta^t\Delta_t = \frac{1}{1-\beta\xi_p}\textstyle\sum_{t=0}^{\infty}\beta^t \Bigg[& \frac{\xi_p\left(\xi_p+\alpha-\alpha\xi_p\right)}{\alpha\left(1-\xi_p\right)}\pi_t^2+\frac{(1-\alpha)\,\xi_p^2}{\alpha\left(1-\xi_p\right)}\beta^2\theta^4\pi_{t-1}^2+ \\ & +\frac{\left(1-\xi_p\right)(1-\alpha)\left(1-\xi_p\beta\right)^2}{\alpha}\left(\psi_a^2 c_t^{\mathcal{B}^2}+\psi_b^2 y_t^2+2\psi_a\psi_b c_t^{\mathcal{B}} y_t\right)+ \\ & +\frac{2(1-\alpha)\left(1-\xi_p\beta\right)\xi_p}{\alpha}\beta\theta^2\pi_{t-1}\left(\psi_a c_t^{\mathcal{B}}+\psi_b y_t\right)-\frac{2(1-\alpha)\,\xi_p^2}{\alpha\left(1-\xi_p\right)}\beta\theta^2\pi_t\pi_{t-1}+ \\ & -\frac{2\xi_p(1-\alpha)\left(1-\xi_p\beta\right)}{\alpha}\pi_t\left(\psi_a c_t^{\mathcal{B}}+\psi_b y_t\right)\Bigg] \end{aligned} \tag{29}$$

where $\psi_a = \frac{\sigma^{-1}}{1+\eta\varepsilon}$ and $\psi_b = \frac{\eta}{1+\eta\varepsilon}$. Considering that $\kappa = \frac{\left(1-\xi_p\right)\left(1-\beta\xi_p\right)\left(\eta+\sigma^{-1}\right)}{\xi_p(1+\varepsilon\eta)}$, the undiscounted price dispersion can be alternatively written, after some simple algebraic manipulation, as

$$var_i\left(\log p_t(i)\right) = \frac{\xi_p}{\left(1-\beta\xi_p\right)\left(1-\xi_p\right)}\pi_t^2 + \frac{(1-\alpha)\,\xi_p^2\left[\pi_t - \beta\theta^2\pi_{t-1} - \kappa\left(\frac{c_t^{\mathcal{B}}+\eta\sigma y_t}{1+\eta\sigma}\right)\right]^2}{\alpha\left(1-\beta\xi_p\right)\left(1-\xi_p\right)} \tag{30}$$

The above expression is derived by defining $P_t \equiv E_i \log p_t(i)$ and $\Delta_t = var_i\left(\log p_t(i)\right)$. In our context the aggregate price level evolves as:

$$P_t = \left(1-\xi_p\right)(1-\alpha)\log p_t^{\mathcal{B}} + (1-\xi_p)\alpha\log p_t^{\mathcal{R}} + \xi_p \log p_{t-1}(i) \tag{31}$$

Subtracting from both sides P_{t-1}:

$$\pi_t = \left(1-\xi_p\right)(1-\alpha)\left(\log p_t^{\mathcal{B}} - P_{t-1}\right) + (1-\xi_p)\alpha\left(\log p_t^{\mathcal{R}} - P_{t-1}\right) \tag{32}$$

We can express (32) as:

$$\pi_t = \left(1-\xi_p\right)\left(P_t^* - P_{t-1}\right) \tag{33}$$

since $P_t^* = (1-\alpha)\log p_t^{\mathcal{B}} + \alpha\log p_t^{\mathcal{R}}$.

Now we compute the variance:

$$\begin{aligned} \Delta_t &= var_i\left[\log p_t(i) - P_{t-1}\right] \\ &= E_i\left[\left(\log p_t(i) - P_{t-1}\right)^2\right] - \left[E_i \log p_t(i) - P_{t-1}\right]^2 \end{aligned} \tag{34}$$

The first term of the right hand side can be rewritten as:

$$\xi_p E_i\left[\left(\log p_{t-1}(i) - P_{t-1}\right)^2\right] + \left(1-\xi_p\right)(1-\alpha)\left(\log p_t^{\mathcal{B}} - P_{t-1}\right)^2 + (1-\xi_p)\alpha\left(\log p_t^{\mathcal{R}} - P_{t-1}\right)^2 \tag{35}$$

The expectation of the boundedly rational agents about the future aggregate price level is:

$$\mathcal{E}_t^{\mathcal{B}} \log p_{t+1} = \left(1-\xi_p\right)\mathcal{E}_t^{\mathcal{B}} \log p_{t+1}^{\mathcal{B}} + \xi_p P_t \tag{36}$$

Adding and subtracting $\left(1-\xi_p\right)P_{t+1}$ to (36) and exploiting the properties of the expectation operator of the non-rational agents we obtain:

$$\frac{\xi_p}{1-\xi_p}\theta^2\pi_{t-1} = \mathcal{E}_t^{\mathcal{B}}\left(\log p_{t+1}^{\mathcal{B}} - P_{t+1}\right) \tag{37}$$

From the model we know that the optimal pricing rule of the bounded agents is:

$$\log p_t^{\mathcal{B}} = \log p_t + \xi_p\beta\theta^2\pi_{t-1} + \left(1-\xi_p\beta\right)\left[\psi_a c_t^{\mathcal{B}} + \psi_b y_t\right] + \xi_p\beta\mathcal{E}_t^{\mathcal{B}}\left(\log p_{t+1}^{\mathcal{B}} - P_{t+1}\right) \tag{38}$$

Substituting (37) in (38):

$$\log p_t^{\mathcal{B}} - P_{t-1} = \pi_t + \frac{\xi_p\beta\theta^2}{1-\xi_p}\pi_{t-1} + \left(1-\xi_p\beta\right)\left[\psi_a c_t^{\mathcal{B}} + \psi_b y_t\right] \tag{39}$$

Price for the rational agents is equal to:

$$\log p_t^{\mathcal{R}} = \frac{P_t^* - (1-\alpha)\log p_t^{\mathcal{B}}}{\alpha} \tag{40}$$

Mixing (39) and (40) and adding and subtracting P_{t-1}:

$$\log p_t^{\mathcal{R}} - P_{t-1} = \frac{1}{\alpha\left(1-\xi_p\right)}\pi_t - \frac{1-\alpha}{\alpha}\left\{\pi_t + \frac{\xi_p\beta\theta^2}{1-\xi_p}\pi_{t-1} + \left(1-\xi_p\beta\right)\left[\psi_a c_t^{\mathcal{B}} + \psi_b y_t\right]\right\} \tag{41}$$

From the combination of (35), (39) and (41), we can rewrite (34) as:

$$\begin{aligned}\Delta_t \quad = \quad & \xi_p\Delta_{t-1} + \left(1-\xi_p\right)(1-\alpha)\left[\pi_t + \frac{\xi_p\beta\theta^2}{1-\xi_p}\pi_{t-1} + \left(1-\xi_p\beta\right)\left[\psi_a c_t^{\mathcal{B}} + \psi_b y_t\right]\right]^2 + \\ & +(1-\xi_p)\alpha\left[\frac{1}{\alpha\left(1-\xi_p\right)}\pi_t - \frac{1-\alpha}{\alpha}\left\{\pi_t + \frac{\xi_p\beta\theta^2}{1-\xi_p}\pi_{t-1} + \left(1-\xi_p\beta\right)\left[\psi_a c_t^{\mathcal{B}} + \psi_b y_t\right]\right\}\right]^2 - \pi_t^2\end{aligned}$$

Finally, after some tedious algebra, we get:

$$
\begin{aligned}
\Delta_t = {} & \xi_p \Delta_{t-1} + \frac{\xi_p \left(\xi_p + \alpha - \alpha \xi_p\right)}{\alpha \left(1-\xi_p\right)} \pi_t^2 + \frac{(1-\alpha)\, \xi_p^2 \beta^2 \theta^4}{\alpha \left(1-\xi_p\right)} \pi_{t-1}^2 + \\
& + \frac{\left(1-\xi_p\right)(1-\alpha)\left(1-\xi_p \beta\right)^2}{\alpha} \left(\psi_a^2 c_t^{\mathcal{B}^2} + \psi_b^2 y_t^2 + 2 \psi_a \psi_b c_t^{\mathcal{B}} y_t\right) + \\
& + \frac{2(1-\alpha)\left(1-\xi_p \beta\right) \xi_p \beta \theta^2}{\alpha} \pi_{t-1} \left(\psi_a c_t^{\mathcal{B}} + \psi_b y_t\right) + \\
& - \frac{2(1-\alpha)\, \xi_p^2 \beta \theta^2}{\alpha \left(1-\xi_p\right)} \pi_t \pi_{t-1} - \frac{2 \xi_p (1-\alpha)\left(1-\xi_p \beta\right)}{\alpha} \pi_t \left(\psi_a c_t^{\mathcal{B}} + \psi_b y_t\right)
\end{aligned} \tag{42}
$$

In the case of $\alpha = 0$ (standard homogenous agents model) it encompasses Calvo price dispersion. Iterating (42) forward, the degree of price dispersion in any period $t \geq 0$ is given by:

$$
\begin{aligned}
\Delta_t = {} & \xi_p^{t+1} \Delta_{-1} + \sum_{s=0}^{t} \xi_p^{t-s} \left[\frac{\xi_p \left(\xi_p + \alpha - \alpha \xi_p\right)}{\alpha \left(1-\xi_p\right)} \pi_s^2 + \frac{(1-\alpha)\, \xi_p^2 \beta^2 \theta^4}{\alpha \left(1-\xi_p\right)} \pi_{s-1}^2 + \right. \\
& + \frac{\left(1-\xi_p\right)(1-\alpha)\left(1-\xi_p \beta\right)^2}{\alpha} \left(\psi_a^2 c_s^{\mathcal{B}^2} + \psi_b^2 y_s^2 + 2 \psi_a \psi_b c_s^{\mathcal{B}} y_s\right) + \\
& + \frac{2(1-\alpha)\left(1-\xi_p \beta\right) \xi_p \beta \theta^2}{\alpha} \pi_{s-1} \left(\psi_a c_s^{\mathcal{B}} + \psi_b y_s\right) + \\
& \left. - \frac{2(1-\alpha)\, \xi_p^2 \beta \theta^2}{\alpha \left(1-\xi_p\right)} \pi_s \pi_{s-1} - \frac{2 \xi_p (1-\alpha)\left(1-\xi_p \beta\right)}{\alpha} \pi_s \left(\psi_a c_s^{\mathcal{B}} + \psi_b y_s\right) \right]
\end{aligned} \tag{43}
$$

We now discount over all periods $t \geq 0$, getting (29).