Fiscal policy, entry and capital accumulation: hump-shaped responses.

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Abstract

In this paper we consider the entry and exit of firms in a Ramsey model with capital and an endogenous labour supply. At the firm level, there is a fixed cost combined with increasing marginal cost, which gives a standard U-shaped cost curve with optimal firm size. The costs of entry (exit) are quadratic in the flow of new firms. The number of firms becomes a second state variable and the entry dynamics gives rise to a richer set of dynamics than in the standard case: in particular, there is likely to be a hump shaped response of output to a fiscal shock with maximum impact after impact and before steady-state is reached. Output and capital per firm are also likely to be hump shaped.

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1 Introduction

Stephen Turnovsky developed the Ramsey (1928) continuous time representative agent model to include an endogenous labour supply[1] in order to explore the effects of fiscal policy[2] (Brock and Turnovsky (1981), Turnovsky (1990), Turnovsky (1995), Turnovsky and Sen (1991)). When applied to fiscal policy, this model is essentially Ricardian in nature: for every government expenditure flow there is an equivalent tax shadow in the form of current or future taxes. This is an income effect which serves to reduce consumption (crowding out) and increase the labour supply (when leisure is normal). If we look at the dynamics, we find that the impact on consumption is greater in the short-run than the long-run: consumption falls a lot on impact and then increases gradually to the new steady-state (this increasing pathway is determined by the fact that the marginal product of capital exceeds the discount rate, but the gap is diminishing as capital is accumulated). This "overshooting" of consumption is also present in the response of output: there is a big initial response, with output falling to its new higher level. If we look at empirical studies of macroeconomic time-series in the form of VARs, we find a different story[3]. The impulse-responses implied by VARs indicate that output follows a hump shaped response: the maximum impact on output is not on impact, but some time later (3-4 quarters) - see for example Mountford and Uhlig (2009). This sort of response is ruled out in the standard Ramsey model, which is clearly missing some vital ingredient. In this paper, we argue that if we include a process of entry (and exit) then we can keep the basic structure of the Ramsey model and move towards understanding the processes giving rise to the sorts of behavior we find in the data as represented by empirical VARs. More specifically, we find that output can have a hump-shaped response to a fiscal shock with the peak effect being after some time. Furthermore, this is not a special case at all, but a general (although not universal) feature. Whilst we do find that consumption can have a non-monotonic path, this is more exceptional.

In this paper we analyze a continuous-time model of entry in which firms produce output with capital and labour. The creation or destruction of firms (both flows) is determined endogenously as in Das and Das (1996) and Datta and Dixon (2002): the cost of entry (exit) is determined by the flow of firms into (out of) the market, so that the equilibrium price of entry (exit) equals the net present value of incumbency.

[1]The model with an exogenous labour supply is often known as the Cass-Koopmans model after Cass (1965) and Koopmans (1965) - see Takayama (1994) for a discussion and exposition.

[2]This mirrored what was also being done in the discrete time real business cycle approach, although fiscal policy was not incorporated until Baxter and King (1993).

[3]Of course, the VAR methodology is inherently reduced form and in that sense it has proven very problematic to uncover and differentiate between exogenous and endogenous fiscal shocks as well as across shocks.

The marginal cost of entry may vary with the flow due to a congestion effect involved in setting up new firms. We use this to explore the response of the economy to fiscal policy in terms of a permanent unanticipated change in government expenditure. We find that the presence of an endogenous labour supply allows for a variety of local dynamics. In particular, the stable eigenvalues can both be real or complex. With real eigenvalues, the adjustment path of either or both capital and the number of firms can be hump-shaped. With complex eigenvalues, the adjustment path of both capital and the number of firms are oscillatory. However, when we analyze the effects of fiscal policy, we find that only the number of firms can be non-monotonic. This is because the "initial position" also matters, and there is a linear relation between the number of firms and the capital stock as Government expenditure varies which restricts the dynamics to be monotonic for capital. However, whilst the dynamics of consumption and capital are monotonic, we find that there can be a non-monotonic "hump shaped" response of output and employment to fiscal policy, taking the form an initial jump response, followed by a smooth hump shaped overshooting of the new steady-state. Moreover, we find that in addition the impact effect of fiscal policy can be greater then or less than the long-run effect. Hence the introduction of entry into the Ramsey model enables a theoretical understanding of output and consumption responses that are more in line with the empirical evidence than the classic Ramsey model.

In the classic Ramsey model, all that matters for determining aggregate output and productivity is the aggregate labour and capital: how it is organized at the firm level does not matter. This approach is justified if there is constant returns to scale at the firm level. However, in this paper we assume an explicit firm-level technology which gives rise to the text-book U-shaped average cost with strictly increasing marginal cost. In the long-run with zero-profits, the aggregate economy will display constant returns in capital and labour[4]. However, in the short-run as we move towards steady-state, output at the firm level will deviate from the efficient minimum AC output. This is important, because it means that capacity[5] utilization and hence the marginal products of labour and capital will vary: for a given level of capital, more firms means less capital per-firm which increases the marginal product of capital. With entry, the dynamics of the number of firms can act to counteract the effect of capital accumulation on reducing the marginal product of capital. It is this crucial difference which enables the model in this paper to display behavior which is more in line with the broad outline of the empirical data.

[4]Since zero-profits in the long-run means that the number of firms rises proportionately with capital and labour, so firms remain at efficient scale with average and marginal cost equalized.

[5]Note that we use capacity utilisation in its standard sense to mean the level of actual output relative to the reference level of efficient output. Some authors have used this to mean capital utilisation, which has to do with the intensity of how capital is used.

The model of entry we employ can be contrasted with the Melitz (2003) model used in trade. In the Melitz model, the cost of entry is constant and does not depend on the flow of entry. However, there is post-entry heterogeneity in the productivity of firms, unlike in the Ramsey framework we adopt where all firms have access to the same technology. Another important difference is that there are increasing returns to scale at the firm level in the Melitz model, not the U-shaped average cost curves in this paper. In our steady-state, all firms are efficient irrespective of the number of firms: inefficiency occurs out of steady-state as (all) firms operate at an output above or below efficient capacity. In the Melitz model, the steady state is not optimal in efficiency terms: there is excess entry due to the monopolistic post-entry equilibrium, and the model is incompatible with perfect competition. The entry model also differs from Jaimovich and Floetotto (2008), where there are no sunk entry costs and there is a zero profit condition that flow operating profits cover the flow overheads. As in Yang and Heijdra (1993) and Linneman (2001), the elasticity of demand varies due to the effect of market share on firm elasticity. Again, this is an essentially monopolistic framework which is not generally efficient in equilibrium: in the absence of love of variety, there will be excess entry and each firm has an increasing returns to scale technology.

It should be stressed that we are providing a *theoretical* framework and not an empirical model: we present numerical calibrations merely to illustrate the general results. If we look at empirical DGSE models that are currently popular, they have many ingredients that are absent in our theoretical model. In Smets and Wouters (2003), Burnside et al. (2004), Christiano et al. (2005) there are a combination of factors determining the dynamic response of policy: habit formation in consumption, capital adjustment costs, capital utilization as well as new Keynesian price and wage nominal rigidity. Indeed, as Woodford (2011) points out, the reaction to monetary policy may be a crucial factor, something which is totally absent in our real model. However, theory is still useful in providing a general understanding that calibrated numerical models cannot (even when they appear to "fit" the data). In particular, we are able to show in our model that a hump shaped response of output is a general (although not universal) feature that can be understood in terms of a phase diagram, and is not just the outcome of particular sets of parameter values.

The outline of the paper is as follows. In section 2 we outline the basic model of the consumer, firm and the entry process. In section 3 we determine the equilibrium in the economy in terms of the social planner's problem. In section 6.1 we analyze the steady-state and present a graphical analysis of the long-run fiscal multiplier. Section 5 provides a general analysis of the local dynamics. Section 6 then applies the previous section to the dynamics of the economy in response to fiscal policy and in particular non-monotonic trajectories in state and non-state variables. Section 6 concludes.

2 The model

In this paper we model a perfectly competitive dynamic general equilibrium economy with endogenous entry/exit of firms and accumulation of capital as a Social Planning optimum. We will first lay out the model in terms of household preferences, firm level and aggregate technology, and the costs of entry and exit. We will then derive the optimality conditions for the first-best optimum.

In each instant, households consume and supply labour $(C(t), L(t))$, with the intertemporal utility function defined by the flow utility and discount rate $\rho > 0$:

$$\bar{U} = \int_0^{\infty} U(C(t), L(t)) e^{-\rho t} dt$$

Assumption 1. ***Representative Household Preferences***. *The instantaneous utility function: is increasing in C, is decreasing in L, is additively separable and concave in (C, L) ($U_C > 0$, $U_L > 0$, $U_{CC} < 0$, $U_{LL} < 0$, and $U_{CL} = 0$); Inada conditions on consumption and labour supply hold: $\lim_{C \to 0^+} U_C = +\infty$, $\lim_{C \to +\infty} U_C = 0$ $\lim_{L \to \infty} U_L = -\infty$, and $\lim_{L \to 0+} U_L = 0$.*

Output is produced by capital and labour which is distributed across a continuum of firms $i \in R_+$. At instant t, there is a measure $n(t)$ such that $i \in [0, n(t)]$ are active (incumbent) and $i > n(t)$ are inactive (potential entrants). An active firm $i \in [0, n(t)]$ incurs a fixed overhead cost and produces output $y(i)$ according to the following technology:

$$y(i,t) = AF(k(i,t), l(i,t)) - \phi$$

Assumption 2. ***Firm level technology***. *The active firm's production function: is increasing in k and l and is concave in (k, l) ($F_k > 0$, $F_l > 0$, $F_{kk} < 0$, $F_{ll} < 0$, $F_{kl} \geq 0$, and $F_{kk}F_{ll} - F_{lk}^2 > 0$); it is homogeneous of degree ν, where $\nu \in (0, 1)$; and standard Inada conditions hold. There is a (flow) overhead cost $\phi > 0$.*

Firms which are *inactive* produce no output, employ no labour and capital and incur no overhead. Note that even if an active firm produces no output, it will still incur the overhead[6]. Under assumption 2, the firm level technology corresponds to the Marshallian $U-$shaped average cost curve with increasing marginal cost. As outlined in Brito and Dixon (2009), the corresponding *efficient* level of production is:

$$y^e = \frac{\phi \nu}{1 - \nu} \tag{1}$$

[6]We do not impose the restriction that $y(i,t) \geq 0$; for example, if the firm hires no labour or capital, it will have to pay the overhead ϕ, which is a negative output corresponding to negative profits. The only way it can avoid the overhead is by exiting the market.

which is increasing in ϕ and ν, and independent of A.

Now, for a given measure of active firms $n(t)$, production is maximized if aggregate capital $K(t)$ and labour $L(t)$ are divided equally between the active firms, so that we have

$$k(i,t) = \frac{K(t)}{n(t)},\ l(i,t) = \frac{L(t)}{n(t)}.$$

Hence the aggregate output produced by active firms (dropping the time subscript):

$$Y = \int_0^n y(i)di = n\left[AF\left(\frac{K}{n}, \frac{L}{n}\right) - \phi\right], \tag{2}$$

where $Y = Y(K, L, n, A, \phi)$ is the aggregate production function. Y is homogenous of degree 1 in (K, L, n) and we have the following marginal products:

$$\begin{aligned} Y_K = Y_K(\underset{-}{K}, \underset{+}{L}, \underset{+}{n}) &= AF_k\left(\frac{K}{n}, \frac{L}{n}\right) \\ Y_L = Y_L(\underset{+}{K}, \underset{-}{L}, \underset{+}{n}) &= AF_l\left(\frac{K}{n}, \frac{L}{n}\right) \\ Y_n = Y_n(\underset{+}{K}, \underset{+}{L}, \underset{-}{n}, \underset{+}{A}, \underset{-}{\phi}) &= (1-\nu)AF\left(\frac{K}{n}, \frac{L}{n}\right) - \phi \end{aligned}$$

The derivatives are in appendix A. Because capital and labour are distributed equally across firms, the marginal products of labour and capital at the aggregate level are equal to the marginal products at the firm level.

The marginal product of n corresponds to the profit or surplus per firm, $\pi = Y_n$: total production less the cost of capital and labour when they are valued at their marginal products, which can be positive or negative. An additional firm means that capital and labour per firm are lower which raises the marginal products; however, an additional overhead is also incurred. The marginal product of n is zero when output per firm is equal to the efficient level (1). Hence zero profits corresponds to an efficient scale of production at the firm level, y^e. Profits are positive when firms produce more than y^e and negative when below. Using this we can define the "efficient" production function:

$$Y^e(K, L, A, \phi) = \max_n Y(K, L, n, A, \phi) \tag{3}$$

Assumption 3. ***Entry****: There is a quadratic adjustment cost in the flow of entry (exit).*

The total cost of entry $Z(t)$ is the integral

$$Z(t) = \gamma \int_0^{e(t)} i\, di = \frac{\gamma}{2} e(t)^2.$$

The model of entry is represented in the decentralized form in Brito and Dixon (2009) and is based on Das and Das (1996) and Datta and Dixon (2002). Here we present it in the equivalent form for the Social Planner. There is a (marginal) cost $q(t)$ to setting up a new firm which is a linear function of the flow of entry $e(t) = \dot{n}(t)$. Likewise, there is a cost of $-q(t)$ to dismantle an existing firm. The marginal cost of entry depends on the flow of entry

$$q(t) = \gamma e(t) \tag{4}$$

This relationship (4) results from the existence of a congestion effect or a fixed factor involved in the process of setting up or dismantling a firm.

The output of firms is used for consumption, government spending and the setting up new firms or dismantling existing ones and investment. We assume for simplicity that there is no depreciation of capital $I(t) = \dot{K}(t)$. Hence:

$$Y(t) = C(t) + I(t) + Z(t) + G(t)$$

Hence from (2) we get the capital accumulation equation:

$$\dot{K} = n\left[AF\left(\frac{K}{n}, \frac{L}{n}\right) - \phi\right] - C - \gamma\frac{e^2}{2} - G. \tag{5}$$

We assume that the government demands output G which is financed by a Lump-sum tax T. Since there is Ricardian equivalence in this model, we assume the budget is balanced instant by instant:

$$T(t) = G(t).$$

3 The social planner's problem

In Brito and Dixon (2009) we analyzed the market (decentralized) equilibrium and showed that it is equivalent to the social planner's problem. In this paper we adopt the social planner's problem of maximizing the utility of the representative household:

$$\max_{C,L,e} \int_0^{+\infty} U(C, L)e^{-\rho t}dt$$

subject to equation (5) and $\dot{n} = e$, given $K(0) = K_0$ and $n(0) = n_0$. There are two state variables (K, n), there are three control (jump) variables (C, L, e) and the usual boundedness properties for the state variables are assumed to hold. We treat A and G as exogenous parameters: later on we will derive how the system responds in the long and short run to variations in G.

As both the utility function and the constraints of the problem are concave functions of the controls, then (if the transversality conditions hold) the Pontriyagin maximum principle will give us necessary and sufficient conditions for optimality. The current value Hamiltonian is,

$$H \equiv U(C,L) + p_K \left\{ n \left[AF\left(\frac{K}{n}, \frac{L}{n}\right) - \phi \right] - C - \gamma \frac{e^2}{2} - G \right\} + p_n e$$

where p_K and p_n are the co-state variables associated to the aggregate capital stock and the number of firms. The static arbitrage condition for labour supply is:

$$U_L(L) + U_C(C) A F_L\left(\frac{K}{n}, \frac{L}{n}\right) = 0 \tag{6}$$

From equation (6) the optimal choice of labour can be represented as a function of (C, K, n, A) [7]

$$\hat{L} = L(\underset{-}{C}, \underset{+}{K}, \underset{+}{n}, \underset{+}{A}). \tag{7}$$

Note that G does not affect L independently of C: for a given wage w, there is an Income Expansion path (IEP) which is strictly monotonic and hence there is a $1-1$ relation of C to L. Changes in G simply move the household up and down the IEP and hence given C there is a unique L.

Using (7), we can represent the marginal product of capital r and the profit per firm π and gross output Y as "reduced form" functions of (C, K, n, A, ϕ) by substituting out the labour supply $L = \hat{L}$:

$$\hat{r} = r(\underset{-}{C}, \underset{-}{K}, \underset{+}{n}, \underset{+}{A}) = AF_k\left(\frac{K}{n}, \frac{\hat{L}}{n}\right) \tag{8}$$

$$\hat{\pi} = \pi(\underset{-}{C}, \underset{+}{K}, \underset{-}{n}, \underset{+}{A}, \underset{-}{\phi}) = (1-\nu) AF\left(\frac{K}{n}, \frac{\hat{L}}{n}\right) - \phi \tag{9}$$

$$\hat{Y} = Y(\underset{-}{C}, \underset{+}{K}, \underset{\pm}{n}, \underset{+}{A}, \underset{-}{\phi}) = n \left[AF\left(\frac{K}{n}, \frac{\hat{L}}{n}\right) - \phi \right] \tag{10}$$

Note that the effect of n on output is in general ambiguous because $Y_n = \pi + w\hat{L}_n$, where π can be negative or positive whilst wL_n is positive. However, around the

[7] By using the implicit function theorem we have: $L_C = -wU_{CC}/H_{LL} < 0$, $L_K = -w_K U_C/H_{LL} > 0$, $L_n = -w_n U_C/H_{LL} > 0$ and $L_A = -w_A U_C/H_{LL} > 0$. where $H_{LL} = U_{LL} + U_C w_L$.

steady-state it is unambiguously (strictly positive) since $\pi = 0$: hence we only have the indirect positive effect of entry on the real wage and labour supply[8].

The optimality condition for entry is

$$\gamma e = q \tag{11}$$

where $q = p_n/p_K$ is the relative cost associated with the number of firms. If we define the inverse of the elasticity of intertemporal substitution $\sigma(C) = -U_{CC}(C)C/U_C(C)$, we get the Euler equation for consumption

$$\dot{C} = \frac{C}{\sigma(C)}(r(C, K, n, A) - \rho). \tag{12}$$

The relative co-state variable associated to the number of firms, verifies the differential equation in q :

$$\dot{q} + \hat{\pi} = r(C, K, n, A)q \tag{13}$$

Integrating this gives us:

$$q(t) = \int_t^{\infty} \pi(s) e^{-\int_t^s r(\tau)d\tau} ds,$$

so that along the optimal path the marginal cost of entry (exit) is equated to the net present value of profits. Furthermore, (13) can be written as:

$$\hat{r} = \frac{\hat{\pi}}{q} - \frac{\dot{q}}{q}.$$

which is an *arbitrage condition* between expanding total capital stock, $K = nk$, through increasing the capital stock per firm or through entry. Investing one unit of the consumption good in capital per firm yields the marginal product of capital r. Investing one unit in a new firm yields a share q^{-1} of the profit flow π of the new firm and a capital gain/loss $\dot{q}$. Along the optimal path, these two will be equated.

From equation (11), then $\dot{e}\gamma = \dot{q}$, and we obtain

$$\dot{e} = r(C, K, n, A)e - \frac{\pi(C, K, n, A, \phi)}{\gamma}. \tag{14}$$

Hence the dynamic general equilibrium is represented by the paths (C, e, K, n) which solve the system (suppressing the parameters (A, ϕ)) (12) -(14) and

$$\dot{K} = Y(C, K, n, A, \phi) - C - \frac{\gamma}{2}e^2 - G, \tag{15}$$

$$\dot{n} = e \tag{16}$$

[8]This indirect effect would not happen if F was homogenous to degree 1 ($\nu = 1$), since then the marginal product would depend only on the ratio of capital to labour at the firm level which would not vary as n varies.

together with the initial conditions $K(0) = K_0$ and $n(0) = n_0$, and the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t} U_C(C(t))K(t) = \lim_{t\to\infty} e^{-\rho t} U_C(C(t))e(t)n(t) = 0. \tag{17}$$

We have the two equations (15) and (16) governing the state variables (K, n),and the two optimality conditions for the controls (C, e) (12) and (14).

Perfect foresight equilibrium trajectories converge to the steady state of the system composed by equations (12), (14), (15) and (16). Therefore, the stable manifold of this system is the set of all equilibrium trajectories. In the next section we determine the steady state and discuss how it changes in fiscal policy, and in section 5 we characterize the stable manifold by presenting a method for dealing with short-run dynamics in the present four-dimensional system. In particular, we establish the conditions under which we will have non-monotonic equilibrium paths.

4 The Steady-State and long-run effects of fiscal policy.

For a given level of Government expenditure, a unique steady state (C^*, e^*, K^*, n^*) exists and satisfies [9]:

$$\begin{aligned} r^* &\equiv r(C^*, K^*, n^*, A) = \rho, && (18)\\ \pi^* &\equiv \pi(C^*, K^*, n^*, A, \phi) = 0, && (19)\\ e^* &= 0, && (20)\\ C^* + G &= Y^* \equiv \hat{Y}(C^*, K^*, n^*, A, \phi), && (21) \end{aligned}$$

with the following relation between the number of firms and total output

$$Y^* = n^* y^* = n^* \left(\frac{\phi\nu}{1-\nu} \right). \tag{22}$$

Therefore, in the steady state profits are zero, there is no net entry, the real rate of return is equated to the rate of time preference and output is only used for (private and government) consumption purposes. Since profits are zero, output per firm is at the efficient level and hence from (1) the number of firms is proportional to output. In order to see how the steady-state operates, it is useful to provide a simple illustration (which we will use in later sections):

[9] A simple proof of the uniqueness of the steady state results from the fact that the partial derivatives of the functions r and π as regards C, K, n are globally non-zero and the application of the inverse function theorem. Equations (18) and (19) can be solved for K and n to get K^* and $n*$ as a function of C. Then substituting in equation (21) we get C^*.

Example: benchmark case We consider a Cobb-Douglas technology and a separable isoelastic instantaneous utility function:

$$F(K,L) = K^{\alpha}L^{\beta}, \ \ 0<\alpha,\beta<1, \tag{23}$$

$$U(C,L) = \frac{C^{1-\sigma}-1}{1-\sigma} - \xi\frac{L^{1+\eta}}{1+\eta}, \tag{24}$$

where $\sigma>0,\ \xi>0,\ \eta\geq 0$ and $\nu=\alpha+\beta\in(0,1)$.

The unique steady state for the benchmark case is given by $K^*=k^*n^*$, $L^*=l^*n^*$ where the firm level stock of capital k^* and labour input l^* are dependent on the technological parameters [10]

$$k^*=\frac{\alpha\phi}{\rho(1-\nu)}\ ,\ l^*=\left(\frac{1}{A}\left(\frac{\rho}{\alpha}\right)^{\alpha}\left(\frac{\phi}{1-\nu}\right)^{1-\alpha}\right)^{1/\beta}\ ,\ y^*=\frac{\nu\phi}{1-\nu}.$$

We determine consumption as a function of n,

$$C(n)=\left(\frac{\beta}{\xi}y^*(l^*)^{-(1+\eta)}(n)^{-\eta}\right)^{1/\sigma}.$$

Then steady state consumption is $C^*=C(n^*)$ where n^* cannot be determined in closed form, but is implicitly given by:

$$n^*=\{n:\ ny^*=C(n)+G\},$$

If $G=0$ then we can determine C^* and n^* explicitly:

$$C^*=\left(\frac{\xi\nu}{\beta l^*}\ (y^*)^{1+\eta}\right)^{1/(\eta+\sigma)}$$

and $n^*=C^*/y^*$.

□

The purpose of this paper is to focus on the dynamic effects of fiscal policy in terms of the effects over the aggregate variables, in particular (C,K,n). However, as a useful preliminary, we will first develop a simple graphical analysis of the long-run (steady-state) effects of changes in G on these variables. The dynamic system has four dimensions with two state variables (K,n) and two controls (e,C). We can look at this graphically by projecting the system onto two-dimensional subspaces. In our analysis of the dynamics we will focus on (K,n). However, for the purposes of

[10] See appendix B for its derivation.

focussing on the steady-state, we also look at consumption-labour space (C, L). The formal derivation of the long-run multipliers is given in Proposition 4 below.

Turning first to (K, n) space, we can take the two equations (18)-(19) and define the two locii:

$$\begin{aligned} r(K, L^*(n, K, G), n, A) &= \rho \\ \pi(K, L^*(K, n, G), n, A, \phi) &= 0 \end{aligned}$$

The first locus $r = \rho$ defines the combinations of (K, n) which equate the marginal product of capital with ρ given steady-state labor supply $L^*(G)$ and technology parameter A. This is upward sloping, since by diminishing marginal productivity at the firm level, an increase in K (given n) *reduces* r, whilst an increase in n *increases* it. The second locus $\pi = 0$ is the combinations of (K, n) which yield zero-profits given $(L^*(G), A, \phi)$. As we will see later, this locus may be positively or negatively sloped in the (K, n)-space when the labour supply is allowed to vary. However, in any case the locus $\pi = 0$ (equivalent to $\dot{e} = 0$) is always less steep than locus $r = \rho$ (equivalent to $\dot{C} = 0$): [11]

$$\left.\frac{dn}{dK}\right|_{r^*=\rho} > \left.\frac{dn}{dK}\right|_{\pi^*=0}.$$

Steady-state occurs where the two locii intersect. See Figure 1 for the benchmark case and for the case featuring a positively slopped $\pi = 0$ locus [12].

Secondly, we look at the steady-state in consumption-labour space to see how labour supply $L(G)$ and consumption $C(G)$ are determined given G. First, note that the steady-state equilibrium conditions (18), (19) define the steady-state real wage w^* (marginal product of labour). This is because output per firm is at the efficient level (19), and the marginal product of capital is equal to ρ. Since the function F is homogeneous of degree ν in $(K/n, L/n)$, (18) determines the capital-labour ratio and hence the marginal product of labour in steady-state. In (C, L) space, we can trace out the steady-state *Income Expansion Path* (IEP) for consumption and labour given w^* at different levels of non-labour income[13]. The IEP is *downward sloping* (since consumption and the absence of work are normal goods). The convex slope of the IEP results from diminishing marginal utility of consumption and increasing disutility of labour with additive separability. The Inada conditions (Assumption 1(b)) ensure that C goes to infinity as L reaches 0 and vice versa.

[11] The inequality is demonstrated in the proof of Lemma 6 in the Appendix A

[12] The derivation of the curves is in Appendix B.

[13] Note that in standard microeconomics, the IEP is defined in terms of leisure and consumption. The format is slightly different here, because we have labour supply as a bad in the utility function without any explicit reference to leisure, and is defined from (6) by $U_L + U_C w = 0$.

Furthermore, given the steady-state capital labour ratio, we also know output-per-unit labour in steady-state Y^*/L^*. Hence, we can depict the aggregate trade-off between consumption and leisure in steady-state, or the *Euler Frontier*[14] *(EF)*[15]: (21) become $C+G = L.(Y^*/L^*)$. The EF is an upward sloping line, with $C = -G$ when $L = 0$. The economy is in steady-state where the IEP and EF intersect, as depicted in Figure 1.

[Figure 1 here]

We can now trace the effects of a permanent increase in G : the initial equilibrium is A, and the equilibrium after the increase is B. Turning first to consumption-leisure space, the EF shifts down by a vertical distance equal to the increase in government expenditure. As in the new Keynesian analysis of fiscal policy (Dixon (1987), Mankiw (1988), Startz (1989)), this income effect leads to a move down the IEP with consumption falling and labour supply (and output) increasing. Hence the fiscal multiplier is strictly positive but less than 1 due to the crowding out of consumption. The function $C^*(G)$ traces out the steady-state relation between C and G, and is strictly decreasing with a derivative above -1. Turning next to (K, n) space, since we know that the increase in G has decreased C^*, and for a given (K, n) the steady-state labour supply $L^*(G)$ has increased. This results in a rightward shift in the $r^* = \rho$ locus and an upward shift in the $\pi^* = 0$ locus. The new equilibrium is at point B, to the north-west A. Indeed, since the steady-state equations define the capital-labour ratio, the steady-state ratio $(n/K)^*$ is also determined. As we vary G, in (K, n) the steady-state moves along a ray through the origin $n = K/k^*$, the dotted line passing thought A and B. We denote this linear steady-state locus "the dG line".

With an exogenous labour supply, fiscal policy is very simple to analyze. In effect, the labour supply function (7) becomes fixed at some level $\hat{L} = \bar{L}$, and all of the derivatives of $\hat{L} = 0$. In terms of consumption-labour space, the IEP becomes vertical at $\bar{L}$. There is 100% crowding out: so the function $C^*(G)$ takes the simple linear form $C^*(G) = C^*(0) - G$ where $C^*(0)$ is the level of consumption when there is no government expenditure. In terms of (K, n) space, the two loci $r = \rho$ and $\pi = 0$ become fixed since it was the change in $L^*(G)$ which shifted them in the endogenous labour case depicted in Figure 2.

[14] We use the term "Euler Frontier" because it means that for a given labour supply L, with zero-profits (efficient production), capital has accumulated to equate the marginal product of capital with ρ. The resultant output can be divided between C and G. Note that the real wage in steady state is *less* than the slope of the *EF*, since wages form only part of total income (the rest being capital's share). Hence the *EF* is *not* tangential to the indifference curve at equilibrium.

[15] See Costa and Dixon (2011)

5 Local Dynamics

We now analyze the behavior of the perfect foresight equilibrium paths away from the steady state, by characterizing the local stable manifold of system (12), (14), (15) and (16) around the steady-state (corresponding to a particular value of G) given by equations (18), (19), (20) and (21). Besides having a high dimensionality, the system does not have a closed form solution (even in the case in which have specific utility and production functions). Next, we will go as far as possible in characterizing the dynamics arbitrarily close to the steady state.

The introduction of the number of firms as an additional state variable while keeping the saddle-path dynamics increases its dimension to two. The stable manifold is two-dimensional. We proved, for the exogenous labour supply case, in Brito and Dixon (2009), that this could be a source of non-monotonic transitional dynamics. In that paper we found that the Jacobian had two negative real eigenvalues, and if the initial values of the state variables, $K(0)$ and $n(0)$, verify certain conditions then a particular hump-shaped transition path can occur, although just for one state variable. The extension to endogenous labour supply allows not only for a hump-shaped transition for a single variable, but also for other different types of non-monotonic transitional dynamics: non-monotonic hump-shaped transitions or oscillatory behavior for *both* state variables. The first case occurs when both eigenvalues of the Jacobian are real and the second occurs when they are complex conjugate. In order to systematically determine the conditions under which all the previous types of non-monotonicity may occur we use and extend the geometrical method for dealing with the mechanics of non-monotonicity within our four-dimensional dynamic system which was introduced in Brito and Dixon (2009). Essentially, the method consists in projecting the transition paths in the space of the state variables (K, n), and partitioning that space according to the initial points that generate several different types of the transition paths. In section 6 we show that this method simplifies a lot the characterization time-response of the model to fiscal policy shocks.

5.1 The local stable manifold

The local stable manifold, W^s, is defined as the set of (C, e, K, n) such that, if the economy starts from there, there is asymptotic convergence to the steady state (C^*, e^*, K^*, n^*). It can be approximated locally by the stable eigenspace, E^s, which is the linear space that is tangent to, W^s, at the steady state. In spite of the dimension of the system we conclude next that the stable manifold is always two-dimensional, which means that the dynamics can be projected in the two-dimensional space of the pre-determined variables (K, n), and we can prove general results concerning the slopes of the equilibrium trajectories in its neighborhood.

The linearized system has the Jacobian

$$J := \begin{pmatrix} C^* r_C^*/\sigma & 0 & C^* r_K^*/\sigma & C^* r_n^*/\sigma \\ -\pi_C^*/\gamma & \rho & -\pi_K^*/\gamma & -\pi_n^*/\gamma \\ Y_C^* - 1 & 0 & Y_K^* & Y_n^* \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{25}$$

where the partial derivatives refer to the capital return, profit and aggregate output optimal functions, i.e, which consider the optimal response of labour supply, evaluated at the steady state[16]

The eigenvalues of Jacobian J are (see Lemma 1 in Appendix A)

$$\lambda_1^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2}\right)^2 - \frac{\mathcal{T}}{2} - \Delta^{\frac{1}{2}} \right]^{\frac{1}{2}}, \quad \lambda_2^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2}\right)^2 - \frac{\mathcal{T}}{2} + \Delta^{\frac{1}{2}} \right]^{\frac{1}{2}} \tag{26}$$

where $\Delta = (\mathcal{T}/2)^2 - \mathcal{D}$ is the "inner" discriminant and $\mathcal{T}$ is the sum of the principal minors of J minus ρ^2 and $\mathcal{D} = \det(J)$. Defining

$$\mathcal{O} \equiv \frac{C^*}{\sigma} \left(r_n^* \pi_K^* + Y_n^* (r_C^* \pi_K^* - r_K^* \pi_C^*) \right), \tag{27}$$

$$\mathcal{Q} \equiv \frac{C^*}{\sigma} \left(r_C^* Y_K^* - r_K^* (Y_C^* - 1) \right) \tag{28}$$

we get

$$\mathcal{T} = \mathcal{Q} + \frac{\pi_n^*}{\gamma}, \quad \mathcal{D} = \frac{\pi_n^*}{\gamma} \mathcal{Q} - \frac{\mathcal{O}}{\gamma}$$

and the "inner" discriminant becomes

$$\Delta = \left[\frac{1}{2} \left(\mathcal{Q} - \frac{\pi_n^*}{\gamma} \right) \right]^2 + \frac{\mathcal{O}}{\gamma}. \tag{29}$$

Because $\pi_n^* < 0$, and $\mathcal{Q} < 0$ [17] we have $\mathcal{T} < 0$, then , $(\rho/2)^2 - \mathcal{T}/2 > (\rho/2)^2$. As the sign of $\mathcal{O}$ is ambiguous then the sign of the "inner" discriminant is ambiguous as well. However, as we show in the proof of Proposition 1, the determinant is always positive, $\mathcal{D} > 0$, which implies that the stable manifold is of dimension two and the discriminant $\Delta < (\mathcal{T}/2)^2$. This was not the case in the exogenous labour case (see Brito and Dixon (2009)) where $\mathcal{O} > 0$ and the "inner" discriminant was always positive. This ambiguity is related to the endogeneity of labour supply and introduces a richer set of dynamics as compared with that model.

Next we prove that the stable manifold is always two dimensional:

[16] For example for the derivatives for the interest rate function we set $r_j^* = \partial \hat{r}_j(C, K, n, .)/\partial j$ for $j = K, n, C$, evaluated at the steady state values $K = K^*$, $n = n^*$ and $C = C^*$.

[17] This is because $r_C^* Y_K^* - r_K^* Y_C^* = L_C (Y_{LK} Y_K - Y_{KK} Y_L) < 0$.

Proposition 1 (**Characterization of the eigenvalues**). *Let assumptions 1, 2 and 3 hold. Then the Jacobian J has always two eigenvalues with negative real parts. In particular: (a) if* $\mathcal{O} < 0$ *and* $\Delta < 0$*, then it has one pair of complex conjugate eigenvalues with negative real part,* $Re(\lambda_2^s) = Re(\lambda_1^s) < 0$*; (b) if* $\mathcal{O} \leq 0$ *and* $\Delta = 0$*, then it has two negative multiple real eigenvalues,* $\lambda_2^s = \lambda_1^s = \rho/2 - ((\rho/2)^2 - \mathcal{T}/2)^{1/2} < 0$*; (c) if* $\mathcal{O} > 0$*, or if* $\mathcal{O} < 0$ *and* $\Delta > 0$*, then it has two real and distinct negative eigenvalues,* $\lambda_2^s < \lambda_1^s < 0$*.*

See Appendix A for the proofs.

This means that in a properly chosen projection space the model behaves like a sink (if the eigenvalues are real) or a stable node (if they are complex conjugate).

The result on the dimension of the stable manifold is explained by the presence of stabilizing forces acting independently over the two state variables, the stock of capital and the number of firms. We may identify, in the expression for $\mathcal{T}$ the three main channels: (1) the negative effect of the number of firms on profits, (2) the negative effect of capital accumulation on the real interest rate, and (3) the interaction between the indirect effects of consumption on labour supply and both the direct and indirect effects of the capital stock over the real interest rate and aggregate output. The first effect acts over the number of firms and the other effects will dampen shocks exerted on the stock of capital. In a model with exogenous labour only the direct components of the first two will be present.

From Proposition 1 we can note that *a necessary* condition for the existence of complex eigenvalues is $\mathcal{O} < 0$, and a *sufficient* condition for the existence of real distinct real eigenvalues is $\mathcal{O} > 0$. If we look inside (27) we see there are two parts. The first effect is $r_n^* \pi_K^*$ captures the stabilizing effects of capital on entry and entry on capital as discussed before: both terms are positive. The second terms $Y_n^*(r_C^* \pi_K^* - r_K^* \pi_C^*)$ explicitly capture the effects of consumption on the labour supply, and hence on profits and the return on capital: from (7) more consumption means less labour which reduces π and r). Hence the sign of the second terms in brackets is ambiguous: it can be negative and so can result in complex eigenvalues. A sufficient condition for the non-existence of complex roots is that $\mathcal{O}$ is positive, when the first effect dominates the second. In an exogenous labour model the second effect is switched off $r_C^* = \pi_C^* = 0$: hence $\mathcal{O} > 0$, $\Delta > 0$ and there can be no complex roots.

We can now understand how an endogenous labour supply can lead to complex dynamics (non-monotonicity in both variables along the equilibrium path). Consumption influences the labour supply, which has an effect of the same sign on both the entry and the return on capital. This can push both (K, n) in the same direction which can overcome (for some time) the natural tendency towards the steady-state. However, as consumption itself converges to the steady-state, its effect will diminish

leading the two state-variables to reverse direction and fall back to steady-state. In the case of fiscal policy, which we examine below in section 6, the impact effect of an increase in government expenditure is to decrease consumption which boosts the labour supply and hence boosts the marginal product of capital and the profitability of entry. This can be sufficient to make total output overshoot the new steady-state and converge back down to it as consumption converges.

The possibility of complex eigenvalues with an endogenous labour supply arises from this effect which is absent with an exogenous labour supply (where eigenvalues are only real Brito and Dixon (2009)).

As there are no eigenvalues with zero real part, the equilibrium dynamics belonging to the stable manifold associated to the unique stationary equilibrium point $(C^*, e^*; K^*, n^*)$, W^s, can be qualitatively approximated by the eigenspace E^s which is tangent to stable manifold.

The trajectories of the variables in the model belonging to space E^s are given by the following equation:

$$\begin{pmatrix} C(t) - C^* \\ e(t) - 0 \\ K(t) - K^* \\ n(t) - n^* \end{pmatrix} = z_1^s \begin{pmatrix} v_{1,1}^s \\ \lambda_1^s \\ v_{3,1}^s \\ 1 \end{pmatrix} e^{\lambda_1^s t} + z_2^s \begin{pmatrix} v_{1,2}^s \\ \lambda_2^s \\ v_{3,2}^s \\ 1 \end{pmatrix} e^{\lambda_2^s t},\ t \geq 0 \tag{30}$$

where (see Lemma 3 in Appendix A)

$$v_{1,1}^s \equiv \frac{\frac{C^*}{\sigma}\left(r_K^* Y_n^* - r_n^* Y_K^* + r_n^* \lambda_1^s\right)}{l_2 - \frac{\pi_n^*}{\gamma}},\ v_{3,1}^s \equiv \frac{\frac{C^*}{\sigma}\left((Y_C^* - 1) r_n^* - Y_n^* r_C^*\right) + Y_n^* \lambda_1^s}{l_2 - \frac{\pi_n^*}{\gamma}} \tag{31}$$

$$v_{1,2}^s \equiv \frac{\frac{C^*}{\sigma}\left(r_K^* Y_n^* - r_n^* Y_K^* + r_n^* \lambda_2^s\right)}{l_1 - \frac{\pi_n^*}{\gamma}},\ v_{3,2}^s \equiv \frac{\frac{C^*}{\sigma}\left((Y_C^* - 1) r_n^* - Y_n^* r_C^*\right) + Y_n^* \lambda_2^s}{l_1 - \frac{\pi_n^*}{\gamma}}, \tag{32}$$

and

$$z_1^s \equiv \frac{(K(0) - K^*) - v_{3,2}^s (n(0) - n^*)}{v_{3,1}^s - v_{3,2}^s},\quad z_2^s \equiv -\frac{(K(0) - K^*) - v_{3,1}^s (n(0) - n^*)}{v_{3,1}^s - v_{3,2}^s}.$$

We denote $l_1 = \lambda_1^s \lambda_1^u$ and $l_2 = \lambda_2^s \lambda_2^u$. Equation (30) is formally valid for any type of eigenvalue with negative real part, real or complex. However, If the eigenvalues are complex then $l_1 = \mathcal{T}/2 + |\Delta|^{1/2} i$ and $l_2 = \mathcal{T}/2 - |\Delta|^{1/2} i$ are complex, and the exponential is a complex function with coefficients dependent on time, because $e^{\lambda_1^s t} = e^{(\theta_s + \vartheta i)t} = e^{\theta_s t}\ (\cos(\vartheta t) + \sin(\vartheta t) i)$ and $e^{\lambda_2^s t} = e^{(\theta_s - \vartheta i)t} = e^{\theta_s t}\ (\cos(\vartheta t) - \sin(\vartheta t) i)$.

Every variable in equation (30) is a weighted sum of the two time-dependent exponential factors, $e^{\lambda_1^s t}$ and $e^{\lambda_2^s t}$. The weighting factors are given by the eigenspaces E_1^s and E_2^s and by two components which ensure that the transversality conditions

z_1^s and z_2^s hold. The first component governs the co-movement of the variables and the second can be used to describe the dynamics by projecting it into the state space (K, n).

The eigenspaces (which can be obtained from the solutions of (30) for $z_2^s = 0$ and $z_1^s = 0$),

$$E_1^s := \{(C, e, K, n) : \ C - C^* = v_{1,1}^s(n - n^*), \ e = \lambda_1^s(n - n^*), \ (K - K^*) = v_{3,1}^s(n - n^*)\}$$

and

$$E_2^s := \{(C, e, K, n) : \ C - C^* = v_{1,2}^s(n - n^*), \ e = \lambda_2^s(n - n^*), \ (K - K^*) = v_{3,2}^s(n - n^*)\}$$

span the stable eigenspace E^s, which is tangent to the stable sub-manifold. As in (Brito and Dixon, 2009, p. 345) equilibrium trajectories starting at an initial position far away from the steady state, will initially be parallel to E_2^s and will converge asymptotically to E_1^s. Therefore, the slope of the two eigenspaces will give a qualitative characterization of the correlation between the variables far away and close to the steady state.

Proposition 2 (**Eigenvalues**). *Let the eigenvalues, λ_1^s and λ_2^s, be real numbers. Then K and n are asymptotically positively correlated if $\mathcal{O} > 0$ or if $\mathcal{O} < 0$ and $\pi_n^* > \gamma\mathcal{Q}$, and they are asymptotically negatively correlated if $\mathcal{O} < 0$ and $\pi_n^* < \gamma\mathcal{Q}$. Far away from the steady state, K and n are negatively correlated if $\mathcal{O} > 0$ or if $\mathcal{O} < 0$ and $\pi_n^* < \gamma\mathcal{Q}$ and are positively correlated if $\mathcal{O} < 0$ and $\pi_n^* > \gamma\mathcal{Q}$.*

5.2 A taxonomy for transitional dynamics

From Propositions 1 and 2 we derive a taxonomy for the main types of transition dynamics. First, we distinguish case **C**, when $\Delta < 0$ and the eigenvalues are complex, from cases **R**, when $\Delta \geq 0$ and the eigenvalues are real. Second, we can split **R** into three different cases: case $\mathbf{R}_1$, if $\mathcal{O} > 0$, in which the the state variables K and n are negatively correlated far away from the steady state and positively correlated asymptotically; case $\mathbf{R}_2$, if $\mathcal{O} < 0$, $\Delta > 0$ and $\pi_n^* > \gamma\mathcal{Q}$, in which the state variables are positively correlated both far away and close to the steady state; and $\mathbf{R}_3$, if $\mathcal{O} < 0$, $\Delta > 0$ and $\pi_n^* < \gamma\mathcal{Q}$, in which they are negatively correlated both away and close to the steady state. In the exogenous labour case in Brito and Dixon (2009) only case $\mathbf{R}_1$ exists.

The sign of the discriminant is an important element in characterizing the different types of transition, by separating complex-eigenvalue generated, non-monotonic,

dynamics from real-eigenvalue, potentially non-monotonic, dynamics. Furthermore, as $\Delta < 0$ only if $\mathcal{O} < 0$ then, in some sense, case **C** separates $\mathbf{R}_2$ from $\mathbf{R}_3$.

The discriminant can be written as a polynomial over $1/\gamma$,

$$\Delta = \frac{1}{(2\gamma)^2} \left\{ \mathcal{Q}^2\gamma^2 + 2(2\mathcal{O} - \pi_n^*\mathcal{Q})\gamma + (\pi_n^*)^2 \right\}$$

We find the following roots of the polynomial equation $\Delta(\gamma) = 0$: if there is only one root

$$\tilde{\gamma} \equiv \left(\frac{\sigma}{C^*}\right)\left(\frac{\pi_n^*}{r_K^*(1 - Y_C^*) + r_C^* Y_K^*}\right) > 0 \tag{33}$$

and, if there are two roots

$$\gamma_{1,2} = (\pi_n^*)^2 \left\{ \pi_n^*\mathcal{Q} - 2\mathcal{O} \pm 2\left[\mathcal{O}(\mathcal{O} - \pi_n^*\mathcal{Q})\right]^{\frac{1}{2}} \right\}^{-1}. \tag{34}$$

Then from (29),(27) and (28): $\Delta < 0$ if and only if $\gamma \in (\gamma_1, \gamma_2)$, $\Delta = 0$ if and only if $\gamma = \tilde{\gamma}$ and $\Delta > 0$ if and only if $\gamma \notin (\gamma_1, \gamma_2)$.

As $\mathcal{O}$ independent from γ, in order to conduct a complete bifurcation analysis we need to choose another parameter or a combination of parameters. A natural choice is σ (the inverse of the elasticity of intertemporal substitution).

Example: benchmark model Let us consider how the parameters (γ, σ) determine the dynamics in the case of the *benchmark model* of the previous section. In the benchmark model, the explicit formula for $\mathcal{O}$ is:

$$\mathcal{O} = \frac{\rho^2(1-\nu)\phi}{(1 - \beta + \eta)^2 n^*}\left(\frac{\nu(1+\eta)^2}{\sigma}\frac{C^*(G)}{C^*(G) + G} - \frac{\phi\beta^2}{\alpha}\right).$$

In order to derive critical conditions for the existence of fluctuations, let us take σ as a first critical parameter and call $\tilde{\sigma}$ to the value of σ such that $\mathcal{O}(\sigma) = 0$, that is

$$\tilde{\sigma} \equiv \left\{ \sigma : \ \sigma\left(\frac{C^*(G,\sigma) + G}{C^*(G,\sigma)}\right) \leq \frac{\nu}{\alpha}\left(\frac{1+\eta}{\beta}\right)^2 \right\}$$

If $G = 0$ then we can determine explicitly

$$\tilde{\sigma} = \alpha\nu\left(\frac{1+\eta}{\beta}\right)^2$$

Then $\mathcal{O} \leq 0$ is equivalent to $\sigma \geq \tilde{\sigma}$, which is a necessary condition for the existence of complex eigenvalues.

The following expressions,

$$\mathcal{Q} = -\frac{\rho^2}{(1-\beta+\eta)\alpha}\left(\frac{C^*(1-\nu)(1-\nu+\eta(1-\alpha))}{n^*\sigma\phi}+\beta\right)<0$$

$$\pi_n^* = \frac{\phi}{n^*}\left(\frac{\beta-\nu(1+\eta)}{1-\beta+\eta}\right)<0$$

and

$$\begin{aligned} 2\mathcal{O}-\pi_n^*\mathcal{Q} &= -\left(\frac{\rho^2}{\alpha n^*(1-\beta+\eta)^2}\right)\{\beta\phi[2\beta(1-\nu)+\alpha+\nu\eta]+ \\ &\quad +\frac{(1-\nu)[\alpha(1+\eta)^2(\nu-1)+\beta\eta(1-\beta+\eta)C^*]}{\sigma n^*}\Bigg\}. \end{aligned}$$

enter the definition of the critical values for γ such that $\Delta = 0$ in equation (34) .

Figure 2 depicts the partitions in the domain of (γ, σ), corresponding to the four cases $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ and $\mathbf{C}$, which are separated by schedules $\mathcal{O} = 0$ and $\Delta = 0$, for the case in which $G = 0$. We have $\mathcal{O} > 0$ below the first curve and $\Delta < 0$ inside the parabola defined by the second curve. The figure also presents the loci where $\mathcal{Q} - \pi^*/\gamma = 0$ and the combinations of parameter values such that it is negative (above) and positive (below).

[Figure 2 here]

In Table 1 we pick parameter values that lead to the dynamics being in each of the four regions, by fixing the technology and preference parameters (α, β, ρ) and by varying all the other parameters $(\sigma, \rho, \xi, \eta, \phi)$.

Table 1: Cobb-Douglas - isoelastic utility case

	γ	ϕ	σ	ξ	η	σ/η	$\mathcal{O}$	Δ	$\mathcal{Q}-\pi^*/\gamma$
$\mathbf{R}_1$	3	0.3	1	0.01	0.5	2	+	+	
$\mathbf{R}_2$	50	0.4	2.2	0.02	0.01	220	-	+	-
$\mathbf{R}_3$	1	0.1	4	0.01	0.3	13.33	-	+	+
$\mathbf{C}$	15	0.2	3	2	0.02	150	-	-	

Common parameters: $\alpha = 0.3$, $\beta = 0.5$ and $\rho = 0.025$

Case $\mathbf{R}_1$ occurs for relatively high elasticity of intertemporal substitution, for any value of γ, given the value of the other parameters. For lower levels of the elasticity

of intertemporal substitution we will tend to have all the other cases. Negative asymptotic correlations between the aggregate stock of capital and the number of firms tend to occur for low levels of the adjustment costs and of the elasticity of intertemporal substitution.γ. If $\xi \to 0$ there is convergence to the exogenous labour case, which corresponds to case $\mathbf{R}_1$. If the disutility of labour ξ increases the complex-eigenvalue case $\mathbf{C}$ becomes likelier.

The dynamics results from a general equilibrium interaction which depends on many factors: technology, preferences, entry costs and so on. However, it is the relationship between consumption and labour supply that creates the possibility of $\mathbf{R}_2$, $\mathbf{R}_3$ and $\mathbf{C}$. The sensitivity of labour supply to consumption depends on the ratio : for a given wage, this is the elasticity of the labour supply to consumption. Hence if this ratio is large, it implies that the labour supply responds a lot to changes in consumption. If we look at Table 1, we can see that this ratio is much higher in cases $\mathbf{R}_2$, $\mathbf{R}_3$ and $\mathbf{C}$ than in case $\mathbf{R}_1$ which is what we would expect. On an empirical level, we would expect a low value of η around 0.2 and a value of σ in the range $2-4$ which gives an elasticity of $10-20$ which is high and consistent with cases other than $\mathbf{R}_1$ [18].

□

5.3 Geometrical characterisation of the solution paths

Next we apply a geometrical method introduced in Brito and Dixon (2009) which allows us to characterise the time paths of the solutions. Although the dynamical system is four-dimensional we use the fact that it has a two-dimensional stable manifold to study qualitatively its dynamics in the two-dimensional space for the pre-determined variables. As in the geometrical theory of differential equations we can characterise, *analytically and qualitatively*, the local dynamics by stratifying the two-dimensional projected space by the (projected) isoclines and the stable eigenspaces. The main results are gathered in Proposition 3 where we show there are four main types of dynamics, depending upon the values of the parameters and the initial values, for the stock of capital and the number of firms. Transitional dynamics can be: oscillatory, hump-shaped for one variable, hump-shaped for two variables, or monotonous.

The solution space to the planner's problem is the stable manifold which is a two-dimensional surface in the four-dimensional domain of (C, e, K, n). If we can find a mapping $(K, n) \mapsto (C, e)$ we can project it into the space (K, n) and the optimal

[18] For instance, if we consider the values for the other parameters for case $\mathbf{R}_1$ in Table 1, with the exception of $\eta = 0.2$ and $\sigma = 3$ and $\phi > 0.4608$, we would have $\mathcal{O} < 0$, which is the value for cases other than $\mathbf{R}_1$.

solutions are obtained from

$$\begin{aligned} \dot{K} &= \dot{K}(K,n) = \dot{K}(C(K,n),K,n) \\ \dot{n} &= \dot{n}(K,n) = e(K,n). \end{aligned}$$

and we can get recursively the optimal trajectories for C and e.

That mapping cannot be determined explicitly. However, we can approximate it locally in the neighborhood of the steady state by the linear system

$$\begin{pmatrix} C - C^* \\ e - e^* \end{pmatrix} = S \begin{pmatrix} K - K^* \\ n - n^* \end{pmatrix}, \; S \equiv \begin{pmatrix} S_{CK} & S_{Cn} \\ S_{eK} & S_{en} \end{pmatrix}. \tag{35}$$

Matrix S is a real matrix and its coefficients satisfy: $S_{CK} > 0$, $S_{en} < 0$, for any value of the parameters, and $S_{Cn} < 0$ and $S_{eK} > 0$ if $\mathcal{O} > 0$, or $S_{Cn} > 0$ and $S_{eK} < 0$ if $\mathcal{O} < 0$. (see Lemma 4 in the Appendix A). Then in case $\mathbf{R}_1$ the coefficients verify $S_{CK} > 0$, $S_{en} < 0$, $S_{Cn} < 0$, and $S_{eK} > 0$ and in cases $\mathbf{R}_2$, $\mathbf{R}_3$ and $\mathbf{C}$ we have $S_{CK} > 0$, $S_{en} < 0$, $S_{Cn} > 0$, and $S_{eK} < 0$. For all cases $\det(S) < 0$.

The projection of the phase diagrams in space (K, n) have generic properties and correspond to sinks for cases $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$, and to a stable node for case $\mathbf{C}$. This is proved in Lemmas 5 and 6 in Appendix A and is illustrated in Figure 3. This Figure displays the projections of the stable eigenspaces and of the isoclines. The projections of the stable eigenspaces E_1^s and E_2^s have slopes given by

$$\left.\frac{dn}{dK}\right|_{E_1^s} = \frac{1}{v_{3,1}^s}, \; \left.\frac{dn}{dK}\right|_{E_2^s} = \frac{1}{v_{3,2}^s}.$$

The transitional dynamics trajectories tend to behave as regards those projections as they do in the phase diagram for two-dimensional ordinary differential equations when the steady state is a sink: trajectories are parallel to E_2^s when they are far away from the steady state and are tangent asymptotically to E_1^s (see Proposition 2).

Figure 3 also displays the two-dimensional projections of the isoclines, which are the geometrical loci where one variable changes the direction of movement. In our case we have: the projections of the tangent to the isoclines $\dot{K} = 0$ and $\dot{n} = 0$, evaluated close to the steady state, have slopes given by

$$\left.\frac{dn}{dK}\right|_{\dot{K}=0} = -\frac{Y_K^* + (Y_C^* - 1)S_{CK}}{Y_n^* + (Y_C^* - 1)S_{Cn}} = \frac{\lambda_2^s v_{3,2}^s - \lambda_1^s v_{3,1}^s}{(\lambda_2^s - \lambda_1^s) v_{3,1}^s v_{3,2}^s}$$

and

$$\left.\frac{dn}{dK}\right|_{\dot{n}=0} = -\frac{S_{eK}}{S_{en}} = \frac{\lambda_2^s - \lambda_1^s}{\lambda_2^s v_{3,1}^s - \lambda_1^s v_{3,2}^s};$$

and the projections of the isoclines $\dot{C}=0$ (or $r(K,n)=\rho$) and $\dot{e}=0$ (or $\pi(K,n)=0$), have slopes given by

$$\begin{aligned}\left.\frac{dn}{dK}\right|_{\dot{C}=0} &= -\frac{r_K^*+r_C^*S_{CK}}{r_n^*+r_C^*S_{Cn}}=\frac{\lambda_2^s v_{1,2}^s-\lambda_1^s v_{1,1}^s}{\lambda_2^s v_{3,1}^s v_{1,2}^s-\lambda_1^s v_{1,1}^s v_{3,2}^s}\\ \left.\frac{dn}{dK}\right|_{\dot{e}=0} &= -\frac{\pi_K^*+\pi_C^*S_{CK}-\rho\gamma S_{eK}}{\pi_n^*+\pi_C^*S_{Cn}-\rho\gamma S_{en}}=\frac{(\lambda_2^s)^2-(\lambda_1^s)^2}{(\lambda_2^s)^2 v_{3,1}^s-(\lambda_1^s)^2 v_{3,2}^s}.\end{aligned}$$

We can also determine and depict the projection for the locus corresponding to a change in the direction of movement of the aggregate product, $\dot{Y}=0$, which has slope

$$\begin{aligned}\left.\frac{dn}{dK}\right|_{\dot{Y}=0} &= -\frac{(Y_K^*+Y_C^*S_{CK})(Y_K^*+(Y_C^*-1)S_{CK})+(Y_n^*+Y_C^*S_{Cn})S_{eK}}{(Y_K^*+Y_C^*S_{CK})(Y_n^*+(Y_C^*-1)S_{Cn})+(Y_n^*+Y_C^*S_{Cn})S_{en}}\\ &= \frac{(1-\nu)Y_n^*(\lambda_2^s-\lambda_1^s)+\gamma n^*(\lambda_1^s(l_1-\mathcal{Q})-\lambda_2^s(l_2-\mathcal{Q}))}{(1-\nu)\gamma Y_n^*\left(n^*(\lambda_2^u-\lambda_1^u)-(1-\nu)(C^*/\sigma)(l_2-l_1)-(1-\nu)Y_n^*(\lambda_1^u(l_2-\mathcal{Q})-\lambda_2^u(l_1-\mathcal{Q}))\right.}\end{aligned}$$

The slopes of the isoclines for K are always positive and the slope for n is positive for case $\mathbf{R}_1$ and is negative for the others. The isocline $\dot{Y}=0$ has a negative slope for case $\mathbf{R}_1$ and is ambiguous for the others[19]. Given any initial value for the two state variables $(K(0),n(0))$, the transition path is initially parallel to E_2^s and converges asymptotically to line E_1^s. If the projection of a trajectory "hits" one of the isoclines, the corresponding variable varies non-monotonically. In case $\mathbf{C}$, depending on how far the initial point is from the steady state, there are several "hits" along the way, for every isocline. and therefore the trajectory is oscillatory for all variables. However, as shown in Brito and Dixon (2009) non-monotonic trajectories may also occur for cases $\mathbf{R}$ associated to sinks. However, in the last cases there can only be one "hit" at one or more isoclines, which produces a hump-shaped trajectory, but only under certain conditions. There are four main types of trajectories when the eigenvalues are real: first, the initial value is lower than the steady state level and the variable increases monotonically over time, second, U hump-shaped non-monotonic trajectories initially decreasing and later increasing over time; third, the initial value is above the steady state level and the variable decreases monotonically over time; and, fourth, an IU hump-shaped non-monotonic trajectories initially increasing and later decreasing over time towards the steady state.

In order to characterize non-monotonicity, we use the same method as in Brito and Dixon (2009). First, we define the following subsets of the domain of (K,n):

[19] We can only sign the denominator for case $\mathbf{R}_2$, which is positive, and the numerator for case $\mathbf{R}_3$, which is negative. For the specific forms and parameters in Figure 3 the slopes are positive for cases in which $\mathcal{O}<0$.

$\mathbf{M}_K = \mathbf{N}_K \cup \mathbf{S}_K$ and $\mathbf{M}_n = \mathbf{E}_n \cup \mathbf{W}_n$ such that

$$\begin{aligned}
\mathbf{N}_K &\equiv \left\{ (K,n):\ n - n^* > \max\left\{ \frac{1}{v^s_{32}} \ , \frac{\lambda^s_2 v^s_{32} - \lambda^s_1 v^s_{31}}{(\lambda^s_2 - \lambda^s_1) v^s_{31} v^s_{32}} \right\} \cdot (K - K^*) \right\} \\
\mathbf{S}_K &\equiv \left\{ (K,n):\ n - n^* < \min\left\{ \frac{1}{v^s_{32}} \ , \frac{\lambda^s_2 v^s_{32} - \lambda^s_1 v^s_{31}}{(\lambda^s_2 - \lambda^s_1) v^s_{31} v^s_{32}} \right\} \cdot (K - K^*) \right\} \\
\mathbf{E}_n &\equiv \left\{ (K,n):\ K - K^* > \max\left\{ v^s_{32} \ , \frac{\lambda^s_2 v^s_{31} - \lambda^s_1 v^s_{32}}{\lambda^s_2 - \lambda^s_1} \right\} \cdot (n - n^*) \right\} \\
\mathbf{W}_n &\equiv \left\{ (K,n):\ K - K^* < \min\left\{ v^s_{32} \ , \frac{\lambda^s_2 v^s_{31} - \lambda^s_1 v^s_{32}}{\lambda^s_2 - \lambda^s_1} \right\} \cdot (n - n^*) \right\}.
\end{aligned}$$

Geometrically set $\mathbf{M}_K$ lies between the projected isocline $\dot{K} = 0$ and the eigenspace E^s_2 and set $\mathbf{M}_n$ lies between the projected isocline $\dot{n} = 0$ and the eigenspace E^s_2.

Then the fundamental result on non-monotonic dynamics follows:

Proposition 3 (**Non-monotonic dynamics**). *Non-monotonic dynamics over space* (K,n). *Let* $(K(0),n(0)) \neq (K^*,n^*)$.

1. *If the eigenvalues are complex then the trajectories are always oscillatory and rotate clockwise for any* $(K(0),n(0))$.

2. *If the eigenvalues are real then: (1) if sets* $\mathbf{M}_K$ *and* $\mathbf{M}_n$ *are disjoint then three cases can occur: (a) if* $(K(0),n(0))$ *does not belong to* $\mathbf{M}_K \cup \mathbf{M}_n$ *then trajectories for both* K *and* n *are monotonic; (b) if* $(K(0),n(0))$ *belongs to* $\mathbf{M}_K$ *then the trajectory for* K *is hump-shaped; (c) if* $(K(0),n(0))$ *belongs to* $\mathbf{M}_n$ *then the trajectory for* n *is hump-shaped; (2) if sets* $\mathbf{M}_K \cap \mathbf{M}_n$ *is not empty then, in addition to the previous three cases, (a), (b) and (c), if* $(K(0),n(0))$ *belongs to* $\mathbf{M}_K \cap \mathbf{M}_n$ *then trajectories for both* K *and* n *are hump-shaped.*

Hence there are sixteen combinations of transition dynamics for variables K and n. Next we highlight the most important cases referring to non-monotonic types of adjustment. We start with the cases involving real eigenvalues. As $\mathbf{M}_K \cup \mathbf{M}_n \subset \mathbb{R}^2_+$ then the complementary set $(\overline{\mathbf{M}_K \cup \mathbf{M}_n})$ is non-empty, which means that there are always solution paths in which both variables K and n converge monotonically. In addition, from Lemma 6 we can see that: if $\mathcal{O} > 0$ (case $\mathbf{R}_1$) then the sets $\mathbf{M}_K$ and $\mathbf{M}_n$ are disjoint but there can be non-empty intersections if $\mathcal{O} < 0$: if $\mathcal{O} < 0$ and $\mathcal{Q} - \pi^*_n/\gamma < 0$ (case $\mathbf{R}_2$) then $\mathbf{E}_n \subset \mathbf{N}_K$ and $\mathbf{W}_n \subset \mathbf{S}_K$; and if $\mathcal{O} < 0$ and $\mathcal{Q} - \pi^*_n/\gamma > 0$ (case $\mathbf{R}_3$) then $\mathbf{N}_K \subset \mathbf{E}_n$ and $\mathbf{S}_K \subset \mathbf{W}_n$.

If $\mathcal{O} > 0$ then, in the transition to the steady state, just one variable may have a humped-shaped trajectory and will change direction only once. This can be seen in the phase diagram for case $\mathbf{R}_1$ depicted in panel NW of Figure 3, which displays

the non-empty and disjoint sets $\mathbf{N}_K$ and $\mathbf{S}_K$, between curves E_2^s and $\dot{K} = 0$, and $\mathbf{E}_n$ and $\mathbf{W}_n$, between curves E_2^s and $\dot{n} = 0$, and the rest of the domain in which no monotonic trajectories are associated. If the initial point is in area $\mathbf{N}_K$ K will follow an IU-hump shape and n will decrease monotonically, if it is in area $\mathbf{S}_K$ K will follow a U-hump shape and n will increase monotonically, if it is in area $\mathbf{W}_n$ K will increase monotonically and n will follow an IU- hump shape adjustment, and if it is in area $\mathbf{E}_n$ K will decrease monotonically and n will follow a U- hump shape adjustment.

Phase diagrams associated to $\mathcal{O} < 0$, also shown in the NE and SW panels of Figure 3 for cases $\mathbf{R}_2$ and $\mathbf{R}_3$, also have the same subsets, with the similar types of dynamics. However, there are two main differences as regards case $\mathbf{R}_1$. First, set $\mathbf{W}_n$ ($\mathbf{E}_n$) is associated with an IU-hump shaped (U-hump shape) adjustment for n. Second, there are non-empty intersections between the previous sets: in the phase diagram for $\mathbf{R}_2$ there are U- or IU-hump-shaped (IU-hump) adjustments for n only if there are U- or IU-hump-shaped adjustments for K (because $\mathbf{E}_n \subset \mathbf{N}_K$ and $\mathbf{W}_n \subset \mathbf{S}_K$), and in the phase diagram for $\mathbf{R}_3$ there are U- or IU-hump-shaped adjustments for K only if there are IU- or U-hump-shaped adjustments for n (because $\mathbf{N}_K \subset \mathbf{E}_n$ and $\mathbf{S}_K \subset \mathbf{W}_n$).

Some representative trajectories are shown in the phase diagrams in Figure 3.

[Figure 3 here]

Phase diagram $\mathbf{R}_3$ is in the of SW panel Figure 3. We can see that all trajectories originating in $\mathbf{N}_K$ and $\mathbf{S}_K$ have both a hump-shaped response first for K and later on for n as well. Along the way they also cross $\dot{Y} = 0$. Take the first case. As both curves E_1^s and E_2^s are negatively sloped, K grows initially while n diminishes, then the capital stock and output will eventually diminish, and finally close to the steady state there will be net entry and the number of firms will increase. Note that as the $\dot{Y} = 0$ curve is less steep than the $\dot{K} = 0$ isocline the hump in capital peaks before the hump in output, or there is no hump for capital if the dynamics originates in $\mathbf{E}_n$ or $\mathbf{W}_n$. In any of this cases the trajectories cross the $\dot{n} = 0$ isocline, indicating an increase in entry close to the steady state. Phase diagram $\mathbf{C}$, associated with the existence of complex eigenvalues, is depicted in Figure 3, SE panel and is qualitatively similar to $\mathbf{R}_3$, however the number of crossings depends upon the distance of the initial point from the steady state. Phase diagram $\mathbf{R}_1$ is depicted in Figure 3, NW panel. Its main features are the following. First, asymptotically the aggregate capital stock and the number of firms are positively related in the neighborhood of the steady state, and will be tangent asymptotically to E_1^s. However, since E_2^s is negatively slopped the initial co-movement is negative. If we take trajectories originating in $\mathbf{N}_K$ a sequence of crossings first of $\dot{Y} = 0$ and then of $\dot{K} = 0$ is possible, meaning that initially output

and the capital stock increase and the number of firms decrease, then output crossed the hump down followed by the capital stock. Asymptotically, the trajectories of n, K, and Y all move together down (this is the same as the exogenous labour case Brito and Dixon (2009)). The number of firms do not have a hump, differently from case $\mathbf{R}_3$. Phase diagram $\mathbf{R}_2$ is depicted in Figure 3, NE panel. It shares some similar properties with the phase diagram $\mathbf{R}_1$, in particular, the positive co-movement of the two state variables and Y in the neighborhood of the steady state. However, the timing of the crossings of $\dot{K} = 0$ and $\dot{Y} = 0$ are the opposite. Far away from the steady state an initial positive correlation is also possible, when the origin is in $\mathbf{W}_n$, as E_2^s has a positive slope.

6 Fiscal policy.

Now we consider unanticipated and permanent variations if public expenditures G which perturbs an economy which is at a steady state.

We consider the locally projected space (K, n) and its perturbation by G

$$\dot{K} = \dot{K}(S_C(K,n), K, n, G) \tag{36}$$
$$\dot{n} = S_e(K,n) \tag{37}$$

Then we can determine if there are hump-shaped or general non-monotonic responses to a permanent unanticipated shock, dG, if (K, n) happens to be initially placed in the subsets $\mathbf{M}_K$ or $\mathbf{M}_n$, relative to the new steady state. We can trace out the location of the initial and by considering the steady state projection

$$dn = \left.\frac{dn}{dK}\right|_{dG} dK = \frac{\partial n/\partial G}{\partial K/\partial G} dK$$

determined from the long run multipliers of the projected system (36)-(37). First we need to determine the long run and impact multipliers.

6.1 Long run and impact responses.

Proposition 4 (**Fiscal policy multipliers**). *Long run multipliers: they are negative for consumption, positive for the stock of capital and the number of firms. Impact multipliers: they are always positive for entry. For consumption they are negative if* $\mathcal{O} < 0$ *and undershoot the long run variation, and they are ambiguous for* $\mathcal{O} > 0$.

Long run multipliers for G (a permanent shock) were described in section : their

explicit expression is given by:

$$\frac{dC^*}{dG} = -\frac{(r_K^*\pi_n^* - r_n^*\pi_K^*)C^*}{\sigma\gamma\mathcal{D}} < 0 \tag{38}$$

$$\frac{de^*}{dG} = 0 \tag{39}$$

$$\frac{dK^*}{dG} = \frac{(r_C^*\pi_n^* - r_n^*\pi_C^*)C^*}{\sigma\gamma\mathcal{D}} > 0 \tag{40}$$

$$\frac{dn^*}{dG} = -\frac{(r_C^*\pi_K^* - r_K^*\pi_C^*)C^*}{\sigma\gamma\mathcal{D}} > 0 \tag{41}$$

This implies the long-run employment and output multipliers:

$$\begin{aligned}\frac{dL^*}{dG} &= L_C\frac{dC^*}{dG} + L_K\frac{dK^*}{dG} + L_n\frac{dn^*}{dG} > 0,\\ \frac{dY^*}{dG} &= Y_K^*\frac{dK^*}{dG} + Y_n^*\,\frac{dn^*}{dG} > 0\end{aligned}$$

which means that $dC^* + dG > 0$ [20].

The Impact multipliers for consumption and entry are

$$\frac{dC(0)}{dG} = \frac{dC^*}{dG} - \left(S_{CK}\frac{dK^*}{dG} + S_{Cn}\frac{dn^*}{dG}\right) \tag{42}$$

$$\frac{de(0)}{dG} = -\left(S_{eK}\frac{dK^*}{dG} + S_{en}\frac{dn^*}{dG}\right) > 0 \tag{43}$$

If $\mathcal{O} < 0$, including the case when we have complex eigenvalues, then from Lemma 4 both coefficients S_{CK} and S_{Cn} associated for the consumption multiplier are positive, whilst S_{eK} and S_{en} associated with entry are negative: therefore we have $\frac{dC(0)}{dG} < \frac{dC^*}{dG} < 0$ and $\frac{de(0)}{dG} > 0$.

If $\mathcal{O} > 0$ and we have real eigenvalues, the coefficients do not have the same sign, and therefore the impact multipliers are potentially ambiguous and the impact effect can be lower or higher than the long-run effect. This possibility is novel and results from the fact that both state variables K and n effect both of the control variables (C, e). Turning first to consumption, the term S_{CK} captures the effect of capital on consumption with the traditional "negative" relationship: as capital moves closer to equilibrium, the gap between the marginal product and the discount rate gets smaller,

[20] If we take the benchmark model, the long run multipliers are: $\frac{dC^*}{dG} = -\frac{\eta C^*}{\sigma Y^* + \eta C^*} < 0$, $\frac{dK^*}{dG} = \frac{\sigma K^*}{\sigma Y^* + \eta C^*} > 0$, $\frac{dn^*}{dG} = \frac{\sigma n^*}{\sigma Y^* + \eta C^*} > 0$, $\frac{dL^*}{dG} = \frac{\sigma L^*}{\sigma Y^* + \eta C^*} > 0$, $\frac{dY^*}{dG} = \frac{\sigma Y^*}{\sigma Y^* + \eta C^*} > 0$, and $\frac{dy^*}{dG} = \frac{dk^*}{dG} = \frac{dl^*}{dG} = 0$.

hence consumption continues to grow but at a slower rate, implying the monotonic trajectory of consumption rising alongside capital accumulation. However, the term S_{Cn} captures the effect of entry on the Euler equation for consumption. More firms can counteract the increase in capital leading to a non-monotonic trajectory in the marginal product of capital (which depends on capital per firm k, not aggregate capital) and hence also non-monotonicity in consumption. This leads to an ambiguous relation between initial consumption and its long-run value if the entry flow is large initially so that capital per firm decreases and the marginal product of capital increases. This sort of *consumption undershooting* response is impossible in the classic Ramsey model where the marginal product depends only on the aggregate capital stock and the impact response of consumption always *overshoots* the long-run response[21].

6.2 Hump-shaped responses.

The short run dynamics is similar to the one generated by equation (30). The existence of non-monotonic adjustment trajectories depend both on the parameters and on the type magnitude of the shock. If we take the after-shock levels of capital and number of firms as K^* and n^* and the pre-shock values as initial values, we can compare the variation with the expressions for the expressions in Proposition 4.

[Figure 4 here]

Proposition 5. *Assume there is a permanent fiscal policy shock. If* $\Delta > 0$ *then the adjustment for* K *is always monotonic and the adjustment for* n *is hump-shaped for case* $\mathbf{R}_3$ *and monotonic for cases* $\mathbf{R}_1$ *and* $\mathbf{R}_2$*. If* $\Delta < 0$ *the adjustment is oscillatory for both variables.*

In Figure 3 we have already shown the four possible cases of short-run dynamics. We see that there is a monotonic adjustment of both (K, n) for cases $\mathbf{R}_1$ and $\mathbf{R}_2$, and a non-monotonic adjustment for cases $\mathbf{C}$ and $\mathbf{R}_3$. Since we are looking at the response of changes to fiscal policy the initial position will lie on the dG line: hence. Figure 4 superposes the dG line. We can see that if we started from an initial steady-state and it is peturbed by dG, the initial steady-state will belong to to sets $\mathbf{W}_n$ and $\mathbf{E}_n$ of the new steady-state. and only the number of firms can adjust non-monotonously. This is an "initial condition effect", and would not hold if we were to consider technology shocks. Whilst for all cases $\mathbf{R}_1$,$\mathbf{R}_2$, $\mathbf{R}_3$ and $\mathbf{C}$ capital will adjust monotonically in response to fiscal policy, capital stock per firm may be non-monotonic in all cases. If

[21] In the classic Ramsey model in (C, K), the saddlepath is upwards sloping so (C, K) move together with an endogenous labour supply.

we turn to output, in case $\mathbf{R}_1$ the $\dot{Y} = 0$ isocline is downward sloping, which implies that output always responds monotonically to fiscal policy. However, the $\dot{Y} = 0$ isocline is upward sloping and flatter than the dG line in the other cases: output will always be non-monotonic in cases $\mathbf{R}_3$ and $\mathbf{C}$ and may be in case $\mathbf{R}_2$.

Phase diagram $\mathbf{R}_3$ is depicted in Figure 3, SW panel. In this case both curves E_1^s and E_2^s are negatively sloped. Since $E_1^s < 0$ there is a *negative co-movement* between the two state variables (K, n) as we approach the steady-state. Furthermore, since dG lies in between the $\dot{K} = 0$ and $\dot{n} = 0$ isoclines, the initial co-movement is *positive*. The trajectories also cross the $\dot{n} = 0$ isocline, indicating an overshooting hump shape for n. The other point to note is that the $\dot{Y} = 0$ line is positively sloped. This means that any trajectory originating from the dG line must pass through $\dot{Y} = 0$: *hence output must have a hump shape response.* Phase diagram $\mathbf{C}$, associated with the existence of complex eigenvalues, is depicted in Figure 4, SE panel and is qualitatively similar to $\mathbf{R}_3$: the reason for this is that all trajectories starting on dG must start off with positive co-movement (in order to get negative co-movement initially, we would need to start from a point the "other side" of the $\dot{K} = 0$ isocline which is off the dG line).

Phase diagram $\mathbf{R}_1$ is depicted in Figure 4, NW panel. Its main features are the following. First, asymptotically the aggregate capital stock and the number of firms are positively related in the neighborhood of the steady state, and will be tangent asymptotically to E_1^s. Since the initial position lies on dG, the initial co-movement will also be positive. Since the trajectory cannot cross any of the $\dot{n}$,= $\dot{K} = \bar{Y} = 0$ lines, along the trajectory of n, K, and Y all move together (this is the same as the exogenous labour case Brito and Dixon (2009)). Phase diagram $\mathbf{R}_2$ is depicted in Figure 4, NE panel. It shares some similar properties with the phase diagram $\mathbf{R}_1$, in particular, the positive co-movement of the two state variables in the neighborhood of the steady state. However, it differs in that trajectories originating from dG may cross the $\dot{Y} = 0$ line. Hence there may be a hump shaped response of output to fiscal policy.

Another way of looking at the dynamics is through the impulse-response functions for the main variables to a fiscal shock in this model, (C, Y, K, L, n, e, k, I) where $k = K/n$ is capital per firm and I is investment in new firms and capital (see Figures 5 to 8). The impulse-response functions for all cases show that capital-per firm k is non-monotonic. However, we also observe that whilst consumption and capital in all cases follow monotonic trajectories (in this benchmark case), this is not so for output Y and employment L. In the case of phase diagram $\mathbf{C}$, where there are two complex eigenvalues and n adjusts non-monotonically, we can see that both output and employment have a hump-shaped response function: output and employment "jump" up, but continue upwards for a time overshooting the long-run effect, peaking and then converging to the new steady-state.

[Figures 5 to 8 here]

Whilst we have not found an example of a non-monotonic trajectory for consumption in the benchmark model, we can determine what it would look like if it were to happen. Since the marginal product of capital will fall as we approach close to the steady-state, we know that consumption and capital will then have a positive co-movement. In the classical Ramsey model, with overshooting, the positive co-movement occurs all along the path. However, if there is undershooting on impact, then consumption will at first fall to below (in the case of a fiscal expansion) the new steady-state, so that it can then increase with capital. There will thus be an initial phase of negative co-movement resulting in a hump shape.

7 Conclusion

In this paper we have analyzed Ricardian fiscal policy in the context of the classic Ramsey model extended to include an endogenous labour supply and a real time entry and exit process. Both of these extensions have been done on their own: entry was developed in Brito and Dixon (2009) and an endogenous labour supply in (Turnovsky, 1995, ch.9). We find that with both extensions together we are able to obtain a much richer set of possible dynamic responses to fiscal policy. Whilst the long-run dynamics are similar to standard models, the short-run can be very different. The dynamics allows for complex eigenvalues so that both state variables (capital and the number of firms) can be non-monotonic.

However, in the analysis of fiscal policy there is an "initial condition" effect that implies that only the number of firms will be non-monotonic as we move from the initial steady-state to the new one. Whilst the response of capital to fiscal policy will be monotonic, we find that there can be a hump shaped response of output in a wide range of cases. This is analyzed in terms of a phase diagram in the state-space of capital and the number of firms, where we are able to define the isocline for output (the combinations of capital and the number of firms at which the time derivative of output is zero). We are able to show that in certain well defined cases the trajectory of the economy in response to a fiscal shock will pass through this isocline and thus exhibit a hump shaped response.

Much of macroeconomics today is thought of in terms of calibrated or estimated numerical models. Whilst theory has its limitations in being limited to shedding light on relatively simple models, theoretical models allow us to understand the general mechanisms underlying macroeconomic relationships in a way that numerical simulation cannot. We have extended the standard Ramsey model by introducing a real time model of entry. Whilst the analysis is complex and four dimensional,

we are able to interpret the results in terms of two dimensional phase diagrams with relatively clear economic interpretation.

The profile of the trajectories generated by a dynamic model is determined by the number of independent dynamic mechanisms, which can be measured by the dimension of the stable manifold. Sometimes calibrated DSGE models are large in terms of the dimension but they are not large in terms of the dimension of the stable manifold. An analytical approach has the advantage of rendering the dynamic mechanisms transparent. In this paper we have learned that in order to have hump-shaped trajectories the dimension of the stable manifold should be two and that the two dynamic driving forces are the decreasing marginal productivity of capital in production and the decreasing marginal profit on entry.

References

Baxter, M. and King, R. G. (1993). Fiscal policy in general equilibrium. *The American Economic Review*, 83(3):pp. 315–334.

Brito, P. and Dixon, H. (2009). Entry and the accumulation of capital: a two state-variable extension to the Ramsey model. *International Journal of Economic Theory*, 5(4):333–357.

Brock, W. A. and Turnovsky, S. J. (1981). The analysis of macroeconomic policies in perfect foresight equilibrium. *Internationl Economic Review*, 22(1):179–209.

Burnside, C., Eichenbaum, M., and Fisher, J. D. M. (2004). Fiscal shocks and their consequences. *Journal of Economic Theory*, 115(1):89 – 117.

Cass, D. (1965). Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies*, 32:233–40.

Christiano, L. J., Eichenbaum, M., and Evans, C. L. (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy*, 13(1):1–45.

Costa, L. and Dixon, H. D. (2011). Fiscal policy under imperfect competition with flexible prices: An overview and survey. *Economics: The Open-Access, Open-Assessment E-Journal*, 5(3).

Das, S. and Das, S. (1996). Dynamics of entry and exit of firms in the presence of entry adjustment costs. *International Journal of Industrial Organization*, 15:217–241.

Datta, B. and Dixon, H. (2002). Technological change, entry, and stock market dynamics: an analysis of transition in a monopolistic economy. *American Economic Review*, 92(2):231–235.

Dixon, H. (1987). A simple model of imperfect competition with Walrasian features. *Oxford Economic Papers*, 39(1):pp. 134–160.

Jaimovich, N. and Floetotto, M. (2008). Firm dynamics, markup variations, and the business cycle. *Journal of Monetary Economics*, 55:1238–1252.

Koopmans, T. (1965). On the concept of optimal economic growth. In *The Econometric Approach to Development Planning*. Pontificiae Acad. Sci., North-Holland.

Linneman, L. (2001). The price index effect, entry, and endogenous markups in a macroeconomic model of monopolistic competition. *Journal of Macroeconomics*, 23:441–458.

Mankiw, N. G. (1988). Imperfect competition and the Keynesian cross. *Economics Letters*, 26(1):7 – 13.

Melitz, M. J. (2003). The impact of trade on intra-industry reallocations and aggregate industry productivity. *Econometrica*, 71(6):1695–1725.

Mountford, A. and Uhlig, H. (2009). What are the effects of fiscal policy shocks? *Journal of Applied Econometrics*, 24(6):960–992.

Ramsey, F. P. (1928). A mathematical theory of saving. *Economic Journal*, 38(Dec):543–59.

Smets, F. and Wouters, R. (2003). An estimated dynamic stochastic general equilibrium model of the Euro area. *Journal of the European Economic Association*, 1(5):1123–1175.

Startz, R. (1989). Monopolistic competition as a foundation for Keynesian macroeconomic models. *The Quarterly Journal of Economics*, 104(4):pp. 737–752.

Takayama, A. (1994). *Analytical Methods in Economics*. Harvester Wheatsheaf.

Turnovsky, S. (1990). The effects of taxes and dividend policy on capital accumulation and macroeconomic behaviour. *Jounal of Economic Dynamics and Control*, 14(3-4):491–521.

Turnovsky, S. (1995). *Methods of Macroeconomic Dynamics*. MIT Press.

Turnovsky, S. and Sen, P. (1991). Fiscal policy, capital accumulation, and debt in an open economy. *Oxford Economic Papers*, 43(1):1–24.

Woodford, M. (2011). Simple analytics of the government expenditure multiplier. *American Economic Journal: Macroeconomics*, 3(January):1–35.

Yang, X. and Heijdra, B. J. (1993). Monopolistic competition and optimum product diversity: comment. *The American Economic Review*, 83(1):295–301.

A Appendix: proofs and auxiliary results

Partial derivatives of function (2) The first partial derivetione of function (2) are

$$Y_K = AF_k\left(\frac{K}{n},\frac{L}{n}\right),\ Y_L = AF_l\left(\frac{K}{n},\frac{L}{n}\right),\ Y_n = (1-\nu)AF\left(\frac{K}{n},\frac{L}{n}\right) - \phi$$

where $Y_K = \partial Y/\partial K$, etc. The last equation results from the application of Euler's Lemma. The second partial derivatives are:

$$Y_{KK} = \frac{AF_{kk}}{n} < 0,\ Y_{KL} = Y_{LK} = \frac{AF_{kl}}{n} > 0,\ Y_{LL} = \frac{AF_{ll}}{n} < 0$$

and

$$Y_{nK} = Y_{Kn} = (1-\nu)\frac{AF_k}{n} > 0,\ Y_{nL} = Y_{Ln} = (1-\nu)\frac{AF_l}{n} > 0,\ Y_{nn} = -\frac{\nu(1-\nu)AF}{n} < 0.$$

Partial derivatives of function (7) Assuming that $U_{CL} \neq 0$ the optimal labour supply function has partial derivatives:

$$\begin{aligned} L_C &= -\frac{Y_L U_{CC}}{U_C Y_{LL} + U_{LL}} < 0 \\ L_K &= -\frac{Y_{LK} U_C}{U_C Y_{LL} + U_{LL}} > 0 \\ L_n &= -\frac{Y_{Ln} U_C}{U_C Y_{LL} + U_{LL}} > 0. \end{aligned}$$

Partial derivations of functions (8)**,** (9) **and** (10) The partial derivatives for the optimal rate of return are

$$\begin{aligned} r_C &= Y_{KL} L_C < 0 \\ r_K &= Y_{KK} + Y_{KL} L_K = \frac{(Y_{KK}Y_{LL} - Y_{KL}Y_{LK})U_C + Y_{KK}U_{LL}}{U_C Y_{LL} + U_{LL}} < 0 \\ r_n &= Y_{Kn} + Y_{KL} L_n > 0 \end{aligned}$$

we use the following facts (resulting from the concavity of the production function)

$$Y_{KK}Y_{LL} - Y_{KL}Y_{LK} = \left(\frac{A}{n}\right)^2 (F_{kk}F_{ll} - F_{kl}^2) > 0$$

The partial derivatives for the optimal profits are

$$\begin{aligned}\pi_C &= Y_{nL}L_C < 0\\ \pi_K &= Y_{nK} + Y_{nL}L_K > 0\\ \pi_n &= Y_{nn} + Y_{nL}L_n = \frac{(Y_{nn}Y_{LL} - Y_{nL}Y_{Ln})U_C + Y_{nn}U_{LL}}{U_C Y_{LL} + U_{LL}} < 0\end{aligned}$$

we use the following facts (resulting from the concavity and rhe homogeneity of degree ν of the production function)

$$Y_{nn}Y_{LL} - Y_{nL}Y_{Ln} = \left(\frac{A}{n}\right)^2 \frac{K}{n}((\nu-1)F_k F_{ll} - (\nu-1)F_l F_{lk}) = \left(\frac{AK}{n^2}\right)^2 (F_{kk}F_{ll} - F_{kl}^2) > 0$$

The partial derivatives for the optimal output are

$$\begin{aligned}Y_C &= Y_L L_C = \hat{w}L_C < 0\\ Y_K &= Y_K + Y_L L_K = \hat{r} + \hat{w}L_K > 0\\ Y_n &= Y_n + Y_L L_n = \hat{\pi} + \hat{w}L_n\end{aligned}$$

which is ambiguous, but $\hat{Y}_n > 0$ at the steady state because $\pi^* = 0$.

Propositions 1 and 2: auxiliary results and proofs

Lemma 1. *Let $\mathcal{D} = \det(J)$, $\mathcal{T} = M_2 - \rho^2$, where M_2 is the sum of the principal minors of order 2 of J Then, the eigenvalues of Jacobian J, are*

$$\lambda_1^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2}\right)^2 - \frac{\mathcal{T}}{2} - \Delta^{\frac{1}{2}}\right]^{\frac{1}{2}}, \quad \lambda_2^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2}\right)^2 - \frac{\mathcal{T}}{2} + \Delta^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{44}$$

where the discriminant is $\Delta \equiv \left(\frac{\mathcal{T}}{2}\right)^2 - \mathcal{D}$.

Proof. The characteristic polynomial for a four dimensional system of ODEs is:

$$c(\lambda) = \lambda^4 - M_1\lambda^3 + M_2\lambda^2 - M_3\lambda + M_4$$

where M_j is the sum of the principal minors of order $j = 1, \ldots, 4$. In infinite horizon optimal control problems with two state variables the following constraints on the value of the sums of the principal minors hold: $M_2 = 2\rho$, which is the trace, and $M_3 = \rho(M_2 - \rho^2)$. Therefore, there are only two independent coefficients involving the sums of the principal minors of even order: $\mathcal{T} \equiv M_2 - \rho^2$ and $\mathcal{D} \equiv M_4 = \det(J)$. Then the characteristic polynomial can be written as

$$c(\lambda) = \left(\frac{\rho}{2}\right)^4 \left[\omega^2 + \left(\left(\frac{\rho}{2}\right)^{-2}\mathcal{T} - 2\right)\omega + \left(\left(\frac{\rho}{2}\right)^{-4} M_4 - \left(\frac{\rho}{2}\right)^{-2}\mathcal{T} + 1\right)\right],$$

where $\omega \equiv \left(\lambda - \frac{\rho}{2}\right)^2 \left(\frac{\rho}{2}\right)^{-2}$. Then we get the eigenvalues as $\left(\lambda - \frac{\rho}{2}\right)_{1,2}^{s,u} = \mp \left[\left(\frac{\rho}{2}\right)\omega_{1,2}\right]^{\frac{1}{2}}$. Then, by solving the polynomial equation $c(\lambda) = 0$, we get the eigenvalues in equation (44). □

Lemma 2. *If the discriminant is non-negative, $\Delta \geq 0$, then the four eigenvalues are real and verify*

$$\lambda_2^s \leq \lambda_1^s < 0 < \lambda_1^u \leq \lambda_2^u.$$

If the discriminant is negative, $\Delta < 0$, then there are two pairs of complex conjugate eigenvalues

$$\lambda_1^s = \theta_s + \vartheta i,\ \lambda_2^s = \theta_s - \vartheta i, \lambda_1^u = \theta_u - \vartheta i, \lambda_2^u = \theta_u + \vartheta i$$

where $i^2 = -1$, and $\theta_s < 0 < \theta_u$:

$$\theta_{s,u} = \frac{\rho}{2} \mp \left(\frac{1}{2}\right)^{1/2} \left\{ \left[\left(\frac{\rho}{2}\right)^4 - \left(\frac{\rho}{2}\right)^2 \mathcal{T} + \mathcal{D} \right]^{1/2} + \left(\frac{\rho}{2}\right)^2 - \frac{\mathcal{T}}{2} \right\}^{1/2}$$

and

$$\vartheta = \left(\frac{1}{2}\right)^{1/2} \left\{ \left[\left(\frac{\rho}{2}\right)^4 - \left(\frac{\rho}{2}\right)^2 \mathcal{T} + \mathcal{D} \right]^{1/2} - \left(\frac{\rho}{2}\right)^2 + \frac{\mathcal{T}}{2} \right\}^{1/2} > 0.$$

Furthermore, the following relationships hold:

1. *$\lambda_i^s + \lambda_i^u = \rho > 0$ (if they are complex we have $\alpha_s + \alpha_u = \rho$);*
2. *$\lambda_1^s \lambda_1^u = l_1$ and $\lambda_2^s \lambda_2^u = l_2$,*

$$l_1 = \begin{cases} \frac{\mathcal{T}}{2} + \Delta^{\frac{1}{2}}, & \text{if } \Delta \geq 0 \\ \frac{\mathcal{T}}{2} + (|\Delta|)^{\frac{1}{2}} i, & \text{if } \Delta < 0 \end{cases},\ l_2 = \begin{cases} \frac{\mathcal{T}}{2} - \Delta^{\frac{1}{2}}, & \text{if } \Delta \geq 0 \\ \frac{\mathcal{T}}{2} - (|\Delta|)^{\frac{1}{2}} i, & \text{if } \Delta < 0 \end{cases}$$

   verify $l_2 \leq l_1 < 0$ are real if $\Delta \geq 0$ and are complex conjugate with a negative real part, if $\Delta < 0$

3. *$l_1 + l_2 = \mathcal{T} < 0$ and $l_1 l_2 = \mathcal{D} > 0$ are real for any type of eigenvalue, real or complex.*

Proof. **Proof of Lemma 2** The first part of the Lemma is obvious from Lemma 1. When $\Delta < 0$ we use the fact that given the complex number $y = a + bi$ it is known that $\sqrt{y} = \sqrt{a + bi} = 1/2 \left\{ \sqrt{r + a} + \text{sign}(b)\sqrt{r - a}i \right\}$ where $r \equiv \sqrt{a^2 + b^2}$. The rest of the proof is obvious. □

Proof. **Proof of Proposition 1** We start by proving that the conditions of Lemma 1 on the sums of the principal minors hold.

First, as $\frac{C^*}{\sigma}Y_{KL}L_C + Y_L L_K = 0$ then $\frac{C^* r_C^*}{\sigma} + Y_K^* = \rho$ which implies that the trace is $M_1 = \rho + \frac{C^* r_C^*}{\sigma} + Y_K^* = 2\rho$. Second, the sum of the principal minors of order two and three are

$$M_2 = \rho\left(\frac{C^* r_C^*}{\sigma} + Y_K^*\right) + \frac{C^*}{\sigma}\left(r_C^* Y_K^* - r_K^*(Y_C^* - 1)\right) + \frac{\pi_n^*}{\gamma}$$

and

$$M_3 = \rho(M_2 - \rho^2) - \frac{1}{\gamma}\left(\frac{C^*}{\sigma} r_n^* \pi_C^* + \pi_K^* Y_n^*\right)$$

But as $\frac{C^*}{\sigma} r_n^* \pi_C^* + \pi_K^* Y_n^* = 0$ then $M_3 = \rho(M_2 - \rho^2)$. Then, we can use Lemma 1 to determine the eigenvalues, noting that

$$\mathcal{T} \equiv M_2 - \rho^2 = \frac{C^*}{\sigma}\left(r_C^* Y_K^* - r_K^* Y_C^*\right) + \frac{C^* r_K^*}{\sigma} + \frac{\pi_n^*}{\gamma} \tag{45}$$

and

$$\mathcal{D} \equiv M_4 = \frac{C^*}{\sigma\gamma}\left[(1 - Y_C^*)(\pi_n^* r_K^* - \pi_K^* r_n^*) + Y_K^*(\pi_n^* r_C^* - \pi_C^* r_n^*) - Y_n^*(\pi_K^* r_C^* - \pi_C^* r_K^*)\right] \tag{46}$$

In order to obtain the dimension of the stable manifold, we have to determine the signs of $\mathcal{T}$ and $\mathcal{D}$. From the signs of the partial derivatives, $r_K^* < 0$, $\pi_n^* < 0$ and as $r_C^* Y_K^* - r_K^* Y_C^* = L_C(Y_{LK}Y_K - Y_{KK}Y_L) < 0$ then $\mathcal{T} < 0$. As

$$\mathcal{D} = \frac{C^*}{\sigma\gamma}\left(\frac{L^*}{n^*}\right)\left(\frac{1}{U_C Y_{LL} + U_{LL}}\right)\left(AFY_L U_{CC} + U_{LL}\frac{L^*}{n^*}\right)\left(Y_{KK}Y_{LL} - Y_{KL}Y_{LK}\right) > 0 \tag{47}$$

From the fundamental theorem of algebra (i.e., on the relationship between the eigenvalues and the coefficients of the characteristic polynomial) it is easy to see that there are two eigenvalues with negative real parts (and always two eigenvalues with positive real parts): $\mathcal{D} > 0$ implies that there are no zero eigenvalues and that the number of eigenvalues with negative real part is even; but as $M_1 > 0$ they should be reduced to zero or two; and, finally as $M_3 = \rho\mathcal{T} < 0$ then we should have one pair of negative eigenvalues (M_2 is ambiguous, in general).

Next, we apply the results in Lemma 2. For finding the conditions under which those three cases can occur let $z \equiv 1/\gamma$. The "inner" discriminant can be written as a quadratic function of z:

$$\Delta(z) = \left(\frac{\pi_n^*}{2}\right)^2\left[z^2 + 2\left(\frac{2\mathcal{O} - \pi_n^*\mathcal{Q}}{(\pi_n^*)^2}\right)z + \left(\frac{\mathcal{Q}}{\pi_n^*}\right)^2\right]. \tag{48}$$

Then $\Delta(z) = 0$ if and only if $\delta(z) = z^2 + a_1 z + a_0 = 0$, such that $z > 0$, where $a_1 \equiv 2\left(\frac{2\mathcal{O}-\pi_n^*\mathcal{Q}}{(\pi_n^*)^2}\right)$ and $a_0 \equiv \left(\frac{\mathcal{Q}}{\pi_n^*}\right)^2$. The sign of a_1 is ambiguous but $a_0 > 0$.

Function $\delta(z)$ has a minimum for $\tilde{z} > 0$ such that $\delta^{'}(\tilde{z}) = 0$. If $\delta(\tilde{z}) = 0$ then if $z = \tilde{z}$ then there will be multiple real eigenvalues and if $z \neq \tilde{z}$ there will be real distinct eigenvalues. If $\delta(\tilde{z}) < 0$ this means that the polynomial has two roots, z_1 and z_2 such that $z_1 > z_2 > 0$. Then there are related values for γ, $\gamma_1 < \gamma_2$. This implies that $\Delta(z) < 0$ for $z_1 > z > z_2$ and the eigenvalues, λ_i^j for $j = s, u$ and $i = 1, 2$ will be complex conjugate.

Therefore, we have only to investigate if there are any positive real roots for the polynomial equation $\delta(z) = 0$. Its roots are

$$\begin{aligned} z_1 &= (\pi_n^*)^{-2}\left\{\pi_n^*\mathcal{Q} - 2\mathcal{O} + 2\left[\mathcal{O}(\mathcal{O} - \pi_n^*\mathcal{Q})\right]^{\frac{1}{2}}\right\}, \\ z_2 &= (\pi_n^*)^{2}\left\{\pi_n^*\mathcal{Q} - 2\mathcal{O} - 2\left[\mathcal{O}(\mathcal{O} - \pi_n^*\mathcal{Q})\right]^{\frac{1}{2}}\right\}. \end{aligned}$$

Observe that as $a_0 > 0$ then the two roots, z_1, z_2, have the same sign, but may be real or complex.

Let $\Delta_z \equiv \mathcal{O}(\mathcal{O} - \pi_n^*\mathcal{Q})$. Three cases may occur:

(a) $\Delta_z = 0$ if $\mathcal{O} = 0$ or $\mathcal{O} = \pi_n^*\mathcal{Q} > 0$. In each of those cases there is only one multiple solution for $\delta(z) = 0$. In the first case, we get $z = \tilde{z} = \mathcal{Q}/\pi_n^* > 0$, given in equation (33). In the second case we get $z = -\mathcal{Q}/\pi_n^* < 0$ which does not belong to the domain of z.

(b) $\Delta_z < 0$ if $\mathcal{O} - \pi_n^*\mathcal{Q} < 0 < \mathcal{O}$ (another formally possible case where $\mathcal{O} - \pi_n^*\mathcal{Q} > 0 > \mathcal{O}$ is not admissible because $\pi_n^*\mathcal{Q} > 0$). In this case z_1 and z_2 are complex conjugate and therefore do not belong to the domain of z.

(c) $\Delta_z > 0$ if $\mathcal{O} < 0$, implying that $\mathcal{O} - \pi_n^*\mathcal{Q} < 0$, or if $\mathcal{O} > \mathcal{O} - \pi_n^*\mathcal{Q} > 0$. In the first case, as a_1 is positive because $\pi_n\mathcal{Q} - 2\mathcal{O} > 0$, and as a_0 is positive, then the two roots for $\delta(z) = 0$ are real and positive. In the second case, as $\pi_n^*\mathcal{Q} - \mathcal{O} < -\mathcal{O} < 0$ then $a_1 < 0$ and the two roots are real but negative, and, therefore do not belong to the domain of z.

Summing up, $\mathcal{O} < 0$ is a necessary condition for $\Delta(z) < 0$ because $z > 0$. This case occurs if and only if $z_1 < z < z_2$. In all the other cases the eigenvalues are real.

The rest of the proof follows from Lemma 2. □

Lemma 3. *If there is no multiplicity then is the eigenvector associated to eigenvalue* λ_j^s, $j = 1, 2$ *of matrix* J *is*

$$V_j^s = (v_{1,j}^s, \lambda_j^s, v_{3,j}^s, 1)^\top, \; j = 1, 2 \tag{49}$$

where

$$v^s_{1,1} \equiv \frac{\frac{C^*}{\sigma}\left(r^*_K Y^*_n - r^*_n Y^*_K + r^*_n \lambda^s_1\right)}{l_2 - \frac{\pi^*_n}{\gamma}},\ v^s_{3,1} \equiv \frac{\frac{C^*}{\sigma}\left((Y^*_C - 1)r^*_n - Y^*_n r^*_C\right) + Y^*_n \lambda^s_1}{l_2 - \frac{\pi^*_n}{\gamma}} \quad (50)$$

$$v^s_{1,2} \equiv \frac{\frac{C^*}{\sigma}\left(r^*_K Y^*_n - r^*_n Y^*_K + r^*_n \lambda^s_2\right)}{l_1 - \frac{\pi^*_n}{\gamma}},\ v^s_{3,2} \equiv \frac{\frac{C^*}{\sigma}\left((Y^*_C - 1)r^*_n - Y^*_n r^*_C\right) + Y^*_n \lambda^s_2}{l_1 - \frac{\pi^*_n}{\gamma}} \quad (51)$$

Proof. Eigenvector V^i_j is obtained as the non-zero solution of the homogeneous system $(J - \lambda^i_j I_4)V^i_j = 0$,

$$V^i_j = \left[\frac{\frac{C^*}{\sigma}\left(r^*_K Y^*_n - r^*_n(Y^*_K - \lambda^i_j)\right)}{(\frac{C^*}{\sigma} - \lambda^i_j)(r^*_K - \lambda^i_j) - \frac{C^*}{\sigma} r^*_K (Y^*_C - 1)}, \lambda^i_j, \frac{\frac{C^*}{\sigma} r^*_n (Y^*_C - 1) - Y^*_n \left(\frac{C^*}{\sigma} r^*_C - \lambda^i_j\right)}{(\frac{C^*}{\sigma} r^*_C - \lambda^i_j)(r^*_K - \lambda^i_j) - \frac{C^*}{\sigma} r^*_K (Y^*_C - 1)}, 1\right],$$

for $i = s, u$ and $j = 1, 2$. But $(\frac{C^*}{\sigma} - \lambda^i_j)(r^*_K - \lambda^i_j) - \frac{C^*}{\sigma} r^*_K (Y^*_C - 1) = (\lambda^i_j)^2 - \lambda^i_j(\frac{C^*}{\sigma} r^*_C + Y^*_K) + \frac{C^*}{\sigma} r^*_C Y^*_K - \frac{C^*}{\sigma} r^*_K (Y^*_C - 1) = \lambda^i_j(\lambda^i_j - \rho) + \mathcal{T} - \frac{\pi^*_n}{\gamma} = -l_i + l_1 + l_2 - \frac{\pi^*_n}{\gamma}$, because $\mathcal{T} = l_1 + l_2$ and $\lambda^s_i + \lambda^u_i = \rho$. Then, for V^s_1 and V^u_1 the denominator becomes $l_2 - \frac{\pi^*_n}{\gamma}$ and for V^s_2 and V^u_2 the denominator becomes $l_1 - \frac{\pi^*_n}{\gamma}$. Then equation (49) results. If the eigenvalues are complex, then $l_1 = \lambda^s_1 \lambda^u_1 = (\theta_s + \vartheta i)(\theta_u - \vartheta i) = \theta_s \theta_u + \vartheta^2 - \vartheta(\theta_u - \theta_s)i = \mathcal{T}/2 + |\Delta|^{1/2} i$ and l_2 is its complex conjugate $l_2 = \mathcal{T}/2 - |\Delta|^{1/2} i$. □

Proof. **Proof of Proposition 2**

We start with the determination of the slopes of the (K, n) projections of the eigenspaces in space (K, n), for the $\Delta > 0$ case: $\frac{dn}{dK}\Big|_{E^s_1} = 1/v^s_{3,1}$ and $\frac{dn}{dK}\Big|_{E^s_2} = 1/v^s_{3,2}$. The numerators of the $v^s_{1,i}$ and $v^s_{3,i}$, for $i = 1, 2$, in equations (50)-(51), are negative. In order to determine the sign of the denominators observe that $l_1 - \frac{\pi^*_n}{\gamma} = \frac{1}{2}\left(\mathcal{Q} - \frac{\pi^*_n}{\gamma} + \Delta^{\frac{1}{2}}\right)$ and $l_2 - \frac{\pi^*_n}{\gamma} = \frac{1}{2}\left(\mathcal{Q} - \frac{\pi^*_n}{\gamma} - \Delta^{\frac{1}{2}}\right)$ where $\mathcal{Q} < 0$, $\pi^*_n < 0$, and $\Delta = \left(\mathcal{Q} - \frac{\pi^*_n}{\gamma}\right)^2 + 4\frac{\mathcal{O}}{\gamma}$. As the sign of $\mathcal{O}$ is ambiguous then the signs of Δ, $l_1 - \frac{\pi^*_n}{\gamma}$ and $l_2 - \frac{\pi^*_n}{\gamma}$ are also ambiguous. However, if we want to guarantee that $\Delta > 0$ we readily conclude that, three cases may occur: (1) if $\mathcal{O} > 0$ then the second member is larger in absolute value and therefore $l_1 - \frac{\pi^*_n}{\gamma} > 0 > l_2 - \frac{\pi^*_n}{\gamma}$. To see this note that we have $\left[\left(\mathcal{Q} - \frac{\pi^*_n}{\gamma}\right)^2 + 4\frac{\mathcal{O}}{\gamma}\right]^{\frac{1}{2}} > \left|\mathcal{Q} - \frac{\pi^*_n}{\gamma}\right| \geq \mathcal{Q} - \frac{\pi^*_n}{\gamma}$. Then $v^s_{1,1} > 0$ and $v^s_{3,1} > 0$, and $v^s_{1,2} < 0$ and $v^s_{3,2} < 0$. If $\mathcal{O} < 0$ and $\Delta > 0$, the first member is larger in absolute value and therefore the sign of the denominators are equal and are the same as the

sign of $\mathcal{Q} - \frac{\pi_n^*}{\gamma}$. Therefore, we have two cases: (2) if $\mathcal{O} < 0$, and $\mathcal{Q} - \frac{\pi_n^*}{\gamma} < 0$, then $v_{1,1}^s > 0$, $v_{3,1}^s > 0$, $v_{1,2}^s > 0$ and $v_{3,2}^s > 0$; or (3) if $\mathcal{O} < 0$, and $\mathcal{Q} - \frac{\pi_n^*}{\gamma} > 0$ then $v_{1,1}^s < 0$, $v_{3,1}^s < 0$, $v_{1,2}^s < 0$ and $v_{3,2}^s < 0$.

Then, the signs of $\frac{dn}{dK}\Big|_{E_1^s}$ and $\frac{dn}{dK}\Big|_{E_2^s}$ are clear. In addition, we have a relationship between those slopes

$$\frac{dn}{dK}\bigg|_{E_1^s} - \frac{dn}{dK}\bigg|_{E_2^s} = \frac{\frac{C^*}{\sigma} r_n^*(l_1 - l_2) + Y_n^*(\lambda_2^s - \lambda_1^s)(\mathcal{Q} - \lambda_1^u \lambda_2^u)}{(\frac{C^*}{\sigma} r_n^* + Y_n^* \lambda_1^u)(\frac{C^*}{\sigma} r_n^* + Y_n^* \lambda_2^u)} > 0$$

for any case of the three previous cases. Then: 1. If $\mathcal{O} > 0$ then $\frac{dn}{dK}\Big|_{E_1^s} > 0$ and $\frac{dn}{dK}\Big|_{E_2^s} < 0$; 2. If $\mathcal{O} < 0$ and $\pi_n^* > \gamma \mathcal{Q}$ then $\frac{dn}{dK}\Big|_{E_1^s} > \frac{dn}{dK}\Big|_{E_2^s} > 0$; 3. If $\mathcal{O} < 0$ and $\pi_n^* < \gamma \mathcal{Q}$ then $0 > \frac{dn}{dK}\Big|_{E_1^s} > \frac{dn}{dK}\Big|_{E_2^s}$. □

Proposition 3: auxiliary results and proof

Lemma 4. *(Local projection of (C, e) on the stable eigenspace). The linear space which is tangent to the stable manifold is given by*

$$E^s \equiv \left\{(C, e, K, n) : \ (C - C^*, e)^\top \ = S(K - K^*, n - n^*)^\top\right\}$$

where

$$S_{CK} = \frac{\frac{C^*}{\sigma}(r_K^* Y_n^* - r_n^* Y_K^*)(l_1 - l_2) + \frac{C^*}{\sigma} r_n^*(\lambda_1^s - \lambda_2^s)(\mathcal{Q} - \lambda_1^s \lambda_2^s)}{\frac{C^*}{\sigma} r_n^*(l_2 - l_1) + Y_n^*(\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u)} > 0 \tag{52}$$

$$S_{Cn} = \frac{\frac{C^*}{\sigma} r_n^*(\lambda_1^s - \lambda_2^s)\mathcal{O}/\gamma}{\frac{\pi_K^*}{\gamma}\ \left[\frac{C^*}{\sigma} r_n^*(l_2 - l_1) + Y_n^*(\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u)\right]} \tag{53}$$

$$S_{eK} = \frac{(\lambda_2^s - \lambda_1^s)\mathcal{O}/\gamma}{\frac{C^*}{\sigma} r_n^*(l_2 - l_1) + Y_n^*(\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u)} \tag{54}$$

$$S_{en} = \frac{\frac{C^*}{\sigma}\left((Y_C^* - 1) r_n^* - Y_n^* r_C^*\right)(\lambda_2^s - \lambda_1^s)(\lambda_1^s \lambda_2^s - \frac{\pi_n^*}{\gamma}) + Y_n^* \lambda_1^s \lambda_2^s (l_1 - l_2)}{\frac{C^*}{\sigma} r_n^*(l_2 - l_1) + Y_n^*(\lambda_2^u - \lambda_1^u)(\mathcal{Q} - \lambda_1^u \lambda_2^u)} < 0 \tag{55}$$

If $\mathcal{O} > 0$ then $S_{Cn} < 0$, $S_{eK} > 0$ and $S_{CK}/S_{Cn} < S_{eK}/S_{en} < 0$. If $\mathcal{O} < 0$ then $S_{Cn} > 0$, $S_{eK} < 0$ and $S_{CK}/S_{Cn} > S_{eK}/S_{en} > 0$.

Proof. Following the same method as in Brito and Dixon (2009) we prove this by solving the last two equations of (30) for the two constants, z_1^s and z_2^s as

$$z_1^s = \left(\frac{K(t) - K^* - v_{3,2}^s(n(t) - n^*)}{v_{3,1}^s - v_{3,2}^s}\right) e^{-\lambda_1^s t}$$

$$z_2^s = \left(\frac{-(K(t) - K^*) + v_{3,1}^s(n(t) - n^*)}{v_{3,1}^s - v_{3,2}^s}\right) e^{-\lambda_2^s t}$$

and substitute in the solutions for C and e in equation (30) to get $(C(t)-C^*, e(t)-e^*)^\top = S(K(t)-K^*, n(t)-n^*)^\top$ where

$$S \equiv \frac{1}{v^s_{3,1}-v^s_{3,2}} \begin{pmatrix} v^s_{1,1}-v^s_{1,2} & v^s_{1,2}v^s_{3,1}-v^s_{3,2}v^s_{1,1} \\ \lambda^s_1-\lambda^s_2 & \lambda^s_2 v^s_{3,1}-\lambda^s_1 v^s_{3,2} \end{pmatrix}.$$

After some algebra we get equations (52) to (55). We already know, from the concavity properties of the production and utility functions, that $r^*_n > 0$, $Y^*_n > 0$, $\pi^*_K > 0$ and $\pi^*_n < 0$. Also $r^*_K Y^*_n - r^*_n Y^*_K < 0$ and $(Y^*_C - 1)r^*_n - Y^*_n r^*_C < 0$ because

$$Y^*_C r^*_n - Y^*_n r^*_C = L_C \left(Y_L(Y_{Kn} + Y_{KL}L_n) - Y_{KL}Y_L L_n\right) = L_C Y_L Y_{Kn} < 0.$$

If the eigenvalues are real then all the denominators are negative because $l_2 - l_1 < 0$, $\mathcal{Q} < 0$, and because of the relationships between the eigenvalues, $\lambda^s_2 < \lambda^s_1 < 0 < \lambda^u_1 < \lambda^u_2$. The numerator of S_{CK} is negative and the numerator of S_{en} is positive. At last, we see that the signs of the numerators of S_{Cn} and S_{eK} are symmetric. Note that, if $\mathcal{O} > 0$ then $l_2 - \mathcal{Q} < 0 < l_1 - \mathcal{Q}$ and if $\mathcal{O} < 0$ then the signs of $l_2 - \mathcal{Q}$ and $l_1 - \mathcal{Q}$ are the same. On the other hand $(l_1 - \mathcal{Q})(l_2 - \mathcal{Q}) = -\mathcal{O}/\gamma$. Then the sign of S_{eK} is the same as $\mathcal{O}$ and is the symmetric of S_{Cn}.

If the eigenvalues are complex we can prove that all the coefficients are of type $(ai)/(bi) = a/b$ and they are also real. The denominator is a complex number with zero real part and a negative coefficient to the complex part $l_2 - l_1 = -2|\Delta|^{1/2}i$ and $(\lambda^u_2 - \lambda^u_1)(\mathcal{Q} - \lambda^u_1\lambda^u_2) = (\mathcal{Q} - (\theta_u)^2 - \vartheta^2)2\vartheta i$ have both negative coefficients. The numerator of S_{CK} is also complex number with negative coefficient to the complex part as $(\lambda^s_1 - \lambda^s_2)(\mathcal{Q} - \lambda^s_1\lambda^s_2) = (\mathcal{Q} - (\theta_s)^2 - \vartheta^2)2\vartheta i$. If we apply the same method to the other coefficients we observe that the numerator of all the other coefficients are complex numbers with zero real part and S_{Cn} has a negative coefficient to the complex part and S_{eK} and S_{en} have positive coefficients to the complex part.

At last, as

$$\begin{aligned} \det(S) &= \frac{\lambda^s_2 v^s_{1,1} - \lambda^s_1 v^s_{1,2}}{v^s_{3,1} - v^s_{3,2}} = \\ &= \frac{\frac{C^*}{\sigma}(r^*_K Y^*_n - r^*_n Y^*_K)((\lambda^s_1 - \lambda^s_2)\frac{\pi^*_n}{\gamma} + \lambda^s_2\lambda^s_1(\lambda^u_1 - \lambda^u_2)) + r^*_n\lambda^s_2\lambda^s_1(l_1 - l_2)}{\frac{C^*}{\sigma}r^*_n(l_2 - l_1) + Y^*_n(\lambda^u_2 - \lambda^u_1)(\mathcal{Q} - \lambda^u_1\lambda^u_2}, \end{aligned}$$

which is negative because

$$r^*_K Y^*_n - r^*_n Y^*_K = L_n(Y_{KK}Y_L - Y_{KL}Y_K) - Y_{Kn}(Y_K + Y_L L_K) < 0,$$

then $\det(S) = S_{CK}S_{en} - S_{Cn}S_{eK} < 0$. □

Lemma 5. *(Slopes of the isoclines in the projected space) The isoclines and their relative slopes in the projected space (K, n) verify:*

1. *if $\mathcal{O} > 0$ then $\frac{dn}{dK}\Big|_{\dot{K}=0} > \frac{dn}{dK}\Big|_{\dot{C}=0} > \frac{dn}{dK}\Big|_{\dot{e}=0} > \frac{dn}{dK}\Big|_{\dot{n}=0} > 0$;*
2. *if $\mathcal{O} < 0$ then $\frac{dn}{dK}\Big|_{\dot{C}=0} > \frac{dn}{dK}\Big|_{\dot{K}=0} > 0 > \frac{dn}{dK}\Big|_{\dot{n}=0} > \frac{dn}{dK}\Big|_{\dot{e}=0}$.*

Proof. The projection of the intersection of the isoclines with the stable eigenspace over the space (K, n) have the following slopes:

$$\begin{aligned}\frac{dn}{dK}\Big|_{\dot{K}=0} &= -\frac{Y_K^* + (Y_C^* - 1)S_{CK}}{Y_n^* + (Y_C^* - 1)S_{Cn}} = \frac{\lambda_2^s v_{3,2}^s - \lambda_1^s v_{3,1}^s}{v_{3,1}^s v_{3,2}^s(\lambda_2^s - \lambda_1^s)} = \\ &= \frac{\frac{C^*}{\sigma} r_n^*(\lambda_1^s - \lambda_2^s)(\mathcal{Q} - \lambda_1^s\lambda_2^s) + Y_n^*\mathcal{Q}(l_1 - l_2)}{(\lambda_2^s - \lambda_1^s)(\frac{C^*}{\sigma} r_n^* + Y_n^*\lambda_2^u)(\frac{C^*}{\sigma} r_n^* + Y_n^*\lambda_1^u)} > 0\end{aligned}$$

$$\begin{aligned}\frac{dn}{dK}\Big|_{\dot{C}=0} &= -\frac{r_K^* + r_C^* S_{CK}}{r_n^* + r_C^* S_{Cn}} = \frac{\lambda_2^s v_{1,2}^s - \lambda_1^s v_{1,1}^s}{\lambda_2^s v_{3,1}^s v_{1,2}^s - \lambda_1^s v_{1,1}^s v_{3,2}^s} = \\ &= \frac{(r_K^* Y_n^* - r_n^* Y_K^*)(\lambda_2^s - \lambda_1^s)(\mathcal{Q} - \lambda_1^s\lambda_2^s) + r_n^*((\lambda_1^s)^2(l_2 - \mathcal{Q}) - (\lambda_2^s)^2(l_1 - \mathcal{Q}))}{(r_K^* Y_n^* - r_n^* Y_K^*)(\lambda_1^s - \lambda_2^s)(\frac{C^*}{\sigma} r_n^*) + r_n^*(\frac{C^*}{\sigma}((\lambda_1^s)^2 - (\lambda_2^s)^2) + Y_n^*(\lambda_2^u(\lambda_1^s)^2 - \lambda_1^u(\lambda_2^s)^2))} > 0\end{aligned}$$

$$\begin{aligned}\frac{dn}{dK}\Big|_{\dot{n}=0} &= -\frac{S_{CK}}{S_{Cn}} = \frac{\lambda_2^s - \lambda_1^s}{\lambda_2^s v_{3,1}^s - v_{3,2}^s \lambda_1^s} = \\ &= \frac{(\lambda_2^s - \lambda_1^s)(l_1 - \mathcal{Q})(l_2 - \mathcal{Q})}{\frac{C^\star}{\sigma}((Y_C^\star - 1)r_n^\star - Y_n^\star r_C^\star)(\lambda_2^s - \lambda_1^s)(\lambda_1^s\lambda_2^s - \frac{\pi_n^\star}{\gamma} + Y_n^\star\lambda_1^s\lambda_2^s(l_1 - l_2)}\end{aligned}$$

and

$$\begin{aligned}\frac{dn}{dK}\Big|_{\dot{e}=0} &= -\frac{\pi_K^* + \pi_C^* S_{CK} - \rho\gamma S_{eK}}{\pi_n^* + \pi_C^* S_{Cn} - \rho\gamma S_{eK}} = \frac{(\lambda_2^s)^2 - (\lambda_1^s)^2}{v_{3,1}^s(\lambda_2^s)^2 - v_{3,2}^s(\lambda_1^s)^2} = \\ &= \frac{((\lambda_1^s)^2 - (\lambda_2^s)^2)(l_1 - \mathcal{Q})(l_2 - \mathcal{Q})}{\frac{C^*}{\sigma} r_n^\star((\lambda_2^s)^2(l_2 - \mathcal{Q}) - (\lambda_1^s)^2(l_1 - \mathcal{Q})) + Y_n^\star(\lambda_1^u(\lambda_2^s)^2(l_2 - \mathcal{Q}) - \lambda_2^u(\lambda_1^s)^2(l_1 - \mathcal{Q}))}\end{aligned}$$

have both the same sign as $-(l_1 - \mathcal{Q})(l_2 - \mathcal{Q})$, then they have both the same sign of $\mathcal{O}$ (see the proof of Lemma 4).

We can also determine the relationships between the projected isoclines. In some cases we can sign unambiguously:

$$\frac{dn}{dK}\Big|_{\dot{K}=0} - \frac{dn}{dK}\Big|_{\dot{n}=0} = -\frac{\lambda_1^s\lambda_2^s(v_{3,1}^s - v_{3,2}^s)^2}{(\lambda_2^s - \lambda_1^s)v_{3,1}^s v_{3,2}^s(\lambda_2^s v_{3,1}^s - \lambda_1^s v_{3,2}^s)} > 0$$

because both the numerator and the denominator are negative because the sign of $v^s_{3,1}v^s_{3,2}(\lambda^s_2 v^s_{3,1} - \lambda^s_1 v^s_{3,2})$ is the same as the sign of $(l_1 - \mathcal{Q})^2(l_2 - \mathcal{Q})^2 > 0$

$$\left.\frac{dn}{dK}\right|_{\dot{C}=0} - \left.\frac{dn}{dK}\right|_{\dot{e}=0} = \frac{\lambda^s_1\lambda^s_2(v^s_{3,1} - v^s_{3,2})(\lambda^s_1 v^s_{1,2} - \lambda^s_2 v^s_{1,1})}{(\lambda^s_2 v^s_{1,2}v^s_{3,1} - \lambda^s_1 v^s_{1,1}v^s_{3,2})((\lambda^s_2)^2 v^s_{3,1} - (\lambda^s_1)^2 v^s_{3,2})} > 0$$

because both the numerator and the denominator are fractions with positive numerators and denominators equal to $(l_1 - \mathcal{Q})^2(l_2 - \mathcal{Q})^2 > 0$,

$$\left.\frac{dn}{dK}\right|_{\dot{C}=0} - \left.\frac{dn}{dK}\right|_{\dot{n}=0} = \frac{\lambda^s_1\lambda^s_2(v^s_{3,1} - v^s_{3,2})(v^s_{1,2} - v^s_{1,1})}{(\lambda^s_2 v^s_{1,2}v^s_{3,1} - \lambda^s_1 v^s_{1,1}v^s_{3,2})(\lambda^s_2 v^s_{3,1} - \lambda^s_1 v^s_{3,2})} > 0$$

because both the numerator and the denominator are fractions with negative numerators and denominators equal to $(l_1 - \mathcal{Q})^2(l_2 - \mathcal{Q})^2 > 0$, and

$$\left.\frac{dn}{dK}\right|_{\dot{K}=0} - \left.\frac{dn}{dK}\right|_{\dot{e}=0} = \frac{\lambda^s_1\lambda^s_2(v^s_{3,1} - v^s_{3,2})(\lambda^s_2 v^s_{3,1} - \lambda^s_1 v^s_{3,2})}{v^s_{3,1}v^s_{3,2}(\lambda^s_1 - \lambda^s_2)((\lambda^s_2)^2 v^s_{3,1} - (\lambda^s_1)^2 v^s_{3,2})} > 0$$

because both the numerator and the denominator are fractions with negative numerators and denominators equal to $(l_1 - \mathcal{Q})^2(l_2 - \mathcal{Q})^2 > 0$.

Other relationships depend on the sign of $\mathcal{O}$:

$$\left.\frac{dn}{dK}\right|_{\dot{C}=0} - \left.\frac{dn}{dK}\right|_{\dot{K}=0} = \frac{\lambda^s_1\lambda^s_2(v^s_{3,1} - v^s_{3,2})(v^s_{1,2}v^s_{3,1} - v^s_{1,1}v^s_{3,2})}{v^s_{3,1}v^s_{3,2}(\lambda^s_2 - \lambda^s_1)(\lambda^s_2 v^s_{1,2}v^s_{3,1} - \lambda^s_1 v^s_{1,1}v^s_{3,2})}$$

and

$$\left.\frac{dn}{dK}\right|_{\dot{e}=0} - \left.\frac{dn}{dK}\right|_{\dot{n}=0} = \frac{\lambda^s_1\lambda^s_2(\lambda^s_2 - \lambda^s_1)(v^s_{3,1} - v^s_{3,2})}{(\lambda^s_2 v^s_{3,1} - \lambda^s_1 v^s_{3,2})((\lambda^s_2)^2 v^s_{3,1} - (\lambda^s_1)^2 v^s_{3,2})}$$

have both the sign of $v^s_{3,1} - v^s_{3,2}$ which has the symmetric sign of $(l_1 - \mathcal{Q})(l_2 - \mathcal{Q}) = -\mathcal{O}/\gamma$ then the difference in the slopes has the sign of $\mathcal{O}$. □

Lemma 6. *(Phase diagrams over the projected space) If the eigenvalues are real, then the projections of the isoclines and of the eigenspaces over space (K, n) verify the following relationships:*

R$_1$ *if $\mathcal{O} > 0$ then $\left.\frac{dn}{dK}\right|_{\dot{K}=0} > \left.\frac{dn}{dK}\right|_{\dot{C}=0} > \left.\frac{dn}{dK}\right|_{E^s_1} > \left.\frac{dn}{dK}\right|_{\dot{e}=0} > \left.\frac{dn}{dK}\right|_{\dot{n}=0} > 0 > \left.\frac{dn}{dK}\right|_{E^s_2}$,*

$\mathbf{R}_2$ *If* $\mathcal{O} < 0$, $\Delta > 0$ *and* $\pi_n^* > \gamma\mathcal{Q}$ *then* $\frac{dn}{dK}\Big|_{\dot{C}=0} > \frac{dn}{dK}\Big|_{\dot{K}=0} > \frac{dn}{dK}\Big|_{E_1^s} > \frac{dn}{dK}\Big|_{E_2^s} > 0 > \frac{dn}{dK}\Big|_{\dot{n}=0} > \frac{dn}{dK}\Big|_{\dot{e}=0}$;

$\mathbf{R}_3$ *If* $\mathcal{O} < 0$, $\Delta > 0$ *and* $\pi_n^* < \gamma\mathcal{Q}$ *then* $\frac{dn}{dK}\Big|_{\dot{C}=0} > \frac{dn}{dK}\Big|_{\dot{K}=0} > 0 > \frac{dn}{dK}\Big|_{\dot{n}=0} > \frac{dn}{dK}\Big|_{\dot{e}=0} > \frac{dn}{dK}\Big|_{E_1^s} > \frac{dn}{dK}\Big|_{E_2^s}$.

Proof. The slopes of the eigenspaces in space (K, n) and the slopes of the projections in (K, n) of the intersections of the isoclines with the stable manifold projections were already determined in Proposition 2, and 5. We use the same method as in the proof of the last result to get their relative position. This can only be done for the case in which the eigenvalues are real, because the eigenvectors associated to the complex eigenvalues are also complex: the slopes of the eigenvectors as regards the projections of the isoclines $\dot{K} = 0$ and $\dot{C} = 0$ can be unambiguously signed,

$$\frac{dn}{dK}\Big|_{\dot{K}=0} - \frac{dn}{dK}\Big|_{E_j^s} = \lambda_j^s \frac{v_{3,2}^s - v_{3,1}^s}{v_{3,1}^s v_{3,2}^s(\lambda_2^s - \lambda_1^s)} > 0,\ j = 1, 2$$

and

$$\frac{dn}{dK}\Big|_{\dot{C}=0} - \frac{dn}{dK}\Big|_{E_j^s} = \frac{\lambda_j^s v_{1,j}^s}{v_{3,j}^s}\left(\frac{v_{3,2}^s - v_{3,1}^s}{\lambda_2^s v_{3,1}^s v_{1,2}^s - \lambda_1^s v_{3,2}^s v_{1,1}^s}\right) > 0,\ j = 1, 2;$$

however, they are ambiguous as regards the projections of the isoclines $\dot{e} = 0$ and $\dot{n} = 0$

$$\frac{dn}{dK}\Big|_{\dot{e}=0} - \frac{dn}{dK}\Big|_{E_j^s} =, \frac{(\lambda_j^s)^2}{v_{3,j}^s}\left(\frac{v_{3,2}^s - v_{3,1}^s}{(\lambda_2^s)^2 v_{3,1}^s - (\lambda_1^s)^2 v_{3,2}^s}\right)\ j = 1, 2$$

and

$$\frac{dn}{dK}\Big|_{\dot{n}=0} - \frac{dn}{dK}\Big|_{E_j^s} =, \frac{\lambda_j^s}{v_{3,j}^s}\left(\frac{v_{3,2}^s - v_{3,1}^s}{\lambda_2^s v_{3,1}^s - \lambda_1^s v_{3,2}^s}\right)\ j = 1, 2$$

however, they have both the sign of $l_j - \pi^*/\gamma$. Then: the differences as regards E_1^s are negative for $\mathcal{O} > 0$ or $\mathcal{O} < 0$ and $\pi^* > \gamma\mathcal{Q}$ (i.e., cases $\mathbf{R}_1$ and $\mathbf{R}_2$) and are positive for $\mathcal{O} < 0$ and $\pi^* < \gamma\mathcal{Q}$ (i.e., case $\mathbf{R}_3$); and he differences as regards E_2^s are positive for $\mathcal{O} > 0$ or $\mathcal{O} < 0$ and $\pi^* < \gamma\mathcal{Q}$ (i.e., cases $\mathbf{R}_1$ and $\mathbf{R}_3$) and are negative for $\mathcal{O} < 0$ and $\pi^* > \gamma\mathcal{Q}$ (i.e., case $\mathbf{R}_2$). □

Proof. **Proof of Proposition 3** Consider equation (30) and the that the initial point is not the steady state $(K(0), n(0)) \neq (K^*, n^*)$. First, if the eigenvalues are complex then the transition dynamics is oscillatory and convergent. As the the coefficient of

the complex part is positive $\vartheta > 0$ then the trajectories rotate clockwise (see proof of Lemma 2). Next we deal with the case in which the eigenvalues are real.

First, consider the equation for K: $K(t) - K^* = z_1^s v_{3,1}^s e^{\lambda_1^s t} + z_2^s v_{3,2}^s e^{\lambda_2^s t}$. As $e^{\lambda_1^s t} > e^{\lambda_2^s t} \geq 0$ if $0 \leq t \leq \infty$ then the trajectories for $K(t)$ are monotonic if $\text{sign}(z_1^s v_{3,1}^s) = \text{sign}(z_2^s v_{3,2}^s)$ and they are non-monotonic if $\text{sign}(z_1^s v_{3,1}^s) \neq \text{sign}(z_2^s v_{3,2}^s)$. There are four types of trajectories: trajectory type I if $K(0) - K^* > 0$ and $K(t) - K^*$ is decreasing towards zero ($K(\infty) = K^*$); trajectory type II if $K(0) - K^*$ is decreasing (it can be positive, zero or negative) initially, reaches a level at time t_K, $K(t_K) - K^* < 0$ and then increases towards zero; trajectory type III if $K(0) - K^* < 0$ and $K(t) - K^*$ is increasing towards zero; and trajectory type IV if $K(0) - K^*$ is increasing (it can be positive, zero or negative) initially, reaches a level at time t_K, $K(t_K) - K^* > 0$ and then decreases towards zero. Type II trajectory is a U hump-shaped trajectory and type IV a IU hump-shaped trajectory. Hump-shaped trajectories only if $t_K \equiv \{t : d(K(t) - K^*)/dt = 0\} > 0$, while type II trajectories are convex at $K(t_K)$ and type IV trajectories are concave at time $K(t_K)$. As

$$t_K = \frac{1}{\lambda_1^s - \lambda_2^s} \ln \left(-\frac{\lambda_2^s z_2^s v_{3,2}^s}{\lambda_1^s z_1^s v_{3,1}^s} \right)$$

and

$$\left. \frac{d^2(K(t) - K^*)}{dt^2} \right|_{t=t_K} = (\lambda_1^s - \lambda_2^s) \lambda_1^s z_1^s v_{3,1}^s e^{\lambda_1^s t_K}$$

then, we have the following conditions for the four types of trajectories:

- trajectory I if $\lambda_1^s z_1^s v_{3,1}^s + \lambda_2^s z_2^s v_{3,2}^s < 0$ and $z_1^s v_{3,1}^s > 0$;
- trajectory II if $\lambda_1^s z_1^s v_{3,1}^s + \lambda_2^s z_2^s v_{3,2}^s < 0$ and $z_1^s v_{3,1}^s < 0$;
- trajectory III if $\lambda_1^s z_1^s v_{3,1}^s + \lambda_2^s z_2^s v_{3,2}^s > 0$ and $z_1^s v_{3,1}^s < 0$;
- trajectory IV if $\lambda_1^s z_1^s v_{3,1}^s + \lambda_2^s z_2^s v_{3,2}^s > 0$ and $z_1^s v_{3,1}^s > 0$.

In order to translate those conditions over a partition of the domain for variables (K, n), two facts are useful: first

$$\lambda_1^s z_1^s v_{3,1}^s + \lambda_2^s z_2^s v_{3,2}^s = \frac{(\lambda_1^s v_{3,1}^s - \lambda_2^s v_{3,2}^s)(K(0) - K^*) + (\lambda_2^s - \lambda_1^s) v_{3,1}^s v_{3,2}^s (n(0) - n^*)}{v_{3,1}^s - v_{3,2}^s} = 0$$

if and only if (see proof of Lemma 5)

$$n(0) - n^* = \frac{\lambda_2^s v_{3,2}^s - \lambda_1^s v_{3,1}^s}{(\lambda_2^s - \lambda_1^s) v_{3,1}^s v_{3,2}^s} (K(0) - K^*) = \left. \frac{dn}{dK} \right|_{\dot{K}=0} (K(0) - K^*)$$

that it $(K(0), n(0))$ are located along the projected isocline for K over space (K, n); second

$$z_1^s v_{3,1}^s = \frac{v_{3,1}^s \left((K(0) - K^*) - v_{3,2}^s (n(0) - n^*) \right)}{v_{3,1}^s - v_{3,2}^s}$$

if and only if (see proof of Proposition 2)

$$n(0) - n^* = \frac{1}{v_{3,2}^s}(K(0) - K^*) = \left. \frac{dn}{dK} \right|_{E_2^s} (K(0) - K^*)$$

that is $(K(0), n(0))$ is located along the projected eigenspace E_2^s over space (K, n). Therefore, the projections of isocline $\dot{K} = 0$ and of the eigenspace E_2^s perform a partition over the domain (K, n) which are associated to the four types of trajectories that were mentioned. Using the previous results, and, in particular, the fact that if $\mathcal{O} > 0$ then $v_{3,1}^s - v_{3,2}^s > 0$ and $v_{3,1}^s > 0$ and $v_{3,2}^s < 0$ and if $\mathcal{O} < 0$ then $v_{3,1}^s - v_{3,2}^s < 0$ and $\text{sign}(v_{3,1}^s) = \text{sign}(v_{3,2}^s)$, then we have:

- trajectory I if $K(0) > K^*$ and $\left. \frac{dn}{dK} \right|_{E_2^s} (K(0) - K^*) < n(0) - n^* < \left. \frac{dn}{dK} \right|_{\dot{K}=0} (K(0) - K^*)$;
- trajectory II if $n(0) - n^* < \min \left\{ \left. \frac{dn}{dK} \right|_{\dot{K}=0}, \left. \frac{dn}{dK} \right|_{E_2^s} \right\} (K(0) - K^*)$;
- trajectory III if $K(0) < K^*$ and $\left. \frac{dn}{dK} \right|_{E_2^s} (K(0) - K^*) > n(0) - n^* > \left. \frac{dn}{dK} \right|_{\dot{K}=0} (K(0) - K^*)$;
- trajectory IV if $n(0) - n^* > \max \left\{ \left. \frac{dn}{dK} \right|_{\dot{K}=0}, \left. \frac{dn}{dK} \right|_{E_2^s} \right\} (K(0) - K^*)$.

Now, we consider the equation for n: $n(t) - n^* = z_1^s e^{\lambda_1^s t} + z_2^s e^{\lambda_2^s t}$. As $e^{\lambda_1^s t} > e^{\lambda_2^s t} \geq 0$ if $0 \leq t \leq \infty$ then the trajectories for $n(t)$ are monotonic if $\text{sign}(z_1^s) = \text{sign}(z_2^s)$ and they are non-monotonic if $\text{sign}(z_1^s) \neq \text{sign}(z_2^s)$. There are four types of trajectories: trajectory type I if $n(0) - n^* > 0$ and $n(t) - n^*$ is decreasing towards zero ($n(\infty) = n^*$); U hump-shaped trajectory of type II if $n(0) - n^*$ is decreasing (it can be positive, zero or negative) initially, reaches a level at time t_n, $n(t_n) - n^* < 0$ and then increases towards zero; trajectory type III if $n(0) - n^* < 0$ and $n(t) - n^*$ is increasing towards zero; and IU hump-shaped of type IV if $n(0) - n^*$ is increasing (it can be positive, zero or negative) initially, reaches a level at time t_n, $n(t_n) - n^* > 0$ and then decreases towards zero. Hump-shaped trajectories only if $t_n \equiv \{t : \, d(n(t) - n^*)/dt = 0\} > 0$, while type II trajectories are convex at $n(t_n)$ and type IV trajectories are concave at time $n(t_n)$. As

$$t_n = \frac{1}{\lambda_1^s - \lambda_2^s} \ln \left(-\frac{\lambda_2^s z_2^s}{\lambda_1^s z_1^s} \right)$$

and

$$\left.\frac{d^2(n(t)-n^*)}{dt^2}\right|_{t=t_K} = (\lambda_1^s - \lambda_2^s)\lambda_1^s z_1^s e^{\lambda_1^s t_K}$$

then, we have the following conditions for the four types of trajectories:

- trajectory I if $\lambda_1^s z_1^s + \lambda_2^s z_2^s < 0$ and $z_1^s > 0$;
- trajectory II if $\lambda_1^s z_1^s + \lambda_2^s z_2^s < 0$ and $z_1^s < 0$;
- trajectory III if $\lambda_1^s z_1^s + \lambda_2^s z_2^s > 0$ and $z_1^s < 0$;
- trajectory IV if $\lambda_1^s z_1^s + \lambda_2^s z_2^s > 0$ and $z_1^s > 0$.

In order to translate those conditions over a partition of the domain for variables (K, n), two facts are useful: first

$$\lambda_1^s z_1^s + \lambda_2^s z_2^s = \frac{(\lambda_1^s - \lambda_2^s)(K(0)-K^*) + (\lambda_2^s v_{3,1}^s - \lambda_1^s v_{3,2}^s)(n(0)-n^*)}{v_{3,1}^s - v_{3,2}^s} = 0$$

if and only if (see proof of Lemma 5)

$$n(0) - n^* = \frac{\lambda_2^s - \lambda_1^s}{\lambda_2^s v_{3,1}^s - \lambda_1^s v_{3,2}^s}(K(0)-K^*) = \left.\frac{dn}{dK}\right|_{\dot{n}=0}(K(0)-K^*)$$

that it $(K(0), n(0))$ are located along the projected isocline for n over space (K, n); second

$$z_1^s = \frac{(K(0)-K^*) - v_{3,2}^s(n(0)-n^*)}{v_{3,1}^s - v_{3,2}^s}$$

if and only if (see proof of Proposition 2)

$$n(0) - n^* = \left.\frac{dn}{dK}\right|_{E_2^s}(K(0)-K^*)$$

that is $(K(0), n(0))$ is located along the projected eigenspace E_2^s over space (K, n). Therefore, the projections of isocline $\dot{n} = 0$ and of the eigenspace E_2^s perform a partition over the domain (K, n) which are associated to the four types of trajectories that were mentioned. Using the previous results, and, in particular, the fact that if $\mathcal{O} > 0$ then $v_{3,1}^s - v_{3,2}^s > 0$ and $v_{3,1}^s > 0$ and $v_{3,2}^s < 0$ and if $\mathcal{O} < 0$ then $v_{3,1}^s - v_{3,2}^s < 0$ and $\text{sign}(v_{3,1}^s) = \text{sign}(v_{3,2}^s)$ (they are positive in case $\mathbf{R}_2$ and negative in case $\mathbf{R}_3$) , then we have trajectory:

- trajectory I if $n(0) > n^*$ and, if we have case $\mathbf{R}_1$ and $\frac{dK}{dn}\big|_{E_2^s}(n(0)-n^*) < K(0)-K^* < \frac{dK}{dn}\big|_{\dot{n}=0}(n(0)-n^*)$ or if we have cases $\mathbf{R}_2$ and $\mathbf{R}_3$ and $\frac{dK}{dn}\big|_{\dot{n}=0}(n(0)-n^*) < K(0)-K^* < \frac{dK}{dn}\big|_{E_2^s}(n(0)-n^*)$;

- trajectory II if we have case $\mathbf{R}_1$ and $K(0)-K^* < \min\left\{\frac{dK}{dn}\big|_{E_2^s}, \frac{dK}{dn}\Big|_{\dot{n}=0}\right\}(n(0)-n^*)$ or if we have cases $\mathbf{R}_2$ and $\mathbf{R}_3$ and $K(0)-K^* > \max\left\{\frac{dK}{dn}\big|_{E_2^s}, \frac{dK}{dn}\Big|_{\dot{n}=0}\right\}(n(0)-n^*)$;

- trajectory III if $n(0) < n^*$ and, if we have case $\mathbf{R}_1$ and $\frac{dK}{dn}\big|_{E_2^s}(n(0)-n^*) > K(0)-K^* > \frac{dK}{dn}\big|_{\dot{n}=0}(n(0)-n^*)$ or if we have cases $\mathbf{R}_2$ and $\mathbf{R}_3$ and $\frac{dK}{dn}\big|_{\dot{n}=0}(n(0)-n^*) > K(0)-K^* > \frac{dK}{dn}\big|_{E_2^s}(n(0)-n^*)$;

- trajectory IV if we have case $\mathbf{R}_1$ and $K(0)-K^* > \max\left\{\frac{dK}{dn}\big|_{E_2^s}, \frac{dK}{dn}\Big|_{\dot{n}=0}\right\}(n(0)-n^*)$ or if we have cases $\mathbf{R}_2$ and $\mathbf{R}_3$ and $K(0)-K^* < \min\left\{\frac{dK}{dn}\big|_{E_2^s}, \frac{dK}{dn}\Big|_{\dot{n}=0}\right\}(n(0)-n^*)$.

If we consider the definitions of sets $\mathbf{N}_K$, $\mathbf{S}_K$, $\mathbf{W}_n$ and $\mathbf{E}_n$, then set $\mathbf{N}_K$ is associated to trajectories of type IV, set $\mathbf{S}_K$ is associated to trajectories of type II, both for variable K, and set $\mathbf{W}_n$ is associated to trajectories of type II for case $\mathbf{R}_1$ and of type IV for cases $\mathbf{R}_2$ and $\mathbf{R}_3$, and set $\mathbf{E}_n$ is associated to trajectories of type IV for case $\mathbf{R}_1$ and of type II for cases $\mathbf{R}_2$ and $\mathbf{R}_3$, for variable n. Using the previous results we also see that: (1) if we have case $\mathbf{R}_1$ the sets $\mathbf{N}_K$, $\mathbf{S}_K$, $\mathbf{W}_n$ and $\mathbf{E}_n$ are all disjoint; (2) if we have case $\mathbf{R}_2$ then $\mathbf{E}_n \subset \mathbf{S}_K$ and $\mathbf{W}_n \subset \mathbf{N}_K$; and (3) if we have case $\mathbf{R}_3$ then $\mathbf{N}_K \subset \mathbf{E}_n$ and $\mathbf{S}_K \subset \mathbf{W}_n$. □

Propositions 4 and 5: proofs

Proof. **Proof of Proposition 4** First, we linearise the system (12), (14), (15) and (16) around an initial steady state and introduce a variation in G, dG, to get the variational system

$$\dot{X}(t) = JdX(t) + J_G dG$$

where $X = (x_1, x_2, x_3, x_4)^\top \equiv (C, e, K, n)^\top$, J is the Jacobian in equation (25) and $J_G = (0, 0, -1, 0)^\top$. Second, given the fact the steady state is hyperbolic, we get the long run multipliers as $dX^*/dG = -J^{-1}J_G$. If we perform the matrix operation and observe that $\det(J) = \mathcal{D} > 0$ then we get the expressions and the signs in equations (38)-(41). Third, to calculate the short-run multipliers we solve the linearised system,

along the stable manifold. We get, for the case in which the eigenvalues are real

$$\frac{dx_i(t)}{dG} = \frac{dx_i^*}{dG} - \frac{v_{i,1}^s}{v_{3,1}^s - v_{3,2}^s}\left(\frac{dK^*}{dG} - v_{3,2}^s\frac{dn^*}{dG}\right)e^{\lambda_1^s t} - \frac{v_{i,2}^s}{v_{3,1}^s - v_{3,2}^s}\left(-\frac{dK^*}{dG} + v_{3,1}^s\frac{dn^*}{dG}\right)e^{\lambda_2^s t},\ i = 1,\ldots,4, \quad (56)$$

and for the case in which the eigenvalues are complex conjugate

$$\frac{dx_i(t)}{dG} = \frac{dx_i^*}{dG} - \frac{1}{v_{3,1}^s - v_{3,2}^s}e^{\theta_s t}\left\{\left((v_{1,1}^s - v_{i,2}^s)\frac{dK^*}{dG} - (v_{i,1}^s v_{3,2}^s - v_{i,2}^s v_{3,1}^s)\frac{dn^*}{dG}\right)\cos(\vartheta t) + \right.$$
$$\left. + \left((v_{i,1}^s + v_{i,2}^s)\frac{dK^*}{dG} - (v_{i,1}^s v_{3,2}^s + v_{i,2}^s v_{3,1}^s)\frac{dn^*}{dG}\right)\sin(\vartheta t)\right\},\ i = 1,\ldots,4 \quad (57)$$

If we evaluate the previous expressions at time $t = 0$, we get the same expressions for both real and complex eigenvalues: $dK(0)/dG = dn(0)/dG = 0$ and equations (42) and (43) for $dC(0)/dG$ and $de(0)/dG$. From Lemma 4, on the signs of the components of matrix S and the signs for the long run multipliers, we readily conclude that: if $\mathcal{O} < 0$ then $S_{CK} > 0$ and $S_{Cn} > 0$ and $S_{eK} < 0$ and $S_{en} < 0$ then $dC(0)/dG < dC^*/dG < 0$ and $de(0)/dG > 0$ and if $\mathcal{O} > 0$ then $S_{CK} > 0$ and $S_{Cn} < 0$ and $S_{eK} > 0$ and $S_{en} < 0$ and then the relationship both the values of the impact multipliers and their relationship with the long run multipliers are ambiguous. □

Proof. **Proof of Proposition 5** Observe that, in the (K, n)-graphs, the pre- and post-shock levels of those variables will be located on the linear schedule $n = K.(n/K)^*$ with a positive slope:

$$\left.\frac{dn}{dK}\right|_{dG} = -\frac{r_C^*\pi_K^* - r_K^*\pi_C^*}{r_C^*\pi_n^* - r_n^*\pi_C^*} > 0,$$

passing through the initial and ex-post steady-states. The pre-shock initial point following a positive (negative) variation of G will be located below (above) the post-shock steady state along the dG line. This means that using the results in Proposition 4 we can have a geometric determination of the existence of non-monotonic effects of fiscal policy by comparing the dG line with the projections of the isoclines for K and n and with curve E_2^s. Interestingly we have:

$$\text{sign}\left(\left.\frac{dn}{dK}\right|_{\dot{K}=0} - \left.\frac{dn}{dK}\right|_{dG}\right) = \text{sign}\left(1 - (Y_C^* - 1)\frac{dC(0)}{dG}\right)$$

$$\text{sign}\left(\left.\frac{dn}{dK}\right|_{\dot{n}=0} - \left.\frac{dn}{dK}\right|_{dG}\right) = -\text{sign}\left(\frac{de(0)}{dG}\right)$$

We find that

$$\left.\frac{dn}{dK}\right|_{dG} > \left.\frac{dn}{dK}\right|_{\dot{n}=0}, \ \left.\frac{dn}{dK}\right|_{dG} < \left.\frac{dn}{dK}\right|_{\dot{K}=0}, \ \left.\frac{dn}{dK}\right|_{dG} > \left.\frac{dn}{dK}\right|_{E_2^s}$$

That is: the long-run relation represented by the dG line is less steep than the projection of the isocline $\dot{K} = 0$ but is steeper than the projection of the isocline $\dot{n} = 0$ and the eigenspace E_2^s. This means that we that schedule $dn/dK|_{dG}$ can only belong to sets $\mathbf{W}_n$ and $\mathbf{E}_n$, for case $\mathbf{R}_3$ that is we will only find the case in which the number of firms can evolves non-monotonically. Capital will always respond in a monotonic manner to fiscal policy, except for the case in which there are complex eigenvalues, case $\mathbf{C}$. □

B A benchmark model

In this section we assume the benchmark case where production function is Cobb-Douglas and the benchmark utility function as in equations (23)-(24).

B.1 Steady state

The steady state $(C^*, K^*, L^*, n^*, e^*)$ is determined from equations (6), (12)- (16) , when the time derivatives are set to zero. Then $e^* = 0$ and the other variables solve the system

$$\begin{aligned}
\beta A K^{\alpha} L^{\beta-1-\eta} n^{1-\alpha-\beta} &= \xi C^{\sigma} && (58)\\
\alpha A K^{\alpha-1} L^{\beta} n^{1-\alpha-\beta} &= \rho && (59)\\
(1-\alpha-\beta) A K^{\alpha} L^{\beta} n^{-(\alpha+\beta)} &= \phi && (60)\\
n\left(A K^{\alpha} L^{\beta} n^{-(\alpha+\beta)} - \phi\right) &= C + G && (61)
\end{aligned}$$

From equation (59) and (60) we obtain $k^* = K^*/n^*$ and $l^* = L^*/n^*$, as

$$\begin{aligned}
k^* &= \left(\frac{\phi}{1-\nu}\right)\frac{\alpha}{\rho}\\
l^* &= \left(\frac{1}{A}\left(\frac{\phi}{1-\nu}\right)^{1-\alpha}\left(\frac{\alpha}{\rho}\right)^{\alpha}\right)^{1/\beta}.
\end{aligned}$$

Then

$$y^* = A k^* l^* - \phi = \frac{\nu\phi}{1-\nu}.$$

If we substitute k^* and l^* in equation (58) we get the steady state consumption level as a function of n,

$$C^* = C(n) = \left(\frac{\beta\nu\phi}{\xi(1-\nu)} (l^*)^{-(1+\eta)} \right)^{1/\sigma} n^{-\eta/\sigma}.$$

The steady state number of firms can be determined from the good market equilibrium condition, equation (61),

$$n^* = \{n : \ \nu\phi n = (1-\nu)\left(C(n) + G\right)\}.$$

B.2 Derivation of the income expansion path and of the Euler frontier

In Figure 1 we trace out two curves. In this section we derive them. The long run income expansion path

$$IEP \equiv \{(L,C) : \ U_L(L) = U_C(C) w(K,L,n), \ r(K,n,L) = \rho, \ \pi(K,n,L) = 0\}$$

is given by the schedule

$$C = \mu_C L^{-\eta/\sigma}$$

where

$$\mu_C = \left\{ \ \frac{\beta A}{\xi} \mu_K^{\alpha} \ \mu_n^{1-\alpha-\beta} \right\}^{1/\sigma}$$

and

$$\mu_K = \left[\frac{\phi}{A(1-\alpha-\beta)} \mu_n^{\alpha+\beta} \ \right]^{1/\alpha}$$

and

$$\mu_n = \left[\ \left(\frac{\phi}{A(1-\alpha-\beta)} \ \right)^{1/\alpha} \left(\frac{\rho}{\alpha A} \ \right)^{1/(1-\alpha)} \right]^{\alpha(\alpha-1)/\beta}$$

The short run income expansion path is defined for a given pair (C, L) as

$$IEP(K,n) \equiv \{(L,C) : \ U_L(L) = U_C(C) w(K,L,n)\}$$

is parameterized by (K, n) and is

$$C = \left(\frac{\beta A}{\xi} \ K^{\alpha} n^{1-\nu} \right)^{1/\sigma} L^{-(1+\eta-\beta)/\sigma}$$

The Euler frontier is

$$EF(G) = \{\ (L, C) : U_L(L) = U_C(C)w(K, L, n),\ r(K, n, L) = \rho,\ Y(L, K, n) = C + G\}$$

becomes

$$C = A\left(\frac{\alpha A}{\rho}\right)^{1/(1-\alpha)} L - G$$

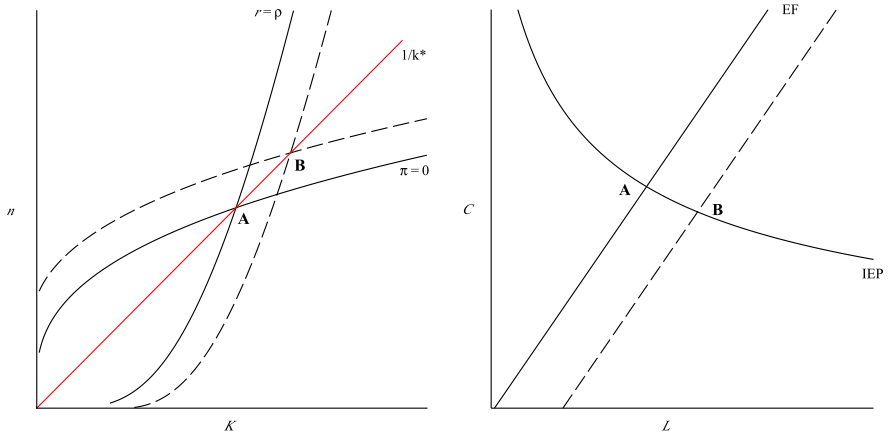

Figure 1: Steady state in (K, n) and (L, C) before (**A**) and after (**B**) a permanent increase in G for the case in which the isocline $\pi = 0$ has a positive slope at the steady state.

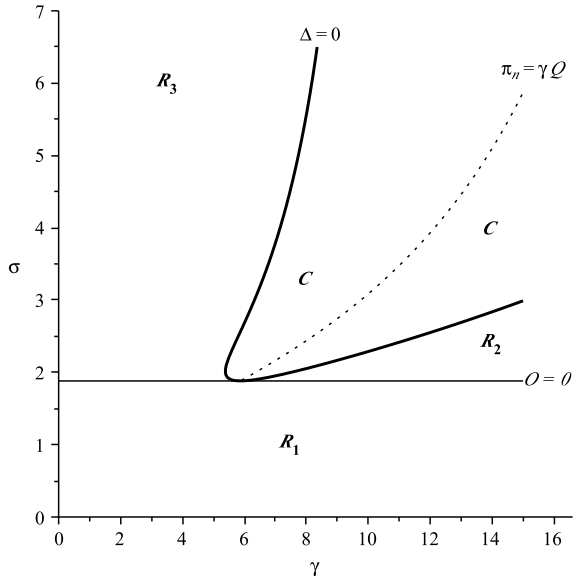

Figure 2: Bifurcation diagram on the (γ, σ)-space for $\alpha = 0.3$, $\beta = 0.5$, $\rho = 0.025$, $\phi = 0.1$, $\xi = 0.01$, $\eta = 0.4$ and $G = 0$, displaying the values for the parameters associated with the four main types of phase diagrams: phase diagram $\mathbf{R}_1$, for $\mathcal{O} > 0$, phase diagram $\mathbf{R}_2$ for $\mathcal{O} < 0$, $\Delta > 0$ and $\pi_n^* > \gamma\mathcal{Q}$, phase diagram $\mathbf{R}_3$ for $\mathcal{O} < 0$, $\Delta > 0$ and $\pi_n^* < \gamma\mathcal{Q}$ and $\mathbf{C}$ for $\Delta < 0$.

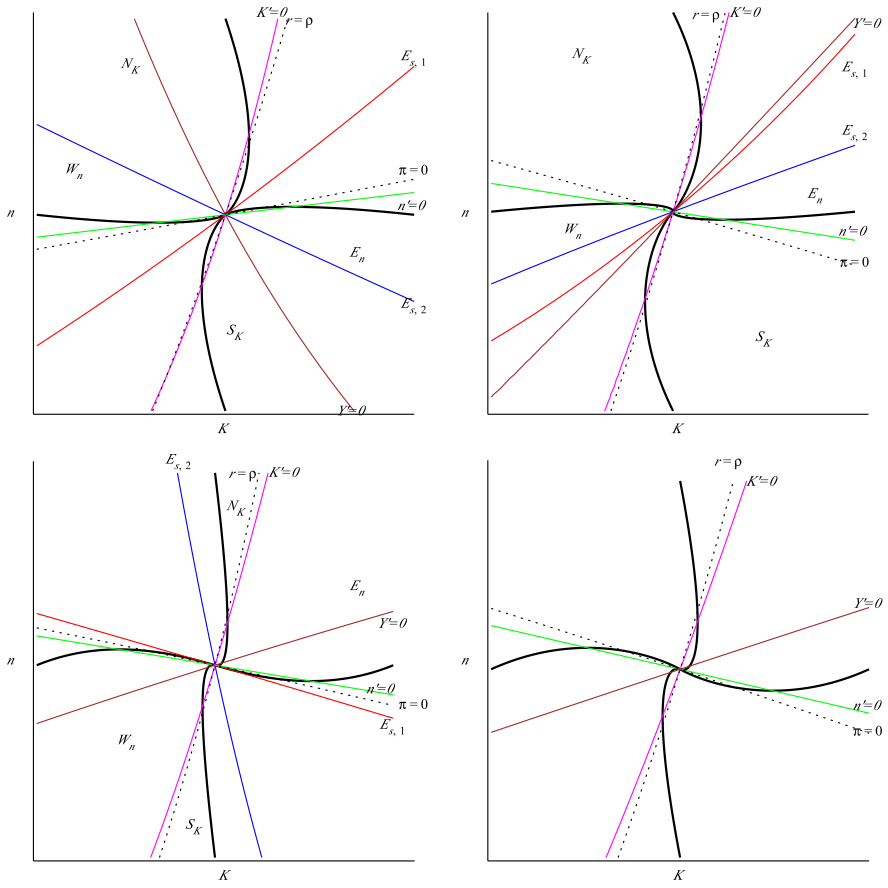

Figure 3: Two-dimensional projections of the phase diagrams in (K, n)-space, for cases $\mathbf{R}_1$, $\mathbf{R}_2$ in the upper panels, $\mathbf{R}_3$ and $\mathbf{C}$, in the lower panel. Each panel displays the isoclines $\dot{K} = 0$, $\dot{n} = 0$ for the state variables and the isoclines $\dot{C} = 0$ $(r = \rho)$ and $\dot{e} = 0$ $(\pi = 0)$, the projections of the two eigenspaces E_1^s and E_2^s and the four areas $\mathbf{N}_k$ and $\mathbf{S}_k$, between isocline for K and E_2^s and $\mathbf{W}_n$ and $\mathbf{E}_n$, between isocline for n and E_2^s.

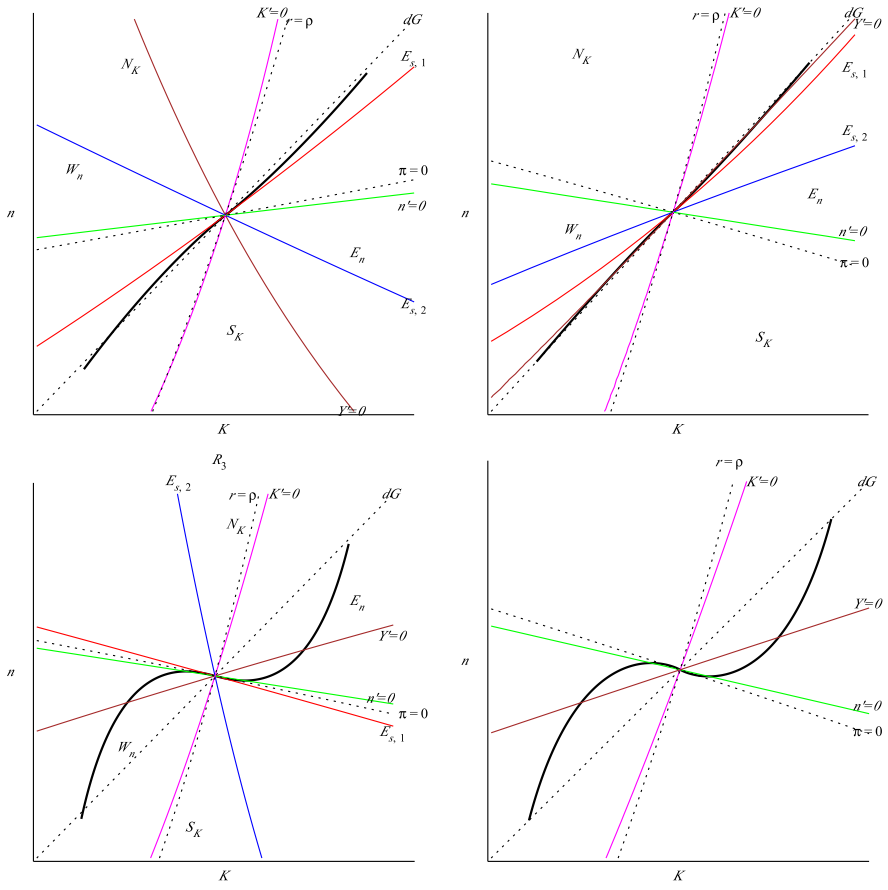

Figure 4: Two-dimensional projections of the phase diagrams in (K, n)-space and the effects of a fiscal shock: monotonous adjustment for cases $\mathbf{R}_1$ and $\mathbf{R}_2$ in the upper panels, hump-shaped adjustment for n in case $\mathbf{R}_3$ and hump-shaped adjustment for K and n in case $\mathbf{C}$.

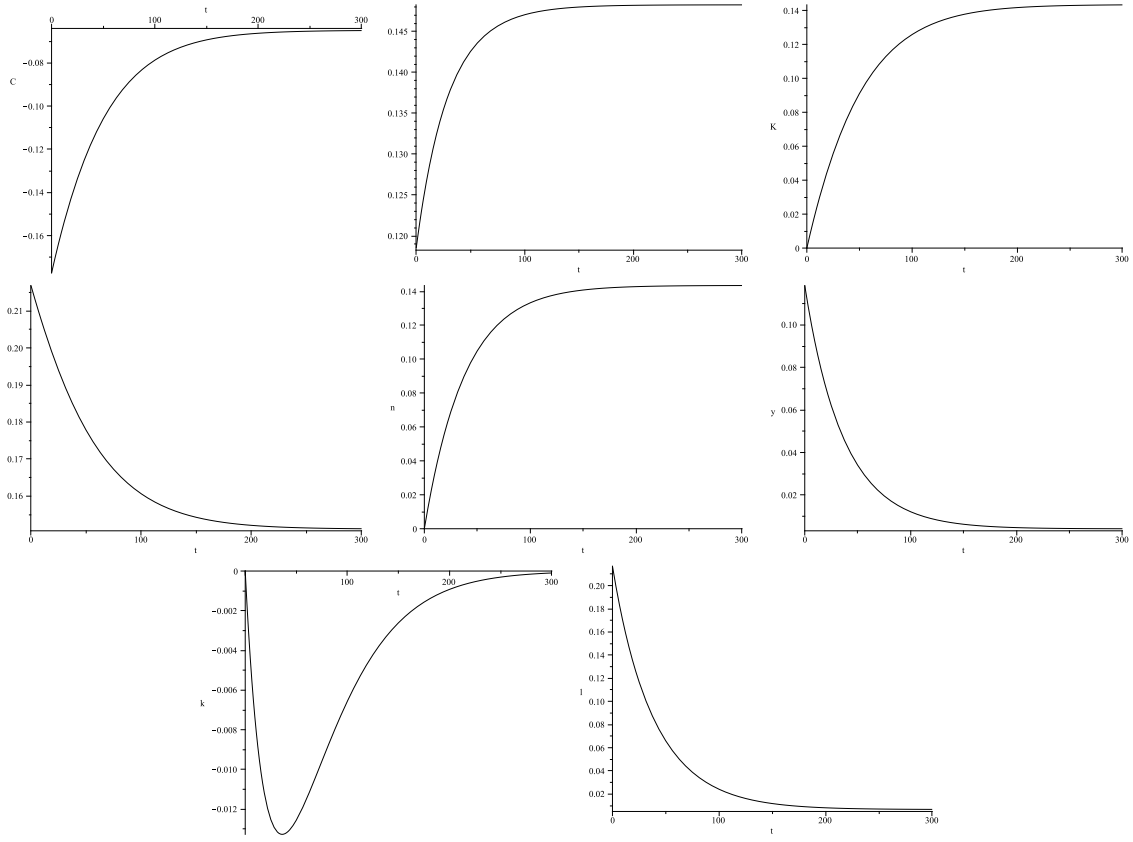

Figure 5: Impulse response for fiscal policy, case $\mathbf{R}_1$

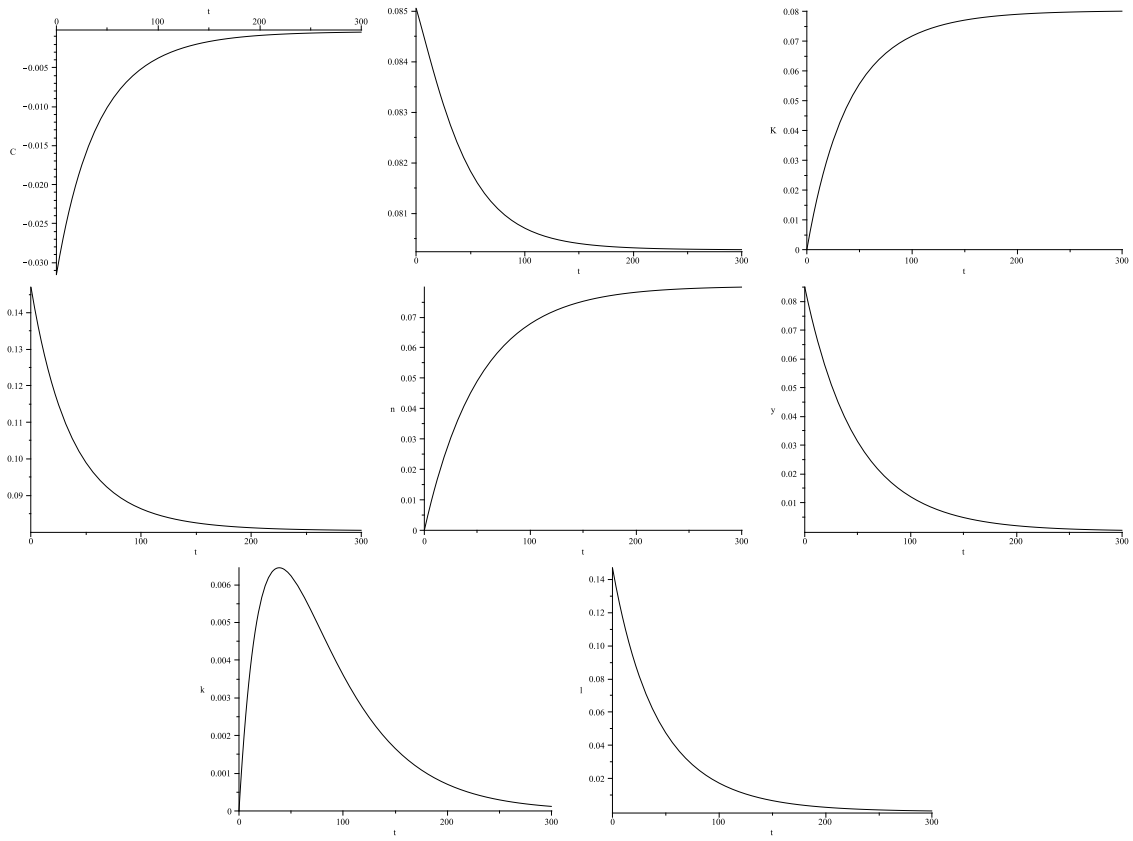

Figure 6: Impulse response for fiscal policy, case $\mathbf{R}_2$

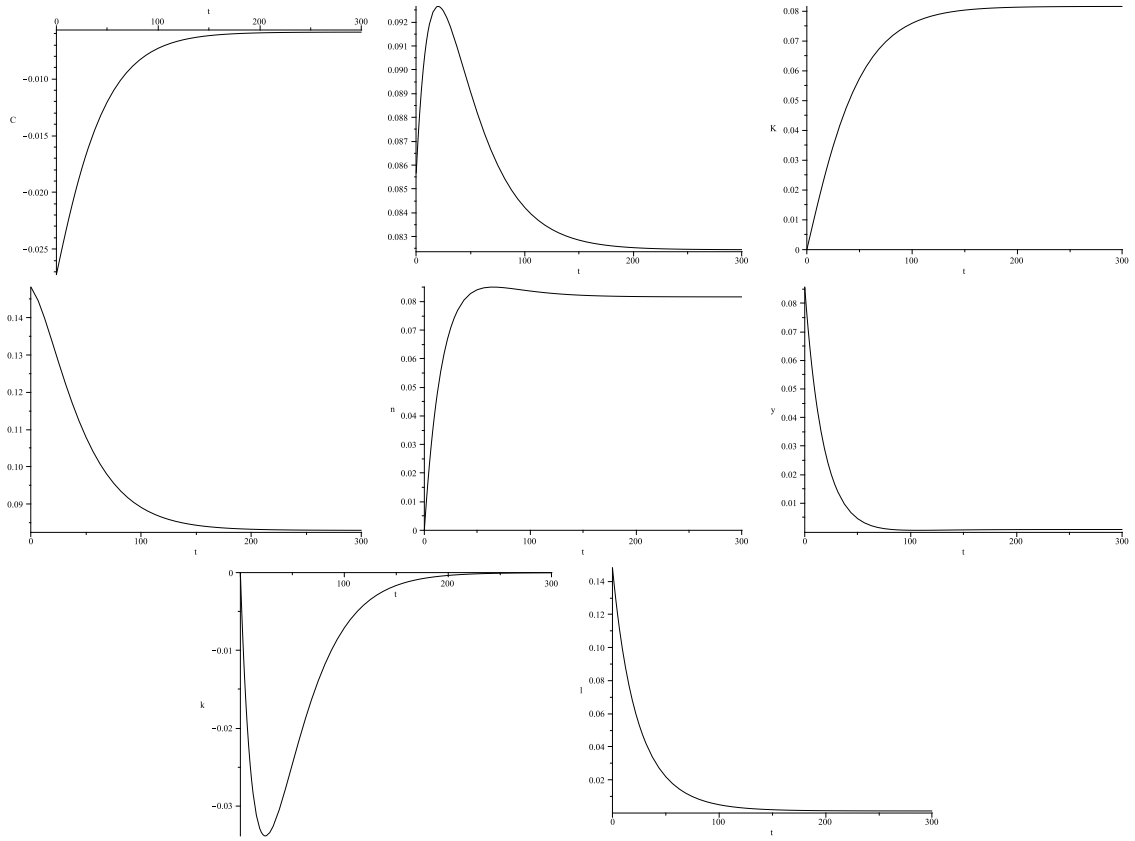

Figure 7: Impulse response for fiscal policy, case $\mathbf{R}_3$

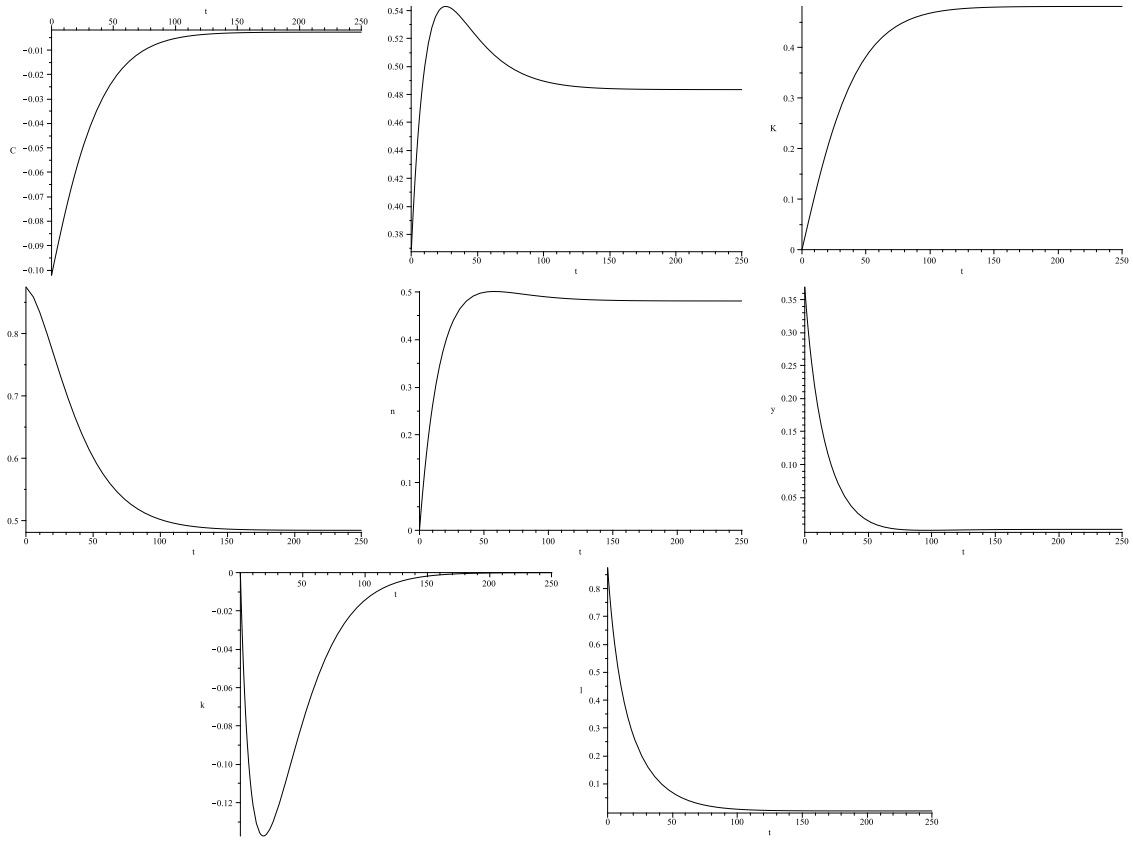

Figure 8: Impulse response for fiscal policy, case **C**