Monthly US Business Cycle Indicators: A New Multivariate Approach Based on a Band–Pass Filter

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Abstract

This article proposes a new multivariate method to construct business cycle indicators. The method is based on a decomposition into trend–cycle and irregular. To derive the cycle, a band–pass filter is applied to the estimated trend–cycle. The whole procedure is fully model–based. Using a set of monthly and quarterly US time series, two monthly business cycle indicators are obtained for the US. They are represented by the cycles of real GDP and the industrial production index. Both indicators can reproduce previous recessions very well. Series contributing to the construction of both indicators are allowed to be leading, lagging or coincident relative to the business cycle. Their behavior is assessed by means of spectral concepts after cycle estimation. The proposed method can serve as an attractive tool for policy making, in particular due to its good forecasting performance and quite simple setting without involving elaborate mechanisms that account for, e.g., volatility changes.

JEL Classification: E32, E37, C18, C32

Keywords: Business cycle, multivariate structural time series model, univariate band–pass filter, forecasts, phase angle

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1 Introduction

Economic policy is a subject which often sparks off an active public debate. For example, policy makers pursuing stabilization policy are expected to take appropriate actions to stimulate the economy if it is on the brink of a crisis, or to prevent the overheating of the economy if an expansion is likely to take place. However, such measures are risky since wrong decisions entail high costs for the society. It is therefore all the more important to have reliable information in the decision making process. Moreover, the decisions must be often made early enough and thus under uncertainty about the future state of the economy. Information available at high frequencies can thus prove helpful in revealing the stage of the business cycle. The aim of this article is to develop a methodology that can both provide reliable information on the course of the economy and reduce the lag in the recognition of its future state.

To identify the course of the economy on the basis of macroeconomic data, a clear signal supposed to represent the business cycle has to be extracted. For that purpose, it is necessary to separate out long–term movements and noisy elements from the data. The question as to how to accomplish this constitutes the central question of business cycle analysis and has been investigated since the seminal work by Burns and Mitchell (1946). They for the first time gave a more narrow definition of business cycles as fluctuations in the economic activity that last between 1.5 and 8 years. The following research attempted to construct business cycle indicators characterized by these periodicities. Some studies focus on univariate approaches, like the filters proposed by Hodrick and Prescott (1997) and Baxter and King (1999) that have become very popular mostly because of their relatively simple implementation.

Among the univariate approaches, an alternative to these ad hoc filtering methods are the unobserved components models that take the stochastic properties of the data into account. As regards this signal extraction approach, two tendencies have emerged in the literature. One direction corresponds to the structural time series models proposed by, e.g., Harvey (1989) or their generalized version allowing for smoother cycles (see Harvey and Trimbur, 2003). The other direction is determined by the ARIMA–model–based approach (see, e.g., Box et al., 1978) combined with the canonical decomposition (see Cleveland and Tiao, 1976).

Since in the univariate approach only one series, typically real GDP or industrial production, can serve as a basis for the construction of a business cycle indicator, the informational content of other macroeconomic time series cannot be exploited. In contrast, the multivariate framework takes the contribution of different time series into account. This advantage of a multivariate setting has been recognized by, e.g., Forni et al. (2000) who develop a Euro area business cycle indicator in a generalized dynamic–factor model using a large panel of macroeconomic indicators. The indicator of Valle e Azevedo et al. (2006) for the Euro area is designed with a structural model including a common cycle, and extracted using a moderate set of series. Creal et al. (2010) extend their approach by taking time–varying volatility into account and adopt this method for the US.

In this article, we propose another multivariate method which is also based on a structural time series model. However, because of the well–known difficulties in modeling cycles directly, a model consisting only on trend plus irregular is initially specified. In this model, the trend is assumed to capture transitory movements and to have a common slope. For this reason, it is more appropriately referred to as a trend–cycle. After estimating the trend–cycle, we apply to it a multivariate band–pass filter to estimate the cycle following the methodology proposed by Gómez (2001). In fact, the filter is designed for univariate series, but then it is extended to multivariate series using diagonal matrices. The whole procedure is fully model–based and is applied to the same set of 11 monthly and quarterly US time series as in Creal et al. (2010). The extracted cycles of real GDP and the industrial production index can act as two alternative monthly business cycle indicators.

The proposed approach exhibits very appealing properties. From the modeling point of view, it provides indicators of the economic activity which conform to the idea of the business cycle featuring periodicities between 1.5 and 8 years. Hence, one can be sure that these indicators are not contaminated with higher– or lower–frequency movements. In addition, the model is flexible since only a few restrictions are imposed, and yet quite simple in that it does not involve special constructs, like time–variant parameters, to capture specific behavior of the series components. The complexity of the proposed method is kept at a rather low level also due to the fact that a dataset with small or moderate number of series is sufficient in the implementation of the procedure. Moreover, the algorithms used for this method are able to deal with data recorded at different frequencies, and can handle missing values straightforwardly.

As regards the policy relevance of the methodology, it is shown that not only previous recessions can be spotted by the resulting business cycle indicators, but also future recessions can be very well predicted. As a reliable forecasting framework, this model can perform better than univariate methods and some elaborate multivariate models. Further, the indicator represented by the real GDP cycle and its predictions are given on a monthly basis even though real GDP itself is recorded quarterly. This leads to a more precise picture on the economic situation and makes it possible to detect changes in the economic course early. To summarize, with its quite simple setup, good forecasting performance and the ability to generate realtime forecasts not distorted by, e.g., highly volatile movements, this method proves to be a well–suited tool for policy makers.

As the information stemming from different time series helps to build the business cycle indicators, it may be of interest to know how these series are related to the business cycle. In contrast to the idea by Stock and Watson (1989), they are not constrained to be coincident indicators only. The behavior of the included series is examined by drawing on the concepts of the phase angle and the mean phase angle. These spectral measures allow for classifying the series as leading, lagging or coincident indicators as well as identifying procyclical or countercyclical patterns.

The remainder of the article is organized as follows. In Section 2, we present the multivariate monthly model. The model is then applied to the US data described in Section 3.1. The resulting business cycle indicators and the behavior of other indicators with respect to the business cycle are analyzed in Section 3.2. Section 3.3 focusses on the forecasting performance of the proposed approach. Section 4 concludes.

2 Multivariate Monthly Model

Given a multivariate monthly time series $\{y_t\}$, $t = 1, \ldots, n$ with $y_t = (y_{1t}, \ldots, y_{kt})'$, the decomposition of y_t is based on a trend plus noise model, i.e.

$$y_t = \mu_t + \epsilon_t, \tag{1}$$

where $\text{Var}(\epsilon_t) = D_\epsilon$ is a diagonal matrix. In the presence of a cycle, μ_t is not seen as a smooth trend but rather as a component containing cyclical movements too. Therefore, it will hereafter be referred to as the trend–cycle.

Alternatively, it would be possible to add a cycle component to model (1) to explicitly take cyclical movements into account. However, it is well known that cycles are not easy to model and that most of the time one ends up fixing some parameters in the cycle model to obtain sensible results (see, e.g., Valle e Azevedo et al., 2006). The difficulty of modeling cycles is also apparent in the univariate case when one starts with an ARIMA model fitted to the series and the models for the components are specified according to the canonical decomposition (see, e.g., Cleveland and Tiao, 1976). In this case, a model for the cycle cannot usually be found using traditional tools of ARIMA modeling, such as graphs or correlograms.

The approach proposed in this paper avoids these problems. It consists of applying a fixed band–pass filter to the trend–cycle component, μ_t in model (1), following the methodology proposed by Gómez (2001). The filter is designed to extract the business cycle fluctuations that correspond to the periods between 1.5 and 8 years. One might wonder why a two–step approach is utilized here instead of simply applying the filter to the original series. The advantage of the two–step framework is that it allows for a complete decomposition of the series into trend, cycle and irregular. The procedure is fully model–based and will be described in the following subsections.

2.1 Model for the Trend–Cycle Component

The trend–cycle component μ_t follows the model

$$\begin{aligned} \mu_{t+1} &= \mu_t + K\beta_t + \zeta_t \\ \beta_{t+1} &= \beta_t + \eta_t, \end{aligned} \tag{2}$$

where $K\beta_t$ denotes the slope of μ_t, and $\mathrm{Var}(\zeta_t) = D_\zeta$ and $\mathrm{Var}(\eta_t) = D_\eta$ are diagonal matrices. Moreover, by assuming $K = [1, b_{21}, ..., b_{K1}]$, $\nabla\mu_{t+1} = \mu_{t+1} - \mu_t$ is allowed to be driven by one common slope, β_t. This common slope acts as a common factor in a common factor model. The so–called multivariate local linear trend model (2) has been proposed by Harvey (1989, p. 452).

The rationale for imposing a common slope in model (2) is motivated by the assumption that the different elements of the series $\{y_t\}$ have the same or a similar cyclical behavior. The intuition behind the assumption of a common slope is, however, not as straightforward as that of a common cycle. The common slope, β_t, should not be mistakenly seen here as a substitute for a common cycle. Since the cycle is not explicitly modeled in eq. (1), an assumption of a common slope helps to account for a similar short–term evolution of the different elements of the series $\{y_t\}$. This can be illustrated by considering the first difference of the series

$$\nabla y_t = K\beta_{t-1} + \eta_{t-1} + \nabla \epsilon_t$$

As in most macroeconomic applications the time series are expressed in logs, ∇y_t corresponds in such cases to the growth rate of y_t. The growth rate of a series is thereby related to the notion of the business cycle inasmuch as negative growth rates accompany economic downturns whereas positive growth rates occur during boom phases. Since $\eta_{t-1} + \nabla \epsilon_t$ is stationary, it becomes apparent that the growth rate of y_t is strongly affected by the common slope, β_t.

To estimate the trend–cycle μ_t, model (1) along with the trend–cycle specification (2) can be first put into the state space form as described in Appendix A.1. Then, the Kalman filter is applied to this state space form to estimate the the unknown parameters of the state space model. Finally, the Kalman smoother yields the estimate of μ_t. Details on these filtering and smoothing methodologies are given in Appendix B.

The estimated trend–cycle is used in the second step for cycle estimation. The whole procedure is model–based, meaning that, first, the model for the trend–cycle serves as a basis to derive the models for the trend and the cycle. Second, the parameters of the trend–cycle model estimated in the first step are used in the estimation of the cycle. As will be seen in the next subsection, we draw on the reduced–form model of the trend–cycle in the derivation of the models for the trend and cycle components. A starting point to arrive at the reduced–form is the following equation derived from model (2):

$$\nabla^2 \mu_{t+1} = K\eta_{t-1} + \nabla \zeta_t$$

Taking into account that for any square matrix M, its square root is defined as any matrix $M^{1/2}$ satisfying $M^{1/2}M^{1/2'} = M$, we let $\eta_t = D_\eta^{1/2} u_t^\eta$ and $\zeta_t = D_\zeta^{1/2} u_t^\zeta$. Then, the previous equation can be rewritten as

$$\begin{aligned}\nabla^2\mu_t &= K\eta_{t-2} + (\zeta_{t-1} - \zeta_{t-2}) \\ &= KD_\eta^{1/2}u_{t-1}^\eta + D_\zeta^{1/2}u_t^\zeta - D_\zeta^{1/2}u_{t-1}^\zeta,\end{aligned}$$

where $\mathrm{Var}([u_t^{\zeta'}, u_t^{\eta'}]') = I$. Thus, by defining $v_t = [u_t^{\zeta'}, u_t^{\eta'}]'$, the following reduced–form model for μ_t can be obtained:

$$\begin{aligned}\nabla^2\mu_t &= C_0 v_t + C_1 v_{t-1} \\ &= C(B)v_t,\end{aligned} \tag{3}$$

where B is the backshift operator such that $Bv_t = v_{t-1}$, and $C(B) = C_0 + C_1 B$ is a matrix polynomial in B with

$$C_0 = \begin{bmatrix} D_\zeta^{1/2} & 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} -D_\zeta^{1/2} & KD_\eta^{1/2} \end{bmatrix} \tag{4}$$

2.2 Cycle Estimation

In order to extract the cycle, a fixed band–pass filter is applied to the estimated trend–cycle component, μ_t. The filter is in this article referred to as the multivariate filter but its use amounts to the application of the same univariate filter to each individual trend–cycle component, μ_{lt}, $l = 1, ...k$. We design a two–sided version of a univariate band–pass Butterworth filter based on the tangent function using the specification parameters δ_1, δ_2, x_{p1}, x_{p2}, x_{s1} and x_{s2} (see Gómez, 2001). The values of these parameters determine the shape of the gain function of the filter, $G(x)$, where x denotes the angular frequency. To be more specific, it holds that $1 - \delta_1 < G(x) \leq 1$ for $x \in [x_{p1}, x_{p2}]$ and $0 \leq G(x) < \delta_2$ for $x \in [0, x_{s1}]$ and $x \in [x_{s2}, \pi]$.

It is possible and convenient to first design a low–pass filter and then, by means of a transformation, to derive from it its band–pass version (see Oppenheim and Schafer, 1989, pp. 430–434). While designing the low–pass filter, we let $x_p = x_{p2} - x_{p1}$ and $x_s = x_{s2} - x_{p1}$ so that the gain function of the low–pass filter, $G_{lp}(x)$, satisfies $1 - \delta_1 < G_{lp}(x) \leq 1$ for $x \in [0, x_p]$ and $0 \leq G_{lp}(x) < \delta_2$ for $x \in [x_s, \pi]$. For such a choice of the parameters x_p and x_s, the appropriate transformation from a low–pass to a band–pass filter is $z = -s(s - \alpha)/(1 - \alpha s)$, where $\alpha = \cos((x_{p2} + x_{p1})/2)/\cos((x_{p2} - x_{p1})/2)$ and $-1 < \alpha < 1$.

It is shown in Gómez (2001) that the band–pass filters obtained from Butterworth filters based on the tangent function admit a model–based interpretation. According to this interpretation, the considered band–pass filter is the Wiener–Kolmogorov filter that estimates the signal in the signal plus noise model

$$z_t = s_t + n_t, \tag{5}$$

where the signal, s_t, follows the model

$$(1 - 2\alpha B + B^2)^d s_t = (1 - B^2)^d b_t \tag{6}$$

The parameters d, α and the quotient of the standard deviations of n_t and b_t, $\lambda = \sigma_n/\sigma_b$, depend on the specification parameters δ_1, δ_2, x_p, and x_s.[1] The reduced–form model for z_t in (5) is

$$(1 - 2\alpha B + B^2)^d z_t = \theta_z(B)a_t,$$

where $\theta_z(B)$ is of degree $2d$. Letting $\delta_z(B) = (1 - 2\alpha B + B^2)^d$, the Wiener–Kolmogorov filters to estimate s_t and n_t in (5) are

$$h_s = \frac{\sigma_b^2}{\sigma_a^2}\frac{(1-B^2)^d(1-F^2)^d}{\theta_z(B)\theta_z(F)}, \qquad h_n = \frac{\sigma_n^2}{\sigma_a^2}\frac{\delta_z(B)\delta_z(F)}{\theta_z(B)\theta_z(F)},$$

respectively, where F is the forward operator such that $Fz_t = z_{t+1}$, $\sigma_b^2 = \text{Var}(b_t)$, $\sigma_n^2 = \text{Var}(n_t)$ and $\sigma_a^2 = \text{Var}(a_t)$.[2]

The previous considerations allow for the integration of the fixed band–pass filter described earlier into a multivariate model–based approach. To show this, we first consider the pseudo covariance generating function (CGF) of μ_t. Denoted by f_μ, the CGF of μ_t can be decomposed as follows:

$$\begin{aligned} f_\mu &= h_s f_\mu + (1 - h_s) f_\mu \\ &= f_c + f_p, \end{aligned}$$

[1] The parameters d and λ can be computed using the low–pass version of the filter as explained in Gómez (2001, p. 372). It should be thereby taken into account that $\lambda = 1/\tan^d(x_c/2)$, where x_c is a frequency such that $G_{lp}(x_c) = 1/2$. For the filter used in this article, the values for the parameters in (6) are $d = 3$, $\alpha = 0.9921$ and $\lambda = 437.19$.

[2] The derivation of the polynomial $\theta_z(B)$ and the variance σ_a^2 is provided by Gómez (2001, p. 371). Without loss of generality, we set for the filter used in this article $\sigma_b^2 = 1$. Then, for this filter $\sigma_n = 437.19$ and $\sigma_a = 568.58$.

where $f_c = h_s f_\mu$ and $f_p = (1-h_s)f_\mu$. This decomposition defines the decomposition of μ_t into two orthogonal unobserved components, c_t and p_t, with CGFs f_c and f_p, respectively. Since the Wiener–Kolmogorov filter to estimate c_t in the model $\mu_t = c_t + p_t$ is the band–pass filter $h_s = f_c/f_\mu$, the subcomponent c_t is considered as the cycle, whereas the other subcomponent, p_t, represents the trend.

The models for c_t and p_t are obtained from their CGFs. Using the reduced–form model for μ_t in eq. (3), the CGF of c_t can be written as

$$\begin{aligned} f_c &= h_s f_\mu \\ &= \frac{1}{(1-B)^2}(C_0 + C_1 B)(C_0' + C_1' F)\frac{1}{(1-F)^2}\frac{\sigma_b^2}{\sigma_a^2}\frac{(1-B^2)^d(1-F^2)^d}{\theta_z(B)\theta_z(F)} \\ &= \frac{(1-B)^{d-2}(1+B)^d}{\theta_z(B)}(C_0 + C_1 B)\frac{\sigma_b^2}{\sigma_a^2}(C_0' + C_1' F)\frac{(1-F)^{d-2}(1+F)^d}{\theta_z(F)}, \end{aligned}$$

where C_0 and C_1 are as in (4). From this, it follows that the model for c_t is

$$\theta_z(B)c_t = (1-B)^{d-2}(1+B)^d \overline{C}(B)\overline{v}_t, \tag{7}$$

where $\overline{C}(B) = (\sigma_b/\sigma_a)C(B)$ and $\mathrm{Var}(\overline{v}_t) = I$. In a similar way, it can be shown that the model for p_t is

$$(1-B)^2\theta_z(B)p_t = \delta_z(B)\widetilde{C}(B)\widetilde{v}_t, \tag{8}$$

where $\widetilde{C}(B) = (\sigma_n/\sigma_a)C(B)$ and $\mathrm{Var}(\widetilde{v}_t) = I$.

Knowing the models for c_t and p_t, the cycle can be estimated using the state space framework. The state space model is set up by taking into account decomposition (1) and the decomposition of μ_t into c_t and p_t. Details on this state space representation are provided in Appendix A.2. The matrices of this state space form contain the parameters of the trend–cycle mode as well as the parameters of the band–pass filter. The former have been estimated as described in the previous subsection whereas the values of the filter parameters have been selected so as to extract the waves corresponding to business cycle frequencies. Therefore, the matrices of the state state representation of the total model do not have to be estimated. The covariance square root Kalman smoother applied to this state space model yields the estimated cycle.

3 Empirical Results

3.1 Data With Mixed Frequencies and Missing Values

In this section, the proposed methodology is used to construct US business cycle indicators on the basis of a set of US macroeconomic time series. To assess the performance of this method, the results in Creal et al. (2010) are considered as a benchmark. For notational convenience, we will use the acronym CKZ when referring to this study. To make the comparison as reliable as possible, the same dataset consisting of 11 seasonally adjusted time series from 1953.M4 to 2007.M9 is used (for details see Creal et al., 2010, p. 702). The monthly series are: the industrial production index (IPI), the unemployment rate, average weekly working hours in manufacturing, and two series from the retail sales category. One of them, retail sales, is discontinued in 2001.M4 whereas the other one, retail sales and food services, is observed between 1991.M1 and 2007.M9. The remaining series, i.e. real GDP, consumer price index inflation, consumption, investment, productivity of the non–farm business sector and hours of the non–farm business sector are available on a quarterly basis. All series except for the unemployment rate and inflation are in logs and multiplied by 100.

An important property of the dataset is the presence of missing values. This, however, poses no problem because the Kalman filter can easily handle missing observations. Another feature of the data is the different observation frequency. Even though the models presented in the previous section as well as their corresponding state space forms are formulated for monthly data, quarterly data can be accommodated in this framework in a straightforward manner.

It is to be noted that different time aggregation patterns apply depending on whether the variables are stocks, time–averaged stocks or flows. It would be possible to account for these different types of variables by incorporating the so–called cumulator variables (see Harvey, 1989, pp. 306–239). They are defined in terms of variables not being transformed so that the correct use of the cumulator variables in the case of series in logs would imply non–linear state space models. Proietti and Moauro (2006) offered an estimation and signal extraction approach for these models. If, instead, the definitions of the cumulator variables are assumed to hold also for series in logs as in Mariano and Murasawa (2003), this can lead to inaccuracies in the components estimates.

In this study, we proceed as Valle e Azevedo et al. (2006) and Creal et al. (2010). For simplicity, we disregard the different time aggregation schemes and treat quarterly data as monthly data with two missing observations added between two consecutive quarterly observations. In this way, non–linearities and larger model dimensions caused by the cumulator variables can be avoided. Since we are primarily interested in capturing the cycle dynamics, this simplification should not have a great impact on the objective of this study.

3.2 Business Cycle Indicators

Figure 1 depicts the business cycle indicators, the IPI and real GDP cycles, estimated in the multivariate framework.[3] It is apparent that the recessions implied by both cycles are in line with the recessions dated by NBER. The IPI cycle is undoubtedly much more volatile than the GDP cycle. Whereas the standard deviation of the GDP cycle is equal to 1.59, the corresponding value for the IPI case is 3.31, more than twice as high. However, both cycles show a very similar pattern. This observation can be also confirmed by their contemporary correlation of 0.945. The high degree of synchronization let them act as alternative recession indicators. The most remarkable deviation in values of each cycle within a single recession can be observed between 1973 and 1975. The strong fall from high positive to high negative values suggests the most severe downturn in the analyzed time span. A further, very sharp decline in the economic activity occurs in the early 1980s and is a result of two recessions separated by a peak in 1981, as is evident from Figure 1. Beside the dips classified by NBER as recessions, both cycles exhibit three smaller dips: the first one in the late 1960s, the second one between 1984 and 1987 and the third one in the mid–1990s. The IPI and GDP cycles are not only able to reproduce the previous US history of downturns, as is made clear by Figure 1, but they also nearly coincide with the respective cycles extracted by Creal et al. (2010).

Given the business cycle indicator, the remaining series can be classified as leading, lagging or coincident indices depending on how they are shifted relative to the business

[3]All computations have been performed with Matlab R2012b (64–bit) using the SSMMatlab toolbox by Gómez (2012) and procedures written by Víctor Gómez.

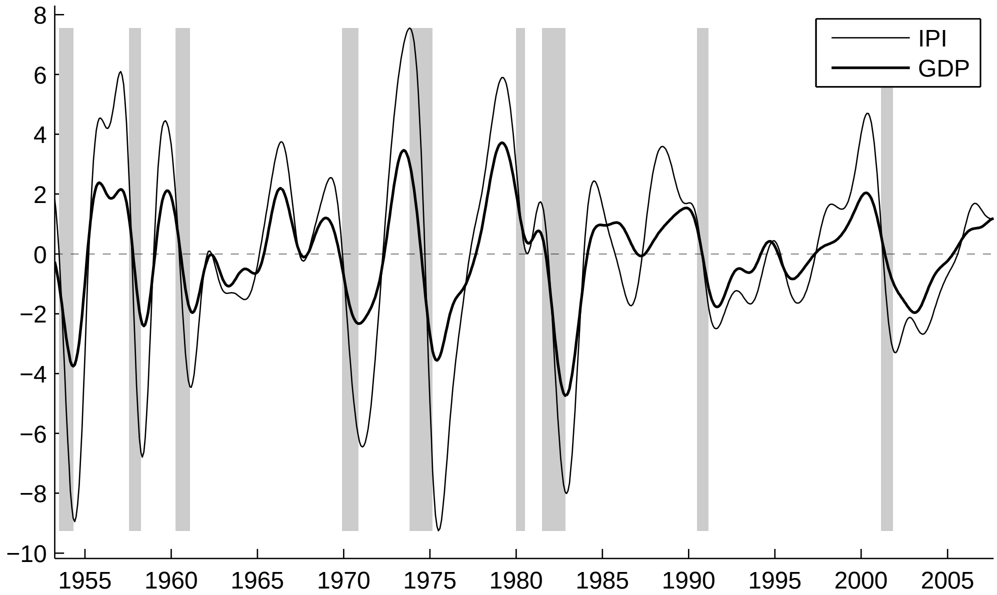


Figure 1: Cycles of the industrial production index (IPI) and real GDP as the business cycle indicators

Note: NBER recession dates are represented by the vertical bands.

cycle. If the cycle is explicitly modeled in a multivariate structural model, a possible way to identify the lead–lag pattern is to directly incorporate phase shifts into the model with a common cycle according to the approach of Rünstler (2004) that has been applied in Creal et al. (2010) and Valle e Azevedo et al. (2006). This has an advantage that the inherent feature of business cycle dynamics is accounted for in the generalized structural model in a consistent way. On the other hand, this procedure also increases the number of parameters to be estimated. In order to keep the model tractable, Valle e Azevedo et al. (2006) fixed the frequency of the common cycle to a specific value so that the inclusion of the shift parameter necessitates additional restrictions. The Bayesian approach for parameter estimation adopted by Creal et al. (2010) per se involves choosing prior distributions for the parameters. The classification procedure we follow in this article has the advantage that it does not increase the model complexity nor does it require certain assumptions. It relies on the concept of phase angle. This measure is well suited to establish the lead–lag relation of two time series as well as the direction (positive or negative) of their relationship. By means of the phase angle, the behavior of the particular cycle with respect to the business cycle can be examined.

If the value of the phase angle at the angular frequency ω, $\theta(\omega)$, lies between 0 and π, the particular series is said to lag the business cycle at ω. The opposite case is implied by $-\pi < \theta(\omega) < 0$. The particular series is defined as coincident at ω, if $\theta(\omega)$ equals zero. Moreover, values of the phase angle ranging between $(-\pi/2, \pi/2)$ point to a positive relation between the particular cycle and the business cycle (procyclical behavior/in–phase movement), whereas the values of $\theta(\omega)$ in the interval $[-\pi, -\pi/2)$ or $(\pi/2, \pi]$ indicate a negative relationship (countercyclical behavior/anti–phase movement) between them.[4]

Judgement of the overall behavior can be made based on the phase angle value with respect to a reference frequency. In the case of the CKZ model, it is the frequency of the common cycle. It corresponds to the largest mass of the spectrum of the common cycle and is thus the same for all series under consideration. In contrast, we allow the cycles to have different spectral densities. The natural counterpart of the reference frequency in the CKZ model therefore seems to be the frequency associated with the strongest relationship between the business cycle and the particular cycle. The strength of their frequency–by–frequency relationship is here measured using the concept of coherence. Though the lead–lag classification approach resting on the strongest coherence creates a link to the CKZ phase shift modeling, it can disregard possible countervailing patterns in the business cycle frequency interval. This problem becomes severe, especially if the spectrum or, in this case, the coherence displays more than one peak and contrasting patterns can be identified among some of them. To avoid this potential problem, it may be useful to analyze the overall behavior of the particular series by evaluating the mean phase angle in the whole business cycle frequency interval $[2\pi/96, 2\pi/18]$. For that purpose, we employ the concept of a mean appropriate for data measured on the angular scale (see Fisher and Lewis, 1983).

The results of the lead–lag analysis pertaining to the IPI cycle as a business cycle indicator can be found in Table 1. In addition to the single estimates of the phase angle based on the reference frequency, $\theta(\omega_h)$, and the mean phase angles $\bar{\theta}$, the respective confidence

[4]Note that the range of the phase angle is constrained to the interval $[-\pi, -\pi]$. The rationale for this common practice and a comprehensive discussion on the values of the phase angle are provided by Marczak and Beissinger (2012).

intervals are reported.[5] It is evident that manufacturing working hours, productivity and investment are leading the business cycle at the 5% significance level. According to the statistically significant negative value of the mean phase angle, consumption can be also classified as a leading indicator. Similar observation can be made for both series from the retail sales category. All these results confirm the CKZ findings. One of the few divergences relative to the CKZ results pertains to the unemployment rate. From the significance of the negative values of $\theta(\omega_h)$ and $\bar{\theta}$, it can be inferred that this series is leading the business cycle. However, the values of $\theta(\omega_h)$ and $\bar{\theta}$ are both very close to π, a value for which the unemployment rate could be characterized as leading or as lagging the business cycle. In fact, it can be observed that the unemployment rate increases before the business cycle reaches its peak, but it also rises after a trough in the economic activity. In the real GDP case, a coincident behavior cannot be ruled out whereas the CKZ findings suggest a leading behavior of real GDP instead. The remaining series, inflation and, as opposed to the CKZ results, hours in the non–farm business are lagging the business cycle at the 5% significance level.

As regards the movements with or against the business cycle, almost all indicators exhibit a statistically significant procyclical pattern. Only the unemployment rate is statistically significant countercyclical. It is worth noting that the similar cyclical behavior for both series, retail sales and retail sales with food services, is not a consequence of any restrictions.

Analogously to Table 1 related to the IPI cycle, Table 2 summarizes the results related to the GDP cycle as a business cycle indicator. It can be noticed that they do not qualitatively differ from the ones corresponding to the IPI cycle. Hence, both business cycle indicators can in this case serve as equivalent reference measures.

[5] The confidence bounds for the estimates of the phase angle and the mean phase angles have been constructed as described in Koopmans (1974, pp. 285–287) and Fisher and Lewis (1983), respectively. All computations for the lead–lag analysis have been performed with Matlab R2012b (64–bit) using the Spectran toolbox by Marczak and Gómez (2012).

Table 1: Leading, lagging and coincident indicators relative to the IPI cycle[a)]

IPI and	Period τ_h in years [b)]	$\theta(\omega_h)$	95% Conf. interval for $\theta(\omega_h)$		$\bar{\theta}$ [c)]	95% Conf. interval for $\bar{\theta}$	
Unemployment	3.41	−0.920	−0.975	−0.865	−0.929	−0.942	−0.916
Manufacturing	3.63	−0.170	−0.237	−0.103	−0.157	−0.173	−0.140
Inflation	5.45	0.329	0.243	0.415	0.166	0.108	0.224
Retail	4.54	−0.070	−0.180	0.040	−0.086	−0.131	−0.041
Retail/food	3.41	−0.050	−0.242	0.142	−0.061	−0.089	−0.033
Productivity	4.19	−0.388	−0.508	−0.267	−0.241	−0.287	−0.195
Real GDP	7.79	−0.030	−0.076	0.016	0.007	−0.014	0.028
Hours	3.03	0.096	0.050	0.141	0.111	0.096	0.126
Consumption	4.54	0.006	−0.109	0.121	−0.146	−0.213	−0.079
Investment	7.79	−0.151	−0.210	−0.092	−0.002	−0.028	0.025

[a)] Angular measures are expressed in terms of shares of π.

[b)] τ_h corresponds to the frequency ω_h at which the coherence between the business cycle indicator and the respective series attains the highest value.

[c)] $\bar{\theta}$ denotes the mean phase angle computed in the frequency interval $[2\pi/96, 2\pi/18]$.

3.3 Forecasting

3.3.1 Forecasts of the Recessions

Apart from providing stylized facts about the past and the current state of the economy, a method for extracting a business cycle indicator should perform well with respect to forecasting. Accurate forecasts of the economic activity in the near future are of a vital importance for economic policy. What is more, the timeliness of the forecasts also plays an essential role in the decision making process, as the information at a higher frequency, e.g. on a monthly basis, gives a more detailed picture on the future economic situation. This aspect has become a motivation for the recently growing literature on the so–called nowcasting dealing with real–time data (see, e.g., Giannone et al., 2008; Bańbura et al., 2012). From the computational point of view, a simple model is advantageous over an elaborate one since it is easier to understand, implement and adjust, and it possibly requires less restrictions. In this section, we show that the multivariate method proposed in this article embodies all these features of a good forecasting model as it is able to yield

Table 2: Leading, lagging and coincident indicators relative to the GDP cycle[a)]

Real GDP and	Period τ_h in years [b)]	$\theta(\omega_h)$	95% Conf. interval for $\theta(\omega_h)$		$\bar{\theta}$ [c)]	95% Conf. interval for $\bar{\theta}$	
Unemployment	4.54	−0.930	−0.986	−0.874	−0.931	−0.947	−0.915
Manufacturing	6.81	−0.180	−0.255	−0.105	−0.151	−0.167	−0.135
Inflation	6.06	0.308	0.191	0.425	0.241	0.197	0.286
Retail	2.27	−0.121	−0.205	−0.036	−0.080	−0.108	−0.052
Retail/food	1.56	−0.222	−0.438	−0.006	−0.054	−0.084	−0.025
Productivity	3.63	−0.293	−0.384	−0.201	−0.217	−0.249	−0.186
IPI	7.79	0.030	−0.016	0.076	−0.007	−0.028	0.014
Hours	3.41	0.131	0.083	0.178	0.110	0.098	0.123
Consumption	4.54	−0.009	−0.095	0.077	−0.066	−0.105	−0.028
Investment	5.45	−0.101	−0.128	−0.075	−0.010	−0.030	0.011

[a)] Angular measures are expressed in terms of shares of π.

[b)] τ_h corresponds to the frequency ω_h at which the coherence between the business cycle indicator and the respective series attains the highest value.

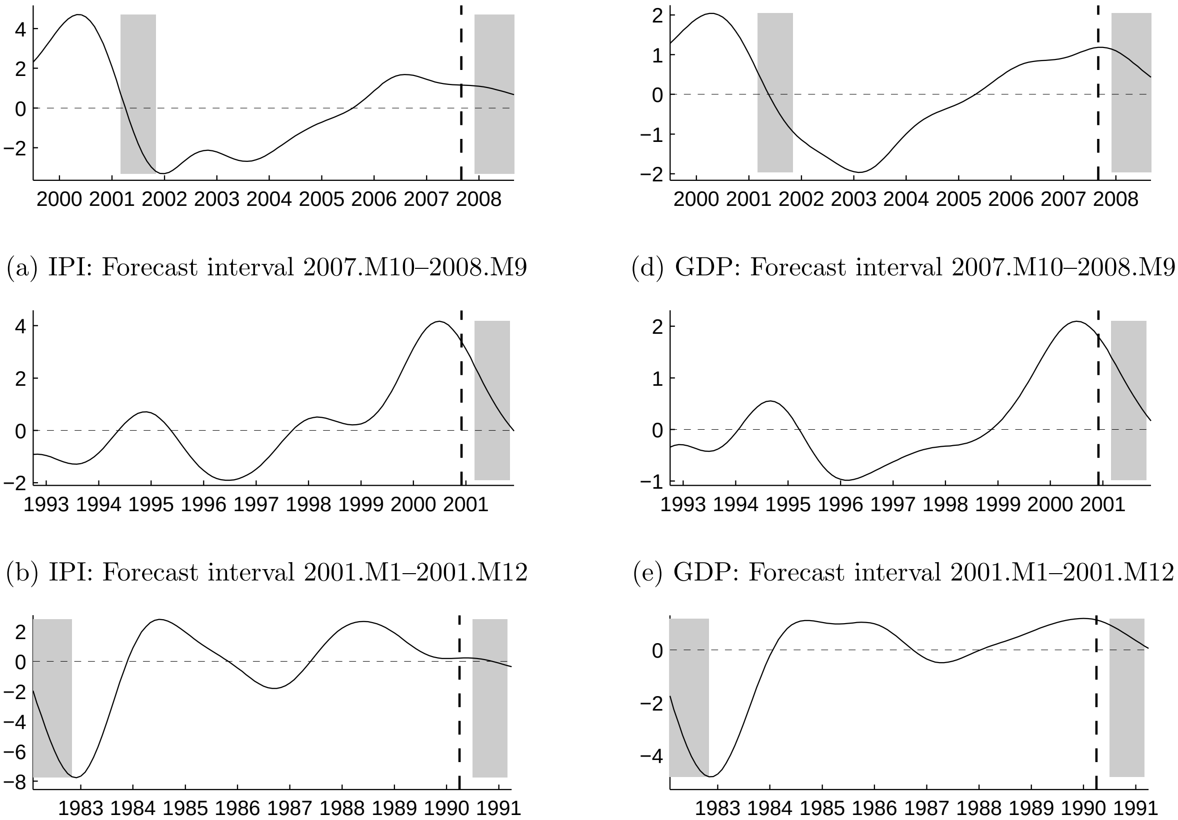
[c)] $\bar{\theta}$ denotes the mean phase angle computed in the frequency interval $[2\pi/96, 2\pi/18]$.

good realtime predictions in a relatively simple modeling framework.

To examine the performance of the presented approach, we first compute one–year forecasts of the IPI and GDP cycle based on the whole sample to check whether the forecasts can reproduce the last recession starting in 2007.M12. Further, the model is estimated with two shorter samples, until 2000.M12 and 1990.M4, respectively. In both cases we also calculate one–year forecasts for both business cycle indicators. In this way, the robustness of this methodology shall be investigated. Figure 2 depicts the smoothed IPI and GDP cycle estimates along with the respective forecasts in three intervals. The results make clear that the proposed method can predict the last three recessions very well.

3.3.2 Comparison with the Model with a Structural Volatility Break

Since the focus of this article lies on developing a reliable, albeit simple, model for the cycle extraction and forecasting, the model presented in Section 2 cannot explicitly take



(a) IPI: Forecast interval 2007.M10–2008.M9

(d) GDP: Forecast interval 2007.M10–2008.M9

(b) IPI: Forecast interval 2001.M1–2001.M12

(e) GDP: Forecast interval 2001.M1–2001.M12

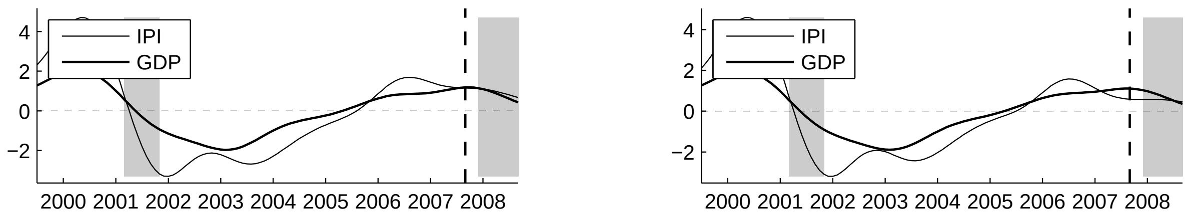
(c) IPI: Forecast interval 1990.M5–1991.M4

(f) GDP: Forecast interval 1990.M5–1991.M4

Figure 2: Smoothed cycle estimates and one–year forecasts for three time intervals

Notes: NBER recession dates are represented by the vertical bands.

into account any possible structural changes present in the data. Indeed, initiated by the studies of Kim and Nelson (1999) and McConnell and Pérez-Quirós (2000), the recent literature provides an evidence of a substantial reduction in the volatility of many macroeconomic time series in the US. There is no consensus whether the moderation has occurred in form of a break, as suggested by Stock and Watson (2002) (or maybe multiple breaks discussed by Sensier and van Dijk, 2004), or rather a gradual change in the volatility, as advocated by Blanchard and Simon (2001). Even though in this part of the study we try to address the issue of the volatility decline, we do not aim to contribute to the literature on the Great Moderation. We rather intend to find out whether accounting for this effect influences the forecast performance. For this reason, a single (one–time) volatility break is considered. we rely on the break time point in 1984.M1 initially detected for output growth by Kim and Nelson (1999) and McConnell and Pérez-Quirós (2000). This single volatility break is incorporated in the common slope and in the multivariate irregular component. We thereby follow the approach proposed by Tsay (1988). For the sake of comparison, Figure 3 presents the IPI and GDP cycles and their forecasts from the model with the volatility break and the base model. The differences between these results refer to the IPI case but are rather small, so that the specification without the volatility break seems to be even better in terms of forecasting than the more complex alternative. In contrast, the stochastic volatility specification in the CKZ model helps correctly predict the last recession.



(a) Base model

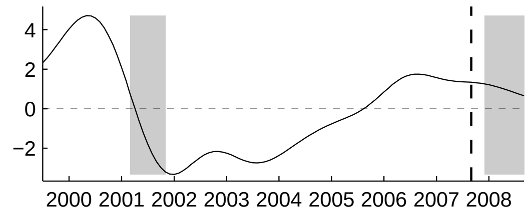
(b) Model with the volatility break

Figure 3: Smoothed cycle estimates and one–year forecasts from 2007.M10 onwards based on the base model and the model with the volatility break in 1984.M1, respectively

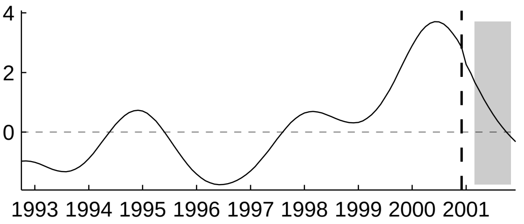
Note: NBER recession dates are represented by the vertical bands.

3.3.3 Comparison with the Univariate Model Based on a Band–Pass Filter

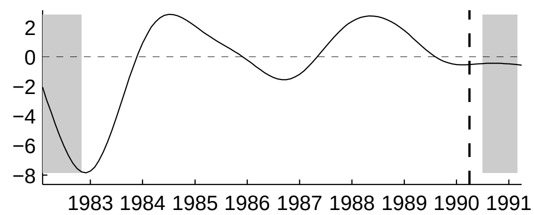
The obvious advantage of a multivariate model over an univariate approach is that it is capable of yielding monthly information on the GDP cycle. Forecasts of the economic situation based on real GDP are in this case more precise in terms of timing than quarterly forecasts resulting from an univariate model. Hence, they represent an alternative to forecasts based on the monthly IPI. The question arises whether, apart from realtime forecasts, the multivariate model presented in Section 2 can as well warrant an improvement in the forecasts quality over univariate methods. To examine this aspect, it seems natural to consider the univariate version of the proposed multivariate model. In so doing, it can be ensured that potential differences in the outcomes are not a consequence of fundamental differences in the modeling principles and thus in the resulting stochastic features. In particular, the univariate structural model with trend–cycle and irregular is estimated for the IPI and real GDP. In the second step, the univariate band–pass filter described in Section 2.2 is applied to the estimated trend–cycle. Similarly to the multivariate counterpart, the procedure is fully model–based. To facilitate the comparison of both approaches, the forecasts are investigated in the same time intervals as in the multivariate case: 2007.M10–2008.M9, 2001.M1–2001.M12 and 1990.M5–1991.M4. For real GDP, these forecasts intervals are translated to the corresponding quarters. The smoothed IPI and GDP cycles along with their forecasts obtained with the univariate model are depicted in Figure 4.



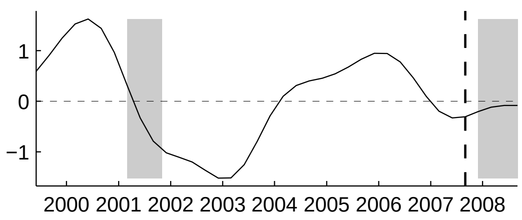
(a) IPI: Forecast interval 2007.M10–2008.M9



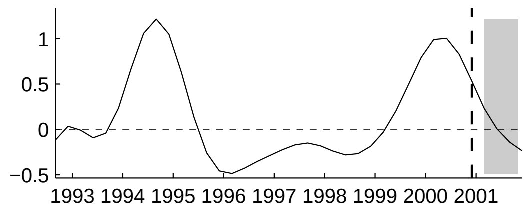
(b) IPI: Forecast interval 2001.M1–2001.M12



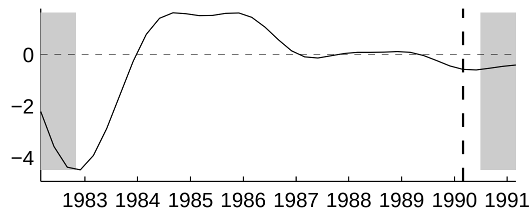
(c) IPI: Forecast interval 1990.M5–1991.M4



(d) GDP: Forecast interval 2007.Q4–2008.Q3



(e) GDP: Forecast interval 2001.Q1–2001.Q4



(f) GDP: Forecast interval 1990.Q2–1991.Q1

Figure 4: Smoothed cycle estimates based on the univariate model and one–year forecasts for three time intervals

Note: NBER recession dates are represented by the vertical bands.

As regards the IPI cycle (Figures 4a, 4b and 4c), the forecasts are almost identical with those resulting from the multivariate model (see Figures 2a, 2b and 2c). In the GDP case, on the other hand, the forecasts misleadingly suggest an expansion in the intervals 2007.Q4–2008.Q3 and 1990.Q2–1991.Q1 as can be seen in Figures 4d and 4f, respectively. This observation is consistent with the finding of Creal et al. (2010). They show that the univariate version of their model (without stochastic volatility) applied to real GDP is not capable of predicting the last recession. The preceding analysis leads to the conclusion that the multivariate model not only can produce forecasts at a frequency higher than the frequency of the data itself, but also offers a better framework for forecasting purposes than the univariate counterpart, at least for real GDP.

4 Conclusions

This article presents a new multivariate model used to construct monthly business cycle indicators for the US. This approach is based on a multivariate structural model and a univariate band–pass filter. It contributes to the literature on the business cycle analysis in several ways. The model allows for considering series observed at different frequencies. Therefore, advantage can be taken of the information contained in several monthly and quarterly macroeconomic indicators which are considered in this article. The two obtained business cycle indicators are, however, given on a monthly basis. They are represented by the cycles of the industrial production index (IPI) and real GDP, respectively. The indicators are smooth and thus consistent with the definition of a business cycle. Moreover, they can reproduce previous recessions very well. Another feature of the approach is its two–step framework as opposed to the single–step framework of the conventional multivariate structural time series models with explicitly modeled cycle. The proposed strategy avoids the problem of modeling the cycle directly so that it reduces the complexity of the structural model in the first step and facilitates its estimation.

The different series used in the proposed procedure are not restricted to be coincident. Their behavior in relation to the business cycle is, however, not explicitly modeled by extra parameters which would increase the complexity of the model. The relationship of other indicators with the real GDP or IPI cycle can still be analyzed after cycle estimation has been performed. For that purpose, the frequency–domain concepts of the phase angle and the mean phase angle are employed. The analysis reveals that the results are virtually the same for both reference cycles. Manufacturing working hours, productivity and retail sales are leading the business cycle at the 5% significance level. Inflation and hours in the non–farm business are statistically significant lagging indicators. For the unemployment rate, the results are somewhat ambiguous. Almost all of the indicators are statistically significant procyclical indicators, whereas the unemployment rate is statistically significant countercyclical.

The greatest strength of the presented approach lies in its forecasting performance. The ability to produce high quality forecasts provided at high frequency can represent a valuable feature for policy making. It is demonstrated that the model is capable of predicting not only the most recent recession but also the two previous ones. No additional assumptions, like changes in the volatility, are needed to achieve such good results. For the sake of completeness, the forecasts obtained with the base model are compared with the forecasts from the model with a volatility break. This comparison cannot uncover any differences. The comparison with the forecasts from the univariate counterpart of the proposed model, on the other hand, shows that the multivariate version performs far better, at least in the real GDP case.

Appendix

A State Space Representations

A.1 Monthly Model With the Trend–Cycle

A state space form for the trend–cycle in eq. (2) is

$$\begin{aligned} \alpha_{t+1} &= T_\mu \alpha_t + H_\mu v_t \\ \mu_t &= Z_\mu \alpha_t, \end{aligned} \tag{9}$$

where $\alpha_t = (\mu_t', \beta_t')'$, v_t is as in eq. (3), and

$$\begin{aligned} T_\mu &= \begin{bmatrix} I_k & K \\ 0 & I_r \end{bmatrix}, \qquad H_\mu = \begin{bmatrix} D_\zeta^{1/2} & 0 \\ 0 & D_\eta^{1/2} \end{bmatrix}, \\ Z_\mu &= \begin{bmatrix} I_k & 0 \end{bmatrix}, \qquad r = 1 \end{aligned} \tag{10}$$

Then, a state space form for the monthly model is

$$\begin{aligned} \alpha_{t+1} &= T\alpha_t + Hu_t \\ y_t &= Z\alpha_t + Gu_t, \qquad t = 1, \ldots, n, \end{aligned}$$

where $u_t = (v_t', \varepsilon_t')'$ with $\mathrm{Var}(u_t) = I$, $T = T_\mu$, $Z = Z_\mu$ and

$$H = \begin{bmatrix} D_\zeta^{1/2} & 0 & 0 \\ 0 & D_\eta^{1/2} & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 & D_\epsilon^{1/2} \end{bmatrix}$$

The initial state vector $\alpha_1 = (\mu_1', \beta_1')'$ is

$$\alpha_1 = A\delta + p,$$

where δ has dimension $k + r$ and is diffuse, A is a suitable nonstochastic matrix, and p has zero mean and a well defined covariance matrix.

A.2 Monthly Model Including the Cycle

For numerical reasons, the model for p_t in eq. (8) is implemented in cascade form as

$$p_t = \left[\theta_z^{-1}(B)\delta_z(B)\right] w_t, \tag{11}$$

where w_t follows the model

$$w_t = \left[(1-B)^{-2}\widetilde{C}(B)\right]\widetilde{v}_t$$

A state space model for w_t can be easily derived from (9), namely

$$\begin{aligned}\gamma_{t+1} &= T_w\gamma_t + H_w\widetilde{v}_t,\\ w_t &= Z_w\gamma_t,\end{aligned}$$

where $T_w = T_\mu$, $Z_w = Z_\mu$ and $H_w = (\sigma_n/\sigma_a)H_\mu$, and the matrices T_μ, Z_μ and H_μ are given in (10). As for p_t in eq. (11), we select the multivariate version of the state space representation used by Gómez and Maravall (1994), which is an extension to the nonstationary case of the approach proposed by Akaike (1974). Thus, the state space representation of (11) is

$$\begin{aligned}\xi_t &= T_v\xi_{t-1} + H_v w_t\\ p_t &= Z_v\xi_t,\end{aligned} \tag{12}$$

where $\xi_t = (p'_t, p'_{t+1|t}, ..., p'_{t+q|t})'$, $\theta_z(B) = 1 + \sum_{i=1}^{q}\theta_{z,i}B^i$, $q = 2d$ is the degree of both polynomials, $\theta_z(B)$ and $\delta_z(B)$,

$$T_v = \begin{bmatrix} 0 & I & 0 & \cdots & 0\\ 0 & 0 & I & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & -\theta_{z,q}I & -\theta_{z,q-1}I & \cdots & -\theta_{z,1}I \end{bmatrix}, \qquad H_v = \begin{bmatrix} I\\ V_1 I\\ \vdots\\ V_q I\end{bmatrix}, \tag{13}$$
$$Z_v = \begin{bmatrix} I & 0 & \cdots & 0\end{bmatrix},$$

and V_i, $i = 0, ..., q$, are the coefficients obtained from $V(B) = \delta_z(B)/\theta_z(B)$. Thus, the state space model for the cascade form of the model for p_t described earlier is

$$\begin{aligned}\varphi_{t+1} &= T_p\varphi_t + H_p\widetilde{v}_{t+1}\\ p_t &= Z_p\varphi_t,\end{aligned} \tag{14}$$

where $\varphi_t = (\xi'_t, \gamma'_{t+1})'$ and

$$T_p = \begin{bmatrix} T_v & H_v Z_w\\ 0 & T_w\end{bmatrix}, \qquad H_p = \begin{bmatrix} 0\\ H_w\end{bmatrix}, \qquad Z_p = \begin{bmatrix} Z_v & 0\end{bmatrix}$$

Similarly to (12), the state space form considered for c_t in eq. (7) is

$$\begin{aligned}\chi_{t+1} &= T_c\chi_t + H_c\overline{v}_{t+1}\\ c_t &= Z_c\chi_t,\end{aligned} \tag{15}$$

where $\chi_t = (c'_t, c'_{t+1|t}, ..., c'_{t+q-1|t})'$,

$$T_c = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\theta_{z,q}I & -\theta_{z,q-1}I & -\theta_{z,q-2I} & \cdots & -\theta_{z,1}I \end{bmatrix}, \qquad H_c = \begin{bmatrix} I \\ Z_1 I \\ \vdots \\ Z_{q-1}I \end{bmatrix},$$

$$Z_c = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$$

and Z_i, $i = 0, ..., q-1$, are the coefficients of the following polynomial

$$Z(B) = (1-B)^{d-2}(1+B)^d \frac{\overline{C}(B)}{\theta_z(B)}$$

Taking models (14) and (15) into account, the state space form for $\mu_t = p_t + c_t$ is

$$\alpha_{t+1} = \begin{bmatrix} T_p & 0 \\ 0 & T_c \end{bmatrix} \alpha_t + \begin{bmatrix} H_p & 0 \\ 0 & H_c \end{bmatrix} \begin{bmatrix} \widetilde{v}_{t+1} \\ \overline{v}_{t+1} \end{bmatrix}$$
$$\mu_t = \begin{bmatrix} Z_p & Z_c \end{bmatrix} \alpha_t,$$

where $\alpha_t = (\varphi'_t, \chi'_t)'$. Thus, the state space form for y_t is

$$\alpha_{t+1} = T\alpha_t + Hu_t$$
$$y_t = Z\alpha_t + Gu_t, \qquad t = 1, \ldots, n,$$

where $u_t = (\widetilde{v}'_{t+1}, \overline{v}'_{t+1}, \varepsilon'_t)'$, $\mathrm{Var}(u_t) = I$, and

$$T = \begin{bmatrix} T_p & 0 \\ 0 & T_c \end{bmatrix}, \qquad H = \begin{bmatrix} H_p & 0 & 0 \\ 0 & H_c & 0 \end{bmatrix},$$
$$Z = \begin{bmatrix} Z_p & Z_c \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 & D_\epsilon^{1/2} \end{bmatrix}$$

The initial state vector $\alpha_1 = (\varphi'_1, \chi'_1)'$, where φ_1 and χ_1 are uncorrelated, is

$$\alpha_1 = \begin{bmatrix} A \\ 0 \end{bmatrix} \delta + \begin{bmatrix} p \\ \chi_1 \end{bmatrix}$$

B Kalman Filter and Covariance Square Root Kalman Smoother

Consider a state space model

$$x_{t+1} = T_t x_t + H_t \epsilon_t$$
$$Y_t = Z_t x_t + G_t \epsilon_t, \qquad t = 1, ..., n$$

where $\mathrm{Var}(\epsilon_t) = I$. The initial state vector x_1 is specified as

$$x_1 = c + a + A\delta,$$

where c has zero mean and covariance matrix Ω, a is a constant vector, δ is diffuse and A is a constant matrix. In the following, it is assumed that $\delta = 0$. Even though the model proposed in this article implies $\delta \neq 0$ (see Appendices A.1 and A.2), this simplifying assumption allows to convey the idea of the applied filtering and smoothing algorithms in a comprehensive way. The Kalman filter is given by the recursions

$$\begin{aligned} E_t &= Y_t - Z_t \hat{x}_{t|t-1}, & \Sigma_t &= Z_t P_t Z_t' + G_t G_t', \\ K_t &= (T_t P_t Z_t' + H_t G_t')\Sigma_t^{-1}, & \hat{x}_{t+1|t} &= T_t \hat{x}_{t|t-1} + K_t E_t, \\ P_{t+1} &= (T_t - K_t Z_t) P_t T_t' + (H_t - K_t G_t) H_t', \end{aligned}$$

initialized with $\hat{x}_{1|0} = a$ and $P_1 = \Omega$. In the general case with $\delta \neq 0$, the so–called diffuse Kalman filter and smoother are applied (see de Jong, 1991).

The formulae for the fixed–interval Kalman smoother are as follows. For $t = n, n-1, \dots, 1$, define the so–called adjoint variable, λ_t, and its covariance matrix, Λ_t, by the recursions

$$\lambda_t = T_{p,t}' \lambda_{t+1} + Z_t' \Sigma_t^{-1} E_t, \quad \Lambda_t = T_{p,t}' \Lambda_{t+1} T_{p,t} + Z_t' \Sigma_t^{-1} Z_t,$$

initialized with $\lambda_{n+1} = 0$ and $\Lambda_{n+1} = 0$, where $T_{p,t} = T_t - K_t Z_t$. Then, for $t = n, n-1, \dots, 1$, the projection, $\hat{x}_{t|n}$, of x_t onto the whole sample $\{Y_t : 1 \leq t \leq n\}$ and its MSE, $P_{t|n}$, satisfy the recursions

$$\hat{x}_{t|n} = \hat{x}_{t|t-1} + P_t \lambda_t, \quad P_{t|n} = P_t - P_t \Lambda_t P_t$$

In this article the covariance square root smoother is applied since it proves to be a stable algorithm if the state vector has a large dimension. For square root smoothing, let $\widehat{Z}_t = \Sigma_t^{-1/2} Z_t$ and $T_{p,t} = T_t - \widehat{K}_t \widehat{Z}_t$, where $\widehat{K}_t = T_t P_t Z_t' + H_t G_t')\Sigma_t^{-1/2'}$. Let the QR algorithm produce an orthogonal matrix U_t such that

$$U_t' \begin{bmatrix} \widehat{Z}_t \\ \Lambda_{t+1}^{1/2'} T_{p,t} \end{bmatrix} = \begin{bmatrix} \Lambda' \\ 0 \end{bmatrix},$$

where Λ' is an upper triangular matrix. Then, $\Lambda = \Lambda_t^{1/2}$ and $\lambda_t = T_{p,t}' \lambda_{t+1} + \widehat{Z}_t' \widehat{E}_t$, where $\widehat{E}_t = \Sigma_t^{-1/2} E_t$. The square root form of the fixed interval smoother used in this article is as follows.

Step 1 In the forward pass, compute and store the quantities $\widehat{E}_t$, $\widehat{K}_t$, $\widehat{Z}_t$, $\hat{x}_{t+1|t}$ and $P_{t+1}^{1/2}$.

Step 2 In the backward pass, compute λ_t recursively by means of the formula $\lambda_t = T'_{p,t}\lambda_{t+1} + \widehat{Z}'_t\widehat{E}_t$. In addition, compute $\Lambda_t^{1/2}$ as explained earlier.

Step 3 Finally, using the output given by steps 1 and 2, compute recursively in the backward pass the fixed interval smoothing quantities

$$\begin{aligned} \hat{x}_{t|n} &= \hat{x}_{t|t-1} + P_t^{1/2}\left(P_t^{1/2'}\lambda_t\right) \\ P_{t|n} &= P_t^{1/2}\left[I - \left(P_t^{1/2'}\Lambda_t^{1/2}\right)\left(\Lambda_t^{1/2'}P_t^{1/2}\right)\right]P_t^{1/2'} \end{aligned}$$

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