# Risk premia in general equilibrium

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**Abstract**

This paper shows that non-linearities can generate time-varying and asymmetric risk premia over the business cycle. These (empirical) key features become relevant and asset market implications improve substantially when we allow for non-normalities in the form of rare disasters. We employ explicit solutions of dynamic stochastic general equilibrium models, including a novel solution with endogenous labor supply, to obtain closed-form expressions for the risk premium in production economies. We find that the curvature of the policy functions affects the risk premium through controlling the individual's effective risk aversion.

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# 1 Introduction

> "... the challenge now is to understand the economic forces that determine the stochastic discount factor, or put another way, the rewards that investors demand for bearing particular risks." (Campbell, 2000, p.1516)

In general equilibrium models, the stochastic discount factor is not only determined by the consumption-based first-order condition, but also linked to business cycle characteristics. In macroeconomics, dynamic stochastic general equilibrium models (DSGE) have been very successful in explaining co-movements in aggregate data, but relatively less effort has been made to understand their asset market implications (recent work includes Jermann, 1998; Tallarini, 2000; Lettau and Uhlig, 2000; Boldrin, Christiano and Fisher, 2001).[1] One main advantage of using general equilibrium models to explain asset market phenomena is that the asset-pricing kernel is consistent with the macro dynamics, which offers an excellent guide to the future development of models in both macroeconomics and finance.

However, little is known about the determinants of the risk premium in DSGE models. Which economic forces determine the risk premium? What are the main implications using production based models compared to the endowment economy? This paper fills this gap by studying asset pricing implications of prototype DSGE models analytically.[2]

In a nutshell, this paper shows that a neoclassical production function alone generates key features of the risk premium. The economic intuition is that individual's effective risk aversion, excluding singular cases, is not constant in a neoclassical production economy.

We use explicit solutions of DSGE models. For this purpose we readopt formulating models in continuous-time, which gives closed-form solutions for a broad class of interesting models and parameter sets (Merton, 1975; Eaton, 1981; Cox, Ingersoll and Ross, 1985).[3] To illustrate our general equilibrium pricing approach, we use Lucas' fruit-tree endowment economy allowing for rare disasters, which subsequently is extended to a production sector and (non-tradable) human wealth with endogenous labor supply. Recent research has emphasized the importance of non-linearities and non-normalities in explaining the business cycle for the US economy (Fernández-Villaverde and Rubio-Ramírez, 2007; Justiniano and Primiceri, 2008; Posch, 2009). Barro (2006, 2009) shows that economic disasters have been sufficiently frequent and large enough to account for the risk-premium puzzle.[4]

[1] There is an increasing interest in DSGE models in finance (cf. Kaltenbrunner and Lochstoer, 2006). A survey of the literature on the intersection between macro and finance is Cochrane (2008, chap. 7).

[2] Our approach differs from the 'analytical' approach of Campbell (1994), as we obtain exact solutions.

[3] Recent contributions of continuous-time DSGE models include e.g. Corsetti (1997), Wälde (1999, 2002), Steger (2005), Turnovsky and Smith (2006), and Posch (2009). An introduction is Turnovsky (2000).

[4] In subsequent papers, Gabaix (2008) and Wachter (2009) suggest variable intensity versions, which generate a time-varying risk premium, as a viable explanation for several macro-finance puzzles.

We find that non-linearities in DSGE models can generate time-varying and asymmetric risk premia over the business cycle.[5] Although these key features of the risk premium are negligible in the standard real business cycle model, we show that they become relevant, and asset market implications improve substantially when we allow for non-normalities in the form of rare disasters (Rietz, 1988; Barro, 2006, 2009). This finding confirms the Barro-Rietz rare disaster hypothesis as being a viable paradigm to reconcile asset pricing implications of DSGE models with the observed data (among other remedies such as capital adjustment cost, habit formation and/or recursive preferences). Our result is based on the finding that the individual's effective risk aversion is not constant for non-homogeneous consumption functions (cf. Carroll and Kimball, 1996).[6] We show that closed-form solutions are important knife-edge cases which shed light on the properties of the risk premium, and we contribute by providing a novel solution for DSGE models with endogenous labor supply.

One caveat of the traditional discrete-time models is the lack of analytical solutions. To some extent, the gap between the literature of asset pricing models using endowment models in finance and typically non-linear production economies in macro is due to the difficulty of solving these models. In particular when focusing on the effects of uncertainty, the traditional linear-quadratic approximation of models about the non-stochastic steady state does not seem to provide an adequate framework. Alternatively, the literature suggests risk-sensitive linear-quadratic objectives to compute approximate solutions (among others Hansen, Sargent and Tallarini, 1999; Tallarini, 2000). Other up-and-coming strategies use perturbation methods and higher-order approximation schemes (Schmitt-Grohé and Uribe, 2004; van Binsbergen, Fernández-Villaverde, Koijen and Rubio-Ramírez, 2009). Although these numerical methods usually are locally highly accurate, the effects of large economic shocks, such as rare disasters on approximation errors, are largely unexplored.

Our continuous-time formulation does not suffer from those limitations. First, we exploit closed-form solutions, which are available for reasonable parametric restrictions, to study the determinants of the risk premium analytically. Second, we use powerful numerical methods to examine the properties of the risk premia for a broader parameter range without relying on local approximations (Posch and Trimborn, 2009). We obtain optimal policy functions allowing for rare events, while the closed-form solutions can be used to ensure accuracy. We propose this formulation as a workable paradigm in the macro-finance literature.

This paper is related to the literature on the determinants of the risk premium. While

[5]While the time-varying feature is well documented empirically (Welch and Goyal, 2008), there is some evidence that the risk premium increases more in bad times than it decreases in good times (Mele, 2008).

[6]Other potential solutions to the Mehra and Prescott (1985) equity-premium puzzle for endowment economies of Epstein and Zin (1989); Abel (1990); Constantinides (1990); Campbell and Cochrane (1999); Bansal and Yaron (2004), are generating time-varying risk aversion through different channels.

it has been recognized that the risk premium in DSGE models is endogenous, only the continuous-time formulation enables us to identify the channels through which non-linearities affect the risk premium in production economies. As a result, our decomposition of the risk premium into (i) default risk, (ii) disaster risk, and (iii) residual risk sheds light on the prices that consumers are willing to pay for avoiding these particular risks.

The remainder of the paper is organized as follows. Section 2 solves in closed form a continuous-time version of Lucas' fruit-tree model with exogenous, stochastic production and obtains the risk premium. Section 3 studies the effects of non-linearities on the risk premium in Merton's neoclassical growth model. Section 4 concludes.

# 2 An endowment economy

This section computes the risk premium from the implicit Euler equation in an endowment economy. It shows how an extension to rare disasters can account for the observed equity premium puzzle, which became known as the Barro-Rietz 'rare disaster hypothesis'.

## 2.1 Lucas' fruit-tree model with rare disasters

Consider a fruit-tree economy (one risky asset or equity), and a riskless asset in normal times but with default risk (government bond) similar to Barro (2006).

### 2.1.1 Description of the economy

*Technology.* Consider an endowment economy (Lucas, 1978). Suppose production is entirely exogenous: no resources are utilized, and there is no possibility of affecting the output of any unit at any time, $Y_t = A_t$ where $A_t$ is the stochastic technology. Output is perishable. The law motion of $A_t$ will be taken to follow a Markov process,

$$dA_t = \bar{\mu} A_t dt + \bar{\sigma} A_t dB_t + \bar{J}_t A_{t-} dN_t, \tag{1}$$

where $B_t$ is a standard Brownian motion, and $N_t$ is a standard Poisson process. The jump size is assumed to be proportional to its value an instant before the jump, $A_{t-}$, ensuring that $A_t$ does not jump negative. For illustration, the independent random variable $\bar{J}_t$ has a degenerated distribution $\bar{J}_t \equiv \exp(\bar{\nu}) - 1$. This assumption is purely for reading convenience and extensions to other distributions of the jump size $\bar{J}_t$ pose no conceptional difficulties but are notationally more cumbersome.

Suppose ownership of fruit-trees with productivity $A_t$ is determined at each instant in a competitive stock market, and the production unit has one outstanding perfectly divisible equity share. A share entitles its owner to all of the unit's instantaneous output in $t$. Shares are traded at a competitively determined price, $p_t$. Suppose that for the risky asset,

$$dp_t = \mu p_t dt + \sigma p_t dB_t + p_{t-} J_t dN_t \tag{2}$$

and for a government bill with default risk

$$dp_0(t) = p_0(t) r dt + p_0(t_-) D_t dN_t, \tag{3}$$

where $D_t$ is a random variable denoting a random default risk in case of a disaster, and $q$ is the probability of default (cf. Barro, 2006). For illustration, we assume

$$D_t = \begin{cases} 0 & \text{with} \quad 1-q \\ \exp(\kappa)-1 & \text{with} \quad q \end{cases},$$

which can be generalized without any difficulty.

Because prices fully reflect all available information, the parameters will be determined in general equilibrium. The objective is to relate exogenous productivity changes to the market determined movements in asset prices. In fact, the evolution of prices ensures that assets are priced such that individuals are indifferent between holding more assets and consuming. Given initial wealth, we are looking for the optimal consumption path.

*Preferences.* Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. The consumer maximizes discounted expected life-time utility

$$U_0 \equiv E \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \ u'' < 0.$$

Assuming no dividend payments, the budget constraint reads

$$dW_t = ((\mu - r) w_t W_t + r W_t - C_t)\, dt + w_t \sigma W_t dB_t + ((J_t - D_t) w_{t-} + D_t) W_{t-} dN_t, \tag{4}$$

where $W_t$ is real financial wealth and $w_t$ denotes a consumer's share holdings.

*Equilibrium properties.* In this economy, it is easy to determine equilibrium quantities of consumption and asset holdings. The economy is closed and all output will be consumed, $C_t = Y_t$, and all shares will be held by capital owners.

### 2.1.2 The short-cut approach

Suppose that the only asset is the *market portfolio*,

$$dp_M(t) = \mu_M p_M(t) dt + \sigma_M p_M(t) dB_t - \zeta_M(t_-) p_M(t_-) dN_t, \tag{5}$$

where $\zeta_M(t)$ is considered as an exogenous stochastic jump-size. With probability $q$ it takes the value $\zeta_M$ and with probability $1-q$ it takes the value $\zeta_M^0$ (no default).

Consider the portfolio choice as an independent decision of the consumption problem. The consumer obtains income and has to finance its consumption stream from wealth,

$$dW_t = (\mu_M W_t - C_t)\,dt + \sigma_M W_t dB_t - \zeta_M(t_-)W_{t-}dN_t. \tag{6}$$

One can think of the original problem with budget constraint (4) as having been reduced to a simple Ramsey problem, in which we seek an optimal consumption rule given that income is generated by the uncertain yield of a (composite) asset (cf. Merton, 1973).

Define the *value function* as

$$V(W_0) \equiv \max_{\{C_t\}_{t=0}^{\infty}} E_0 \int_0^\infty e^{-\rho t}u(C_t)dt, \quad s.t. \quad (6), \quad W_0 > 0. \tag{7}$$

The Bellman equation reads when choosing the control $C_s \in \mathbb{R}_+$ at time $s$

$$\begin{aligned}\rho V(W_s) \quad &= \quad \max_{C_s} \big\{u(C_s) + (\mu_M W_s - C_s)\,V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} \\ &\qquad +(V((1-\zeta_M)W_s)q + V((1-\zeta_M^0)W_s)(1-q) - V(W_s))\lambda\big\}.\end{aligned}$$

Because it is a necessary condition, the *first-order conditions* reads

$$u'(C_s) - V_W(W_s) \quad = \quad 0 \quad \Rightarrow \quad V_W(W_s) = u'(C_s) \tag{8}$$

for any interior solution at any time $s = t \in [0,\infty)$.

It can be shown that the *Euler equation* is (cf. appendix)

$$\begin{aligned}du'(C_t) \quad = \quad &\big((\rho - \mu_M + \lambda)u'(C_t) - \sigma_M^2 W_t u''(C_t)C_W - u'(C((1-\zeta_M)W_t))(1-\zeta_M)q\lambda \\ &-u'(C((1-\zeta_M^0)W_t))(1-\zeta_M^0)(1-q)\lambda\big)dt \\ &+\pi u'(C_t)dB_t + (u'(C((1-\zeta_M(t_-))W_{t-})) - u'(C(W_{t-})))dN_t,\end{aligned} \tag{9}$$

which implicitly determines the optimal consumption path, where the traditional market price of risk can be defined as $\pi \equiv \sigma_M W_t u''(C_t)C_W/u'(C_t)$. We defined $C_W$ as the marginal propensity to consume out of wealth, i.e., the slope of the *consumption function*. Using the inverse function, we are able to determine the path for consumption ($u'' \neq 0$).

To shed some light on the effects of uncertainty, we use the Euler equation (9), and obtain

$$\begin{aligned}\rho - \frac{1}{dt}E\left[\frac{du'(C_t)}{u'(C_t)}\right] \quad = \quad &\mu_M - E\bigg[-\frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma_M^2 + \frac{u'(C((1-\zeta_M)W_t))}{u'(C(W_t))}\zeta_M q\lambda \\ &+\frac{u'(C((1-\zeta_M^0)W_s))}{u'(C(W_t))}\zeta_M^0(1-q)\lambda\bigg],\end{aligned} \tag{10}$$

which defines the *certainty equivalent rate of return* (shadow risk-free rate), i.e., the expected rate of return on saving less the expected *implicit risk premium*,

$$RP_t \equiv -\frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma_M^2 + E^\zeta\left[\frac{u'(C((1-\zeta_M(t))W_t))}{u'(C(W_t))}\zeta_M(t)\lambda\right]. \tag{11}$$

It gives the minimum difference an individual requires to accept an uncertain rate of return between the expected value (conditioned on no disasters) and the certain rate of return the individual is indifferent to. In equilibrium, this equals the expected cost of forgone consumption, i.e., the rate of time preference, and the expected rate of change of marginal utility in (10). The full expected rate of return on the market portfolio is $\mu_M - E(\zeta_M(t))\lambda$.

The implicit risk premium as from (11) extends the 'proportional probability premium' in static utility-of-wealth models (Pratt, 1964) to dynamic consumption-portfolio models. It is related to the effective relative risk aversion of the indirect utility function,

$$RRA_W = -\frac{V_{WW}W_t}{V_W} = -\frac{u''(C_t)C_W W_t}{u'(C_t)}. \tag{12}$$

Hence, the indirect utility function, i.e., the value function, and the utility function are intimately linked by the optimality condition (8). This condition demands that the marginal utility of consumption equals the marginal utility of wealth (cf. Breeden, 1979, p.274).

### 2.1.3 A more comprehensive approach

Define the *value function* as

$$V(W_0) \equiv \max_{\{(w_t,C_t)\}_{t=0}^\infty} E_0 \int_0^\infty e^{-\rho t}u(C_t)dt, \quad s.t. \quad (4), \quad W_0 > 0. \tag{13}$$

The Bellman equation reads when choosing the control $(w_s, C_s) \in \mathbb{R}\times\mathbb{R}_+$ at time $s$

$$\begin{aligned}\rho V(W_s) &= \max_{(w_s,C_s)} \big\{u(C_s) + ((\mu - r)w_s W_s + rW_s - C_s)V_W + \tfrac{1}{2}w_s^2\sigma^2 W_s^2 V_{WW} \\ &\quad + \big(V((e^\kappa + (e^{\nu_1} - e^\kappa)w_s)W_s)q \\ &\quad + V((1+(e^{\nu_2}-1)w_s)W_s)(1-q) - V(W_s)\big)\lambda\big\}.\end{aligned}$$

Because it is a necessary condition, the *first-order conditions* are

$$u'(C_s) - V_W = 0 \quad \Rightarrow \quad V_W = u'(C_s), \tag{14}$$

$$\begin{aligned} 0 &= (\mu - r)W_s V_W + w_s\sigma^2 W_s^2 V_{WW} + V_W((e^\kappa + (e^{\nu_1}-e^\kappa)w_s)W_s)(e^{\nu_1}-e^\kappa)W_s q\lambda \\ &\quad + V_W((1+(e^{\nu_2}-1)w_s)W_s)(1-q)(e^{\nu_2}-1)W_s\lambda \\ \Rightarrow \quad w_s &= -\frac{V_W(W_s)}{V_{WW}(W_s)W_s}\frac{\mu - r}{\sigma^2} - \frac{V_W((e^\kappa + (e^{\nu_1}-e^\kappa)w_s)W_s)}{V_{WW}(W_s)W_s}\frac{e^{\nu_1}-e^\kappa}{\sigma^2}q\lambda \\ &\quad - \frac{V_W((1+(e^{\nu_2}-1)w_s)W_s)}{V_{WW}(W_s)W_s}\frac{e^{\nu_2}-1}{\sigma^2}(1-q)\lambda, \end{aligned} \tag{15}$$

for any interior solution at any time $s = t \in [0, \infty)$. Therefore, without imposing further restrictions an analytical solution for the optimal shares is no longer available.

It can be shown that the *Euler equation* for consumption is (cf. appendix)

$$\begin{aligned} du'(C_t) &= \big((\rho - ((\mu - r)w_t + r) + \lambda)u'(C_t) - w_t^2\sigma^2 W_t V_{WW} \\ &\quad - u'(C((e^\kappa + (e^{\nu_1} - e^\kappa)w_t)W_t))(e^\kappa + (e^{\nu_1} - e^\kappa)w_t)q\lambda \\ &\quad - u'(C((1 + (e^{\nu_2} - 1)w_t)W_t))(1 + (e^{\nu_2} - 1)w_t)(1 - q)\lambda\big)dt + w_t\sigma W_t V_{WW} dB_t \\ &\quad + (u'(C((1 + (J_t - D_t)w_{t-} + D_t)W_{t-})) - u'(C(W_{t-})))dN_t. \end{aligned} \tag{16}$$

Before we proceed, it seems notable that the implicit risk premium from the Euler equation in (16) coincides with the short-cut approach (11) defining

$$\mu_M \equiv (\mu - r)w_t + r, \quad \sigma_M \equiv w_t\sigma, \quad \zeta_M(t) \equiv (D_t - J_t)w_t - D_t. \tag{17}$$

as the expected return and variance on the market portfolio conditioned on no disasters, and the market portfolio jump, respectively.

As shown below, the implicit risk premium comprises the risk premium of the expected market rate over the riskless rate (henceforth *market risk premium*) and the default risk.[7] It is related to the traditional market price of risk (i.e., the Sharpe ratio) through

$$\pi\sigma_M = -\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma_M^2 = RP_t - E^\zeta\left[\frac{u'(C((1 - \zeta_M(t))W_t))}{u'(C(W_t))}\zeta_M(t)\lambda\right]. \tag{18}$$

Both approaches can be employed to obtain the reward that investors demand and consumers implicitly would be willing to pay for bearing/avoiding the systematic market risk.

### 2.1.4 Towards a security market plane

Given demand for the risky asset and market-clearing, one usually obtains the equilibrium relation between expected return on any asset and the expected return on the market. It is straightforward to show that the market price follows

$$\begin{aligned} dp_M(t) &= ((\mu - r)w_t + r)\, p_M(t)dt + w_t\sigma p_M(t)dB_t \\ &\quad + ((J_t - D_t)w_{t-} + D_t)p_M(t_-)dN_t. \end{aligned} \tag{19}$$

Conditioned on no disasters, we define the instantaneous expected percentage change $\mu_M \equiv (\mu - r)w_t + r$, and its variance, $\sigma_M^2 \equiv w_t^2\sigma^2$, whereas $\zeta_M(t) \equiv (D_t - J_t)w_t - D_t$. With probability $q$ the portfolio jump is $\zeta_M \equiv (\zeta_M(t)|D_t = \exp(\kappa) - 1) = 1 - e^\kappa - (e^{\nu_1} - e^\kappa)w_t$,

[7] Whenever no ambiguity arises, we may use *market premium* and *market risk premium* interchangeably.

and with probability $1-q$ it takes the value $\zeta_M^0 \equiv (\zeta_M(t)|D_t = 0) = (1-e^{\nu_2})w_t$. Hence, the unconditional expected rate of return on the market portfolio, which includes disasters, is

$$\begin{aligned} E\left[\frac{dp_M(t)}{p_M(t_-)}\right] &= \mu_M - (\zeta_M q + \zeta_M^0(1-q))\lambda \\ &= \mu_M - (1-(e^{\nu_1}-e^{\kappa})w_t - e^{\kappa})\, q\lambda - (1-e^{\nu_2})w_t(1-q)\lambda. \end{aligned} \tag{20}$$

Similarly, we obtain expected percentage change on equity, and on government bills,

$$E\left[\frac{dp_t}{p_{t-}}\right] = \mu - (1-e^{\nu_1})q\lambda - (1-e^{\nu_2})(1-q)\lambda, \quad E\left[\frac{dp_0(t)}{p_0(t_-)}\right] = r - (1-e^{\kappa})q\lambda. \tag{21}$$

Given the demand for risky assets, we obtain the conditional market premium. Use the first-order condition for consumption, the optimal portfolio weights in (15) are

$$\begin{aligned} w_t &= -\frac{u'(C(W_t))}{u''(C_t)C_W W_t}\frac{\mu - r}{\sigma^2} - \frac{u'(C((e^{\kappa}+(e^{\nu_1}-e^{\kappa})w_t)W_t))}{u''(C(W_t))C_W W_t}\frac{e^{\nu_1}-e^{\kappa}}{\sigma^2}q\lambda \\ &\quad -\frac{u'(C((1+(e^{\nu_2}-1)w_t)W_t))}{u''(C(W_t))C_W W_t}\frac{e^{\nu_2}-1}{\sigma^2}(1-q)\lambda, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \mu_M - r &= -\frac{u''(C_t)C_W W_t}{u'(C(W_t))}\sigma_M^2 - \frac{u'(C((1-\zeta_M)W_t))}{u'(C(W_t))}(1-\zeta_M-e^{\kappa})q\lambda \\ &\quad +\frac{u'(C((1-\zeta_M^0)W_t))}{u'(C(W_t))}\zeta_M^0(1-q)\lambda, \end{aligned} \tag{22}$$

denoting the *conditional market premium*, i.e., conditioned on no disasters.

### 2.1.5 General equilibrium prices

This section shows that general equilibrium conditions pin down the prices in the economy. From the Euler equation (16), we obtain

$$\begin{aligned} dC_t &= \Big((\rho - \mu_M + \lambda)u'(C_t)/u''(C_t) - \sigma_M^2 W_t C_W - \tfrac{1}{2}u'''(C_t)/u''(C_t)\sigma_M^2 W_t^2 C_W^2 \\ &\quad - E^{\zeta}\left[u'(C((1-\zeta_M(t))W_t))(1-\zeta_M(t))\right]\lambda/u''(C_t)\Big)dt \\ &\quad +\sigma_M W_t C_W dB_t + (C((1-\zeta_M(t))W_{t-}) - C(W_{t-}))dN_t, \end{aligned} \tag{23}$$

where we employed the inverse function $c = g(u'(c))$ which has

$$g'(u'(c)) = 1/u''(c), \quad g''(u'(c)) = -u'''(c)/(u''(c))^3.$$

Economically, concave utility ($u'(c) > 0$, $u''(c) < 0$) implies risk aversion, whereas convex marginal utility, $u'''(c) > 0$, implies a positive precautionary saving motive. Accordingly,

$-u''(c)/u'(c)$ measures absolute risk aversion, whereas $-u'''(c)/u''(c)$ measures the degree of absolute prudence, i.e., the intensity of the precautionary saving motive (Kimball, 1990b).

Because output is perishable, using the market clearing condition $Y_t = C_t = A_t$, and

$$dC_t = \bar{\mu} C_t dt + \bar{\sigma} C_t dB_t + \bar{J}_t C_{t-} dN_t, \tag{24}$$

the risk free rate and both market prices of risk are pinned down in general equilibrium. In particular, we obtain $J_t$ implicitly as function of $\bar{J}_t$, $D_t$ (stochastic investment opportunities), and the curvature of the consumption function, where $\tilde{C}(W_t) \equiv C((1-\zeta_M(t))W_t)/C(W_t)$ defines optimal consumption jumps. In equilibrium, market clearing requires the percentage jump in aggregate consumption to match the size of the disaster, $\bar{J}_t = \tilde{C}(W_t) - 1$, and thus $\bar{J} = C((1+(J_t - D_t)w_t + D_t)W_t)/C(W_t) - 1$ implies a constant jump size. For example, if consumption is linearly homogeneous in wealth, the jump size of the risky asset satisfies[8]

$$\zeta_M = \zeta_M^0 \quad \Rightarrow \quad e^{\nu_1} - e^{\kappa} = \frac{e^{\bar{\nu}} - e^{\kappa}}{e^{\bar{\nu}} - 1}(e^{\nu_2} - 1). \tag{25}$$

Without loss in generality, using the condition that the optimal jump in consumption is constant, $C((1-\zeta_M(t))W_t) = e^{\bar{\nu}} C(W_t)$, optimal portfolio weights satisfy

$$w_t = -\frac{u'(C_t)}{u''(C_t) C_W W_t} \frac{\mu - r}{\sigma^2} - \frac{(e^{\nu_1} - e^{\kappa})q + (e^{\nu_2} - 1)(1-q)}{\sigma^2} \frac{u'(e^{\bar{\nu}} C(W_t))}{u''(C(W_t)) C_W W_t} \lambda,$$

where the first term is the usual myopic demand for the risky asset, whereas the second term reflects the demand for the risky asset as a vehicle to *hedge* against the disaster risk. This illustrates that the market clearing condition in fact identifies the jump risk by restricting admissible market portfolios and thus provides a closed-form solution for optimal weights. Similarly, the market clearing condition pins down $\sigma_M W_t C_W = \bar{\sigma} C_t$, and

$$\mu_M - r = -\frac{u''(C_t) C_W W_t}{u'(C(W_t))} \sigma_M^2 - \frac{u'(e^{\bar{\nu}} C(W_t))}{u'(C(W_t))} \left((1 - e^{\kappa})q - \zeta_M\right) \lambda.$$

Inserting our results back into (23), we obtain that consumption follows,

$$\begin{aligned} dC_t &= (\rho - r + \lambda) \frac{u'(C_t)}{u''(C_t)} dt - \tfrac{1}{2} \frac{u'''(C_t)}{u''(C_t)} \sigma_M^2 W_t^2 C_W^2 dt - (1 - (1 - e^{\kappa})q) \frac{u'(e^{\bar{\nu}} C_t)}{u''(C_t)} \lambda dt \\ &\quad + \sigma_M W_t C_W dB_t + (C((1-\zeta_M(t))W_{t-}) - C(W_{t-})) dN_t. \end{aligned}$$

This in turn determines the riskless rate of return as

$$r = \rho - \frac{u''(C_t) C_t}{u'(C_t)} \bar{\mu} - \tfrac{1}{2} \frac{u'''(C_t) C_t^2}{u'(C_t)} \bar{\sigma}^2 + \lambda - (1 - (1 - e^{\kappa})q) \frac{u'(e^{\bar{\nu}} C_t)}{u'(C_t)} \lambda. \tag{26}$$

[8] Conditioning on no default, $(\zeta_M(t)|D_t = 0) = \zeta_M^0$, gives $e^{\bar{\nu}} - 1 = (e^{\nu_2} - 1)w_t$. Conditioning on default, $(\zeta_M(t)|D_t = \exp(\kappa) - 1) = \zeta_M$, demands $e^{\bar{\nu}} - 1 = e^{\kappa} - 1 + (e^{\nu_1} - e^{\kappa})w_t$.

As a result, the higher the subjective rate of time preference, $\rho$, the higher is the general equilibrium interest rate to induce individuals to defer consumption (cf. Breeden, 1986). For convex marginal utility (decreasing absolute risk aversion), $u'''(c) > 0$, a lower conditional variance of dividend growth, $\bar{\sigma}^2$, a higher conditional mean of dividend growth, $\bar{\mu}$, and a higher default probability, $q$, decrease the bond price and increases the interest rate.

#### 2.1.6 Components of the risk premium

Observe that the implicit risk premium (11) in general equilibrium simplifies to

$$RP_t = \underbrace{-\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma_M^2}_{\text{residual risk}} + \underbrace{\frac{u'(e^{\bar{\nu}} C(W_t))}{u'(C(W_t))} \zeta_M \lambda}_{\text{total jump risk}} \tag{27}$$

whereas the conditional market premium reads

$$\begin{aligned} \mu_M - r &= \underbrace{-\frac{u''(C_t) C_W W_t}{u'(C(W_t))} \sigma_M^2}_{\text{residual risk}} + \underbrace{(\zeta_M - (1 - e^{\kappa}) q) \frac{u'(e^{\bar{\nu}} C(W_t))}{u'(C(W_t))} \lambda}_{\text{disaster risk}} \\ &= \underbrace{-\frac{u''(C_t) C_W W_t}{u'(C(W_t))} \sigma_M^2}_{\text{residual risk}} + \underbrace{\frac{u'(e^{\bar{\nu}} C(W_s))}{u'(C(W_t))} \zeta_M \lambda}_{\text{total jump risk}} - \underbrace{(1 - e^{\kappa})\, q \frac{u'(e^{\bar{\nu}} C(W_t))}{u'(C(W_t))} \lambda}_{\text{default risk}}. \end{aligned} \tag{28}$$

Notice that $\bar{\nu} < 0$ and $\kappa < 0$ are typical for a 'disaster' hypothesis.

In the presence of default risk, the conditional market premium differs from the implicit risk premium. The reason is that we obtain the implicit risk premium from the certainty equivalent rate of return (shadow risk-free rate), but the government bill has a risk of default. This default risk is not rewarded in the market as there is no truly riskless asset, but is reflected in the implicit risk premium. If there was no default risk, the implicit risk premium would have the usual interpretation of the conditional market premium.

#### 2.1.7 Explicit solutions

As shown in Merton (1971), the standard dynamic consumption and portfolio selection problem has explicit solutions where consumption is a linear function of wealth. For later references, we provide the solution for constant relative risk aversion (CRRA).

**Proposition 2.1 (CRRA preferences)** *If utility exhibits constant relative risk aversion, i.e., $-u''(C_t)C_t/u'(C_t) = \theta$, then the optimal portfolio weights are constant, and the optimal consumption function is proportional to wealth, $C_t = C(W_t) = bW_t$, where*

$$b \equiv \left(\rho + \lambda - (1-\theta)\mu_M - (1-\zeta_M)^{1-\theta}\lambda + (1-\theta)\theta\tfrac{1}{2}\sigma_M^2\right)/\theta.$$

*Optimal portfolio weights are given by*

$$w = \frac{\mu - r}{\theta\sigma^2} + ((e^{\nu_1} - e^{\kappa})q + (e^{\nu_2} - 1)(1-q)) \frac{(1-\zeta_M)^{-\theta}\lambda}{\theta\sigma^2}. \tag{29}$$

**Proof.** see Appendix A.1.4 ■

**Corollary 2.2** *Use the policy function, $C_t = C(W_t) = bW_t$, and the implicit risk premium in general equilibrium (27), to obtain*

$$RP \equiv \theta\sigma_M^2 + e^{-\theta\bar{\nu}}\zeta_M\lambda. \tag{30}$$

*The riskless rate in (26) reads $r = \rho + \theta\bar{\mu} - \frac{1}{2}\theta(1+\theta)\bar{\sigma}^2 + \lambda - (1-(1-e^{\kappa})q)\, e^{-\theta\bar{\nu}}\lambda$. Hence, the conditional market premium (28) and variance of the market portfolio, i.e., for a sample conditioned on no disasters, is given by*

$$\mu_M - r = \theta\sigma_M^2 + e^{-\theta\bar{\nu}}\left(\zeta_M - (1-e^{\kappa})\, q\right)\lambda, \quad \textit{and} \quad \sigma_M = \bar{\sigma}. \tag{31}$$

*The unconditional market premium, i.e., for long samples, is $\mu_M - \zeta_M\lambda - (r - (1-e^{\kappa})q\lambda)$. As a result, for constant relative risk aversion, $RRA_W = \theta$, the risk premium is constant.*

#### 2.1.8 Stochastic discount factor

This section illustrates the link between the implicit risk premium and the stochastic discount factor (SDF). Our approach can be used to compute the SDF in any continuous-time DSGE model. We obtain the SDF from the Euler equation (9), which in general equilibrium is

$$\begin{aligned} du'(C_t) &= (\rho - r)u'(C_t)dt + (1-e^{\kappa})u'(C(e^{\bar{\nu}}W_t))q\lambda dt - (u'(C(e^{\bar{\nu}}W_t)) - u'(C_t))\lambda dt \\ &\quad + \pi u'(C_t)dB_t + (u'(C(e^{\bar{\nu}}W_{t-})) - u'(C(W_{t-})))dN_t, \end{aligned}$$

where the deterministic term consists of, firstly, the difference between the subjective rate of time preference and the riskless rate, secondly, a term which transforms this rate into the certainty equivalent rate of return (shadow risk-free rate) and, thirdly, the compensation which transforms the Poisson process to a martingale. For $s \geq t$, we may write

$$\begin{aligned} d\ln u'(C_t) &= \left(\frac{u''(C_t)C_t}{u'(C_t)}\bar{\mu} + \tfrac{1}{2}\frac{u'''(C_t)C_t^2}{u'(C_t)}\bar{\sigma}^2 - \tfrac{1}{2}\pi^2\right)dt \\ &\quad + \pi dB_t + \left(\ln u'(C(e^{\bar{\nu}}W_{t-})) - \ln u'(C(W_{t-}))\right)dN_t \\ \Leftrightarrow \frac{e^{-(s-t)\rho}u'(C_s)}{u'(C_t)} &= \exp\left(-\int_t^s \left(\rho - \frac{u''(C_v)C_v}{u'(C_v)}\bar{\mu} - \tfrac{1}{2}\frac{u'''(C_v)C_v^2}{u'(C_v)}\bar{\sigma}^2 + \tfrac{1}{2}\pi^2\right)dv\right) \\ &\quad \times \exp\left(\int_t^s \pi dB_v + \int_t^s \left(\ln u'(C(e^{\bar{\nu}}W_{t-})) - \ln u'(C(W_{t-}))\right)dN_v\right), \end{aligned}$$

i.e., equating discounted marginal utility in $s$ and $t$. Therefore,

$$m_s/m_t \;\equiv\; \frac{e^{-(s-t)\rho}u'(C_s)}{u'(C_t)} \tag{32}$$

defines the *stochastic discount factor* (also known as the pricing kernel or state-price density) which can be used to price any asset in this economy. For CRRA preferences, it reads

$$\begin{aligned} m_s/m_t &= \exp\left(-(r - e^{-\theta\bar{\nu}}(1-e^{\kappa})q\lambda + \tfrac{1}{2}(\theta\bar{\sigma})^2 + (e^{-\bar{\nu}\theta}-1)\lambda)(s-t)\right) \\ &\quad \times \exp\left(\theta\bar{\sigma}(B_s - B_t) - \theta\bar{\nu}(N_s - N_t)\right) \\ &= \exp\left(-(\rho + \theta\bar{\mu} - \tfrac{1}{2}\theta\bar{\sigma}^2)(s-t) + \theta\bar{\sigma}(B_s - B_t) - \theta\bar{\nu}(N_s - N_t)\right) \end{aligned}$$

which has the well known properties (cf. Campbell, 2000).

# 3 A neoclassical production economy

This section illustrates that non-linearities in a neoclassical DSGE model imply interesting asset market implications, in particular these can generate a time-varying risk premium. We use a version of Merton' (1975) asymptotic theory of growth under uncertainty.

## 3.1 A model of growth under uncertainty

This section assumes that there is no truly riskless asset. We employ the certainty equivalent rate of return, or the shadow risk-free rate, to obtain the implicit risk premium.

### 3.1.1 Description of the economy

*Technology.* At any time, the economy has some amounts of capital, labor, and knowledge, and these are combined to produce output. The production function is a constant return to scale technology $Y_t = A_t F(K_t, L)$, where $K_t$ is the aggregate capital stock, $L$ is the constant population size, and $A_t$ is the stock of knowledge or total factor productivity (TFP), which in turn is driven by a standard Brownian motion $B_t$

$$dA_t = \bar{\mu}A_t dt + \bar{\sigma}A_t dB_t. \tag{33}$$

$A_t$ has a log-normal distribution with $E_0(\ln A_t) = \ln A_0 + (\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)t$, and $Var_0(\ln A_t) = \bar{\sigma}^2 t$.

The capital stock increases if gross investment exceeds stochastic capital depreciation,

$$dK_t = (I_t - \delta K_t)dt + \sigma K_t dZ_t + J_t K_{t-} dN_t, \tag{34}$$

where $Z_t$ is a standard Brownian motion (uncorrelated with $B_t$), and $N_t$ is a standard Poisson process with arrival rate $\lambda$. Unlike in Merton's (1975) model, the assumption of stochastic depreciation introduces instantaneous riskiness making physical capital indeed a risky asset (for similar examples see Turnovsky, 2000). The fundamental difference to Lucas' endowment economy is that the *shares* follow a stochastic process (i.e., not only the production technology, but the number of trees is stochastic). The jump size is assumed to be proportional where we assume that $J_t \equiv \exp(\nu) - 1$ is a degenerated distribution.

*Preferences.* Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. The consumer maximizes expected life-time utility

$$U_0 \equiv E_0 \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \; u'' < 0 \tag{35}$$

subject to

$$dW_t = ((r_t - \delta) W_t + w_t^L - C_t) dt + \sigma W_t dZ_t + J_t W_{t-} dN_t. \tag{36}$$

$W_t \equiv K_t / L$ denotes individual wealth, $r_t$ is the rental rate of capital, and $w_t^L$ is labor income. The paths of factor rewards are taken as given by the representative consumer.

*Equilibrium properties.* In equilibrium, factors of production are rewarded with value marginal products, $r_t = Y_K$ and $w_t^L = Y_L$. The goods market clearing condition demands

$$Y_t = C_t + I_t. \tag{37}$$

Solving the model requires the aggregate capital accumulation constraint (34), the goods market equilibrium (37), equilibrium factor rewards of perfectly competitive firms, and the first-order condition for consumption. It is a system of stochastic differential equations determining, given initial conditions, the paths of $K_t$, $Y_t$, $r_t$, $w_t^L$ and $C_t$, respectively.

### 3.1.2 The short-cut approach

Define the value of the optimal program as

$$V(W_0, A_0) = \max_{\{C_t\}_{t=0}^\infty} U_0 \quad s.t. \quad (36) \quad \text{and} \quad (33) \tag{38}$$

denoting the present value of expected utility along the optimal program. It can be shown that the first-order condition for the problem is (cf. appendix)

$$u'(C_t) = V_W(W_t, A_t), \tag{39}$$

for any $t \in [0, \infty)$, making consumption a function of the state variables $C_t = C(W_t, A_t)$.

It can be shown that the *Euler equation* is (cf. appendix)

$$\begin{aligned} du'(C_t) &= (\rho - (r_t - \delta) + \lambda) u'(C_t) dt - u'(C(e^\nu W_t, A_t)) e^\nu \lambda dt - \sigma^2 V_{WW} W_t dt + V_{AW} A_t \bar{\sigma} dB_t \\ &\quad + V_{WW} W_t \sigma dZ_t + [u'(C(e^\nu W_{t-}, A_{t-})) - u'(C(W_{t-}, A_{t-}))] dN_t \qquad (40) \\ &= (\rho - (r_t - \delta) + \lambda) u'(C_t) dt - u'(C(e^\nu W_t, A_t)) e^\nu \lambda dt - \sigma^2 u''(C_t) C_W W_t dt \\ &\quad + u''(C_t)(C_A A_t \bar{\sigma} dB_t + C_W W_t \sigma dZ_t) + [u'(C(e^\nu W_{t-}, A_{t-})) - u'(C(W_{t-}, A_{t-}))] dN_t, \end{aligned}$$

which implicitly determines the optimal consumption path. To shed some light on the effects of uncertainty, we use the Euler equation and obtain the (implicit) risk premium from

$$\begin{aligned}\frac{du'(C_t)}{u'(C_{t-})} &= \left(\rho - (r_t - \delta) + \lambda - \frac{u'(C(e^{\nu}W_t, A_t))}{u'(C(W_t, A_t))}e^{\nu}\lambda - \frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma^2\right) dt \\ &\quad + \frac{u''(C_t)}{u'(C_t)}\left(C_A A_t \bar{\sigma} dB_t + C_W W_t \sigma dZ_t\right) + \left[\frac{u'(C(e^{\nu}W_{t-}, A_{t-}))}{u'(C(W_{t-}, A_{t-}))} - 1\right] dN_t \\ \Rightarrow \quad \frac{1}{dt}E\left[\frac{du'(C_t)}{u'(C_t)}\right] &= \rho - E(r_t) + \delta + E\left[\frac{u'(C(e^{\nu}W_t, A_t))}{u'(C(W_t, A_t))}(1 - e^{\nu})\lambda - \frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma^2\right].\end{aligned}$$

Collecting terms, we find that the *certainty equivalent rate of return* equals the expected return net of depreciation, $E(r_t - \delta)$, less the expected *implicit risk premium*,

$$RP_t \equiv -\frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma^2 + \frac{u'(C(e^{\nu}W_t, A_t))}{u'(C(W_t, A_t))}(1 - e^{\nu})\lambda. \tag{41}$$

It is remarkable that the structure is equivalent to the endowment economy (27), but the premium in (41) has quite interesting properties. Hence, the implicit risk premium indeed refers to the rewards that investors demand for bearing the systematic risk, while it does not directly account for the risk of a stochastically changing total factor productivity (33).

### 3.1.3 Explicit solutions

A convenient way to describe the behavior of the economy is in terms of the evolution of $C_t$, $A_t$ and $W_t$. Similar to the endowment economy there are explicit solutions available, due to the non-linearities only for specific parameter restrictions. Below we use two known restrictions where the *policy function* $C_t = C(A_t, W_t)$ (or consumption function) is available, and all economic variables can be solved in closed form.

**Proposition 3.1 (linear-policy-function)** *If the production function is Cobb-Douglas, $Y_t = A_t K_t^{\alpha} L^{1-\alpha}$, utility exhibits constant relative risk aversion, i.e., $-u''(C_t)C_t/u'(C_t) = \theta$, and $\alpha = \theta$, then optimal consumption is linear in wealth.*

$$\begin{aligned}\alpha = \theta \quad &\Rightarrow \quad C_t = C(W_t) = \phi W_t \\ \textit{where} \quad &\phi \equiv (\rho - (e^{(1-\theta)\nu} - 1)\lambda + (1-\theta)\delta)/\theta + \tfrac{1}{2}(1-\theta)\sigma^2\end{aligned} \tag{42}$$

**Proof.** see Appendix A.2.2 ■

**Corollary 3.2** *Use the policy function $C_t = C(W_t) = \phi W_t$ and (41) to obtain*

$$RP = \theta\sigma^2 + e^{-\theta\nu}(1 - e^{\nu})\lambda. \tag{43}$$

**Proposition 3.3 (constant-saving-function)** *If the production function is Cobb-Douglas, $Y_t = A_t K_t^\alpha L^{1-\alpha}$, utility exhibits constant relative risk aversion, i.e., $-u''(C_t)C_t/u'(C_t) = \theta$, and the subjective discount factor is*

$$\bar{\rho} \equiv (e^{(1-\alpha\theta)\nu} - 1)\lambda - \theta\bar{\mu} + \tfrac{1}{2}\left(\theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2\right) - (1-\alpha\theta)\delta,$$

*then optimal consumption is proportional to current income (i.e., non-linear in wealth).*

$$\rho = \bar{\rho} \quad \Rightarrow \quad C_t = C(W_t, A_t) = (1-s)A_t W_t^\alpha, \quad \theta > 1, \quad \textit{where} \quad s \equiv 1/\theta \tag{44}$$

**Proof.** see Appendix A.3.2 ■

**Corollary 3.4** *Use the policy function $C_t = C(W_t, A_t) = (1-s)A_t W_t^\alpha$ and (41) to obtain*

$$RP = \alpha\theta\sigma^2 + e^{-\alpha\theta\nu}(1-e^\nu)\lambda. \tag{45}$$

We are now in a position to understand why the (implicit) risk premium depends on the curvature of the policy function (or consumption function). Any homogenous consumption function, where $C_W(W_t, A_t)W_t = kC(W_t, A_t)$ or equivalently $C(cW_t, A_t) = c^k C(W_t, A_t)$ for $c, k \in \mathbb{R}_+$, implies a constant risk premium. Technically, the policy function is homogenous of degree $k$ in wealth. Because these functions are obtained only for knife-edge restrictions, we conclude that the (implicit) risk premium generally will be dependent on wealth, which in turn implies that a time-varying behavior as wealth is changing stochastically.

Economically, the reason why the risk premium depends on the curvature of the policy function (and can vary over time) is that the optimal response to disasters or shocks depends on the level of wealth. An individual with high levels of financial wealth will adjust its optimal expenditures for consumption differently from an individual that has no financial wealth. Though the utility function has CRRA with respect to consumption, the indirect utility function (the value function), except for the knife-edge cases above, does *not* exhibit CRRA with respect to wealth. This finding is closely related to the link Kimball (1990a) shows between the marginal propensity to consume and the effective risk aversion of the value function. Accordingly, a higher marginal propensity to consume out of gross wealth (inclusive of labor income) raises the effective risk aversion. A concave consumption function implies that the effective level of risk aversion will fall more quickly with wealth than it would if the marginal propensity to consume were constant (Carroll and Kimball, 1996, p.982).

There are two important differences to the earlier work. First, our consumption function is defined by the optimal policy rule which gives consumption as a function of financial wealth (exclusive of labor income), which is the only observable and tradable asset. Hence, it cannot easily be interpreted as a function of gross wealth (inclusive of labor income) or total wealth (i.e., financial and human wealth). Thus, the marginal propensity to consume out of wealth is defined by the slope of the consumption function with respect to financial wealth. In contrast, the consumption and saving rates, in the tradition of the Solow model, refers to current income (i.e., labor and capital income). Second, the effects of uncertainty are studied in a general equilibrium environment which allows us to obtain analytical solutions for linear and strictly concave consumption functions in a DSGE model for mild parametric restrictions, while Carroll and Kimball restrict their focus on partial equilibrium models, leaving the processes for labor income and capital returns exogenous.

Unfortunately, an analytical study of the structural parameters in the general case is not possible. Though clearly being extreme cases, our explicit solutions are important to understand the mechanism that affect the risk premium in DSGE models. Both solutions imply that the consumption function is homogenous and thus the risk premium is constant.[9] Below we study the implications when allowing the parameters to take different values.

### 3.1.4 Numerical solutions

This section implements the algorithm as in Posch and Trimborn (2009) to obtain a numerical solution for the case where $\sigma = \bar{\sigma} = \bar{\mu} = 0$, and with $A = 1$.[10] As it is seen below, this assumption does not affect our conclusions, but reduces the computational burden as the reduced form representing the dynamics of the DSGE model can be summarized as

$$\begin{aligned} dW_t &= ((r_t - \delta)W_t + w_t^L - C_t)dt - (1 - e^{\nu})W_{t-}dN_t, \\ dC_t &= -\frac{u'(C_t)}{u''(C_t)}(r_t - \delta - \rho - \lambda)dt - \frac{u'(C(e^{\nu}W_t))}{u''(C(W_t))}e^{\nu}\lambda dt + [C(e^{\nu}W_{t-}) - C(W_{t-})]dN_t, \end{aligned}$$

where $r_t = Y_K$, and $w_t^L = Y_L$. For the case of Cobb-Douglas production, $Y_t = AK_t^{\alpha}L^{1-\alpha}$, and CRRA preferences with relative risk aversion $\theta$, we obtain from (41)

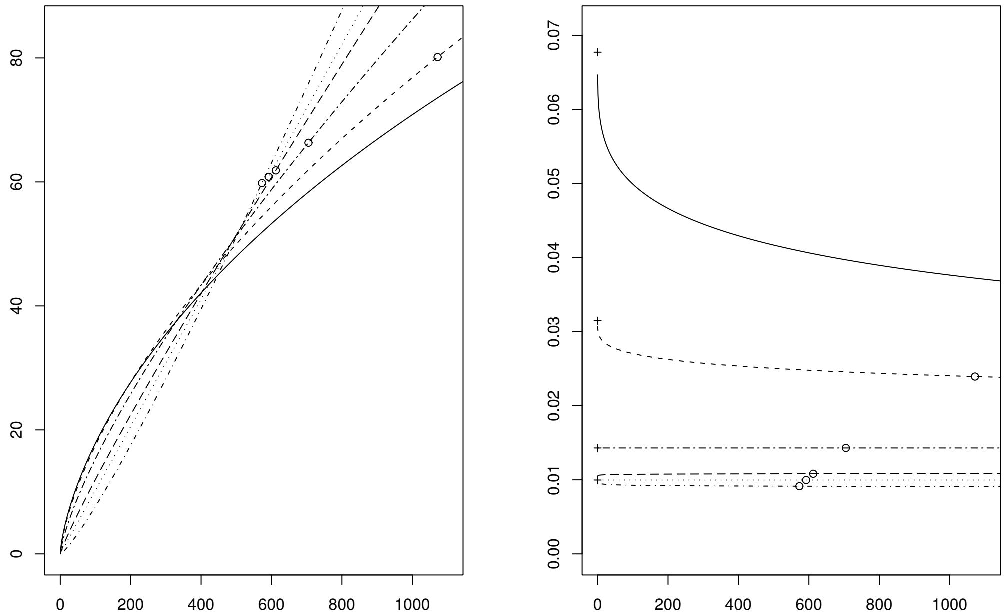
$$RP \equiv \frac{C(e^{\nu}W_t)^{-\theta}}{C(W_t)^{-\theta}}(1 - e^{\nu})\lambda.$$

Figure 1 illustrates the optimal policy function and the resulting (implicit) risk premium for different values for the parameter of relative risk aversion. For $\theta = \alpha$ the policy function is a linear-homogenous function with slope $\phi$, which refers to the analytical solution in (42). In this singular case the risk premium is $e^{-\theta\nu}(1 - e^{\nu})\lambda$, which is equivalent to the risk premium in the fruit-tree model. At each point, a change of the expected proportional decline in

[9] For $\alpha = \theta$, the consumption function becomes a linear function in wealth, i.e., is linearly homogeneous or homogeneous of degree one. In the case of $\rho = \bar{\rho}$, which is only possible for values $\theta > 1$, the consumption function becomes homogeneous of degree $\alpha$.

[10] The numerical solution makes use of Waveform Relaxation, which solves the original (non-linear) system of functional stochastic differential equations of retarded type (cf. Posch and Trimborn, 2009).

Figure 1: Risk premia in a production economy




Notes: These figures illustrate the optimal policy functions (left panel) and the risk premium (right panel) as a function of financial wealth for different levels of relative risk aversion for the case of $\sigma = \bar{\sigma} = \bar{\mu} = 0$. The calibrations of other parameters is $(\rho, \alpha, \theta, \delta, \lambda, 1 - e^{\nu}) = (.05, .75, \cdot, .1, .017, .4)$ where $\theta = .5$ (dotdash), $\theta = .75$ (dotted), $\theta = 1$ (longdash), $\theta = 1.9406$ (twodash) which refers to the knife-edge case $\rho = \bar{\rho}$ in (44) with a constant saving rate, $\theta = 4$ (dashed), and $\theta = 6$ (solid).

marginal utility equals the change in capital rewards, implying a constant risk premium. For $\theta < \alpha$ the policy function is convex, and the marginal propensity to consume increases with wealth, $C(e^{\nu}W_t) < e^{\nu}C(W_t)$. This increase, however, is less rapid than the increase of the consumption-wealth ratio, which lowers the effective level of risk aversion. Hence, the risk premium is convex and has the upper bound $e^{-\alpha\nu}(1 - e^{\nu})\lambda$ for wealth approaching zero. For $\theta > \alpha$, which is the empirically most plausible scenario, the consumption function has the standard form, i.e., (strictly) concave such that the marginal propensity to consume is decreasing with wealth, $C(e^{\nu}W_t) > e^{\nu}C(W_t)$. In this case, the properties of the risk premium depend on whether the pure rate of time preference, $\rho$, exceeds or falls short of $\bar{\rho}$.

At the knife-edge value of $\rho = \bar{\rho}$ the policy function is homogeneous of degree $\alpha$, which refers to the analytical solution in (44), where the savings rate is constant, $s = 1/\theta$, and the risk premium is $e^{-\alpha\theta\nu}(1 - e^{\nu})\lambda$. For $\rho < \bar{\rho}$ the individual prefers a higher savings rate, $s(W_t) > s$, and the marginal propensity to consume out of wealth decreases more rapidly than it would if the saving rate (or consumption-income ratio) were constant, which lowers the effective risk aversion of the value function (Carroll and Kimball, 1996). Because the saving rate is increasing in wealth and bounded by unity, $s < s(W_t) < 1$, the risk premium is convex and has the upper bound $e^{-\alpha\theta\nu}(1-e^{\nu})\lambda$ for wealth approaching zero. For $\rho > \bar{\rho}$ the saving rate is smaller, $s(W_t) < s$, and the marginal propensity to consume out of wealth decreases less rapidly than it would if the saving rate were constant, which raises the effective risk aversion of the value function. Since the saving rate is decreasing in wealth, the risk premium is concave with lower bound $e^{-\max(\theta,1)\alpha\nu}(1-e^{\nu})\lambda$ for wealth approaching zero.

In our numerical study, $\bar{\rho}$ depends on the arrival rate, $\lambda$, the disaster size, $e^{\nu}-1$, the output elasticity of capital, $\alpha$, and the risk aversion, $\theta$, which coincides with the inverse of the intertemporal elasticity of substitution (IES), and the rate of depreciation, $\delta$. For the case $\alpha\theta > 1$, that is when the output elasticity of capital exceeds the IES, this critical value is positive, $\bar{\rho} > 0$, and vice versa. Thus, for the empirically most plausible calibrations, e.g., for $\alpha \approx 0.5$ and $\theta \approx 4$, we find that $\alpha\theta > 1$ and obtain a positive knife-edge value, $\bar{\rho} > 0$.

For the general case, i.e., using the risk premium implicitly obtained from the Euler equation (41), the same analysis could be conducted. Then, the consumption function will be concave in wealth for $\theta \geq \alpha$ and the risk premium, conditional on the state variable $A_t$, will have the same properties as in Figure 1. Moreover, the knife-edge value $\bar{\rho}$ as from the definition in (44) decreases in the mean, $\bar{\mu}$, but increases in the variance $\bar{\sigma}^2$ of TFP growth, and for the case $\alpha\theta > 1$ increases in the variance of stochastic depreciation, $\sigma^2$.

#### 3.1.5 Human wealth and financial wealth

One interpretation is that the individual implicitly solves an optimal consumption problem in a stochastically changing investment opportunity set. In this view, the state variables which determine investment opportunities are the *aggregate* capital stock, $K_t$, and total factor productivity $A_t$, whereas the risky asset returns $r_t = r(A_t, K_t)$ and the wage rate $w_t = w(A_t, K_t)$ depend on the state variables. The DSGE model at hand is a specific case where general equilibrium conditions pin down asset prices, as well as cost of capital and leisure (hours) in the economy (cf. Campbell and Viceira, 2002, chap. 6).

In particular, as shown in Bodie, Merton and Samuelson (1992, p.431) one could think of any optimal decisions of households in terms of financial wealth (physical assets) and human wealth (present value of future labor income), since the individual's human capital is essentially the same as a financial asset, except that it is not traded, but valued by the individual *as if it were a traded asset*. It therefore seems important to allow for flexible labor supply when studying the risk premium. If we allow individuals to choose their amount of leisure optimally, we can study the impact of adding one additional channel to mitigate the presence of risk, which potentially has an effect on the properties of the premium.

## 3.2 An extension: endogenous labor supply

This section allows for elastic labor supply in the neoclassical DSGE model. For reading convenience, this section replicates some of the equations from the previous section.

### 3.2.1 Description of the economy

*Technology.* As before, the production function is a constant return to scale technology, $Y_t = A_t F(K_t, H_t)$, where $K_t$ is the aggregate capital stock, $H_t$ is total hours worked, $L$ is the constant population size, and $A_t$ is total factor productivity, which in turn is driven by a standard Brownian motion $B_t$

$$dA_t = \bar{\mu} A_t dt + \bar{\sigma} A_t dB_t. \tag{46}$$

The capital stock increases if gross investment exceeds stochastic capital depreciation,

$$dK_t = (I_t - \delta K_t)dt + \sigma K_t dZ_t + J_t K_{t-} dN_t, \tag{47}$$

where $Z_t$ is a standard Brownian motion (uncorrelated with $B_t$), and $N_t$ is a standard Poisson process with arrival rate $\lambda$. The jump size is assumed to be proportional where we assume that $J_t \equiv \exp(\nu) - 1$ is a degenerated distribution.

*Preferences.* Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers, such that $C_t = Lc_t = c_t$ and $H_t = 1 - l_t$. The consumer maximizes expected life-time utility

$$U_0 \equiv E_0 \int_0^\infty e^{-\rho t} u(C_t, H_t) dt, \quad u_C > 0,\ u_H < 0,\ u_{CC} \leq 0,\ u_{CC} u_{HH} - (u_{CH})^2 \geq 0, \tag{48}$$

subject to

$$dW_t = ((r_t - \delta)W_t + H_t w_t^H - C_t)dt + \sigma W_t dZ_t + J_t W_{t-} dN_t. \tag{49}$$

$W_t \equiv K_t / L$ denotes individual wealth, $r_t$ is the rental rate of capital, and $H_t w_t^H$ is labor income. The paths of factor rewards are taken as given by the representative consumer.

*Equilibrium properties.* In equilibrium, factors of production are rewarded with value marginal products, $r_t = Y_K$ and $w_t^H = Y_H$. The goods market clearing condition demands

$$Y_t = C_t + I_t. \tag{50}$$

Solving the model requires the aggregate constraints for factor productivity (46), capital (47), the goods market equilibrium (50), equilibrium factor rewards of perfectly competitive firms, and the first-order condition for consumption and hours. It is a system of equations determining, given initial conditions, the paths of $K_t$, $Y_t$, $r_t$, $w_t^H$, $C_t$ and $H_t$, respectively.

### 3.2.2 The short-cut approach

Define the value of the optimal program as

$$V(W_0, A_0) = \max_{\{C_t, H_t\}_{t=0}^{\infty}} U_0 \quad s.t. \quad (49) \quad \text{and} \quad (46), \tag{51}$$

denoting the present value of expected utility along the optimal program. It can be shown that the first-order conditions for any interior solution are (cf. appendix)

$$\begin{aligned} u_C(C_t, H_t) &= V_W(W_t, A_t), & (52)\\ -u_H(C_t, H_t) &= w_t^H V_W(W_t, A_t), & (53) \end{aligned}$$

for any $t \in [0, \infty)$, making optimal consumption and hours functions of the state variables $C_t = C(W_t, A_t)$ and $H_t = H(W_t, A_t)$, respectively (cf. Bodie et al., 1992, p.435). Specifically it pins down the price (or opportunity cost) of leisure,

$$w_t^H = -\frac{u_H(C_t, H_t)}{u_C(C_t, H_t)}, \tag{54}$$

which determines the amount of leisure the individual 'purchases'.

It can be shown that the *Euler equation* for consumption is (cf. appendix)

$$\begin{aligned} du_C = {} & (\rho - (r_t - \delta) + \lambda)u_C dt - u_C(C(e^{\nu}W_t, A_t), H(e^{\nu}W_t, A_t))e^{\nu}\lambda dt \\ & -\sigma^2 \left(u_{CC}(C_t, H_t)C_W + u_{CH}(C_t, H_t)H_W\right) W_t dt \\ & +(C_A A_t \bar{\sigma} dB_t + C_W W_t \sigma dZ_t)u_{CC} + (H_A A_t \bar{\sigma} dB_t + H_W W_t \sigma dZ_t)u_{CH} \\ & + \left[\frac{u_C(C(e^{\nu}W_{t-}, A_{t-}), H(e^{\nu}W_{t-}, A_{t-}))}{u_C(C(W_{t-}, A_{t-}), H(W_{t-}, A_{t-}))} - 1\right] u_C(C_{t-}, H_{t-})dN_t, \end{aligned} \tag{55}$$

which implicitly determines the optimal consumption path. To shed some light on the effects of uncertainty, we may use this equation and obtain the (implicit) risk premium from

$$\begin{aligned} \frac{du_C(C_t, H_t)}{u_C(C_{t-}, H_{t-})} = {} & \left(\rho - (r_t - \delta) + \lambda - \frac{u_C(C(e^{\nu}W_t, A_t), H(e^{\nu}W_t, A_t))}{u_C(C(W_t, A_t), H(W_t, A_t))}e^{\nu}\lambda\right) dt \\ & -\frac{u_{CC}(C_t, H_t)C_W + u_{CH}(C_t, H_t)H_W}{u_C(C_t, H_t)} W_t \sigma^2 dt \\ & +(\cdot)dB_t + (\cdot)dZ_t + \left[\frac{u_C(C(e^{\nu}W_{t-}, A_{t-}), H(e^{\nu}W_{t-}, A_{t-}))}{u_C(C(W_{t-}, A_{t-}), H(W_{t-}, A_{t-}))} - 1\right] dN_t, \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{dt}E\left[\frac{du_C(C_t, H_t)}{u_C(C_t, H_t)}\right] = {} & \rho - E(r_t - \delta) + E\left[\frac{u_C(C(e^{\nu}W_t, A_t), H(e^{\nu}W_t, A_t))}{u_C(C(W_t, A_t), H(W_t, A_t))}(1 - e^{\nu})\lambda\right] \\ & -E\left[\frac{u_{CC}(C_t, H_t)C_W + u_{CH}(C_t, H_t)H_W}{u_C(C_t, H_t)} W_t \sigma^2\right]. \end{aligned}$$

Collecting terms we obtain that the *certainty equivalent rate of return* equals the expected return net of depreciation, $E(r_t - \delta)$, less the expected *implicit risk premium*,

$$RP_t \equiv -\frac{u_{CC}C_W + u_{CH}H_W}{u_C(C_t, H_t)}W_t\sigma^2 + \frac{u_C(C(e^\nu W_t, A_t), H(e^\nu W_t, A_t))}{u_C(C(W_t, A_t), H(W_t, A_t))}(1 - e^\nu)\lambda. \quad (56)$$

Observe that the structure is equivalent to (41), with the notable difference that the curvature of both the consumption function and the policy function for optimal hours matters for effective risk aversion and therefore the risk premium.

### 3.2.3 Explicit solutions

As before, a convenient way to describe the behavior of the economy is in terms of the evolution of $C_t$, $H_t$, $A_t$ and $W_t$. Similar to the endowment economy there exist explicit solutions, however, due to the non-linearities only for specific parameter restrictions. Below we use one restriction where the *policy functions* $C_t = C(A_t, W_t)$ and $H_t = H(A_t, W_t)$, and most economic variables of interest can be solved in closed form.

In what follows, we restrict our attention to the class of utility functions which exhibits constant relative risk aversion with respect to consumption $RRA_C = -u_{CC}C_t/u_C = \theta$, and leisure $RRA_L = u_{HH}H_t/u_H = 1 - (1-\theta)\psi$,

$$u(C_t, H_t) = \frac{(C_t(1 - H_t)^\psi)^{1-\theta}}{1-\theta}, \quad \theta > 0, \ \psi \geq 0. \quad (57)$$

Similar to Turnovsky and Smith (2006), the parameter $\psi$ measures the preference for leisure. To ensure concavity, we restrict $\theta - (1-\theta)\psi \geq 0$. Observe that positive risk aversion with respect to leisure requires $(1-\theta)\psi < 1$. For the case where $\psi = 0$, the explicit solutions in Proposition 3.1 (linear policy-function) and Proposition 3.5 (constant-saving-function) apply accordingly. For the broader case where $\psi > 0$, a closed-form solution can be obtained where optimal hours are constant. In contrast to the previous cases, both this solution and the numerical solution techniques used here are novel in the macro literature.

**Proposition 3.5 (constant-saving-function)** *If the production function is Cobb-Douglas,* $Y_t = A_t K_t^\alpha H_t^{1-\alpha}$ *and the rate of time preference is*

$$\bar{\rho} \equiv (e^{\nu(1-\alpha\theta)} - 1)\lambda - (1-\alpha\theta)\delta - \theta\bar{\mu} + \tfrac{1}{2}\left(\theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\sigma)\sigma^2\right),$$

*then optimal consumption is proportional to income, and optimal hours are constant.*

$$\rho = \bar{\rho} \quad \Rightarrow \quad C_t = C(W_t, A_t) = (1-s)A_t W_t^\alpha H^{1-\alpha}, \quad \theta > 1, \quad \psi \neq 0$$

$$\textit{where} \quad H = \frac{\theta(1-\alpha)}{\theta(1-\alpha) - \psi(1-\theta)}, \quad s \equiv 1/\theta.$$

**Proof.** see Appendix A.3.2 ■

**Corollary 3.6** *Use the policy function $C_t = C(W_t, A_t) = (1-s)A_t W_t^{\alpha} H^{1-\alpha}$ and (56) to get*

$$RP = e^{-\alpha\theta\nu}(1-e^{\nu})\lambda + \alpha\theta\sigma^2, \tag{58}$$

*where we use $C_W = \alpha(1-s)A_t W_t^{\alpha-1} H^{1-\alpha}$ and*

$$u_C = C_t^{-\theta}(1-H)^{(1-\theta)\psi}, \quad u_{CC} = -\theta C_t^{-\theta-1}(1-H)^{(1-\theta)\psi}.$$

This particular rate of time preference, $\bar{\rho}$, clearly is a knife-edge condition which ensures that the optimal leisure, the saving rate and the implicit risk premium are constant. In this singular case, the parameter measuring the preference for leisure, $\psi$, does not affect the risk premium or the saving rate, though it affects hours. To study the dynamic effects of labor supply flexibility for a broader parameter set, we employ numerical solutions.

#### 3.2.4 Numerical solutions

This section again uses the algorithm as in Posch and Trimborn (2009) to obtain a numerical solution for the case $\sigma = \bar{\sigma} = \bar{\mu} = 0$ and $A = 1$. This procedure requires a reduced form system of controlled stochastic differential equations under Poisson uncertainty. We employ both first-order conditions (52) and (53) to substitute the costate $V_W$ in the evolution of the costate variable and to obtain *Euler equations* for optimal consumption and optimal hours. This was done for consumption in (55), which for our simplifying assumptions reduces to

$$\begin{aligned} du_C &= (\rho - r_t + \delta + \lambda)u_C dt - u_C(C(e^{\nu}W_t), H(e^{\nu}W_t))e^{\nu}\lambda dt \\ &\quad + \left(u_C(C(e^{\nu}W_{t-}), 1 - H(e^{\nu}W_{t-})) - u_C(C_{t-}, H_{t-})\right) dN_t \\ \Leftrightarrow\ dC_t &= \frac{u_C}{u_{CC}}(\rho - (r_t - \delta) + \lambda)dt - \frac{u_C}{u_{CC}} \frac{u_C(C(e^{\nu}W_t), H(e^{\nu}W_t))}{u_C(C(W_t), H(W_t))} e^{\nu}\lambda dt \\ &\quad - \frac{u_{CH}}{u_{CC}}(dH_t - (H(e^{\nu}W_{t-}) - H(W_{t-}))dN_t) + (C(e^{\nu}W_{t-}) - C(W_{t-}))dN_t. \end{aligned}$$

Similarly, we use the first-order condition for hours, and replace $V_W$ by

$$\begin{aligned} d(u_H/Y_H) &= (\rho - (r_t - \delta) + \lambda)u_H/Y_H dt - u_H(C(e^{\nu}W_t), H(e^{\nu}W_t))/Y_H(e^{\nu}W_t, H(e^{\nu}W_t))e^{\nu}\lambda dt \\ &\quad + \left[\frac{u_H(C(e^{\nu}W_{t-}), H(e^{\nu}W_{t-}))}{Y_H(e^{\nu}W_{t-}, H(e^{\nu}W_{t-}))} - \frac{u_H(C(W_{t-}), H(W_{t-}))}{Y_H(W_{t-}, H(W_{t-}))}\right] dN_t \\ \Leftrightarrow\ du_H &= (\rho - (r_t - \delta) + \lambda)u_H dt - u_H(C(e^{\nu}W_t), H(e^{\nu}W_t))\frac{Y_H(W_t, H_t)}{Y_H(e^{\nu}W_t, H(e^{\nu}W_t))} e^{\nu}\lambda dt \\ &\quad + u_H/Y_H(dY_H - (Y_H(W_t, H_t) - Y_H(W_{t-}, H_{t-}))dN_t) \\ &\quad + (u_H(C_t, H_t) - u_H(C_{t-}, H_{t-}))dN_t, \end{aligned}$$

where, because of the wage rate, $Y_H = Y_H(W_t, H_t)$,

$$\begin{aligned} dY_H &= Y_{HH}(dH_t - (H_t - H_{t-})dN_t) + Y_{HK}(dW_t - (W_t - W_{t-})dN_t) \\ &\quad + (Y_H(W_t, H_t) - Y_H(W_{t-}, H_{t-}))dN_t. \end{aligned}$$

Defining $\bar{u} = \bar{u}(C, H) \equiv u_{CH}u_{HC} - u_{CC}u_{HH}$ gives, after some tedious algebra,

$$\begin{aligned} dH_t &= \frac{u_{HC}u_C - u_{CC}u_H}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}}(\rho - (r_t - \delta) + \lambda)dt \\ &\quad - \frac{u_{CC}u_H Y_{HK}/Y_H}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}}((r_t - \delta)W_t + H_t w_t^H - C_t)dt \\ &\quad - \frac{u_{HC}u_C}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}} \frac{u_C(C(e^\nu W_t), H(e^\nu W_t))}{u_C(C(W_t), H(W_t))} e^\nu \lambda dt \\ &\quad + \frac{u_{CC}u_H}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}} \frac{u_H(C(e^\nu W_t), H(e^\nu W_t))}{u_H(C(W_t), H(W_t))} \frac{Y_H(W_t, H_t)}{Y_H(e^\nu W_t, H(e^\nu W_t))} e^\nu \lambda dt \\ &\quad + (H_t - H_{t-})dN_t. \end{aligned}$$

Hence, our problem reduces to solving the controlled system of SDEs,

$$\begin{aligned} dC_t &= -\frac{u_C}{u_{CC}}\left(r_t - \rho - \delta - \lambda + \frac{u_C(C(e^\nu W_t), H(e^\nu W_t))}{u_C(C(W_t), H(W_t))} e^\nu \lambda\right) dt \\ &\quad - \frac{u_{CH}}{u_{CC}}(dH_t - (H(e^\nu W_{t-}) - H(W_{t-}))dN_t) + (C(e^\nu W_{t-}) - C(W_{t-}))dN_t, \\ dH_t &= \frac{u_{HC}u_C - u_{CC}u_H}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}}(\rho - (r_t - \delta) + \lambda)dt - \frac{u_{CC}u_H Y_{HK}/Y_H}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}}((r_t - \delta)W_t + H_t w_t^H - C_t)dt \\ &\quad - \frac{u_{HC}u_C}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}} \frac{u_C(C(e^\nu W_t), H(e^\nu W_t))}{u_C(C(W_t), H(W_t))} e^\nu \lambda dt + (H(e^\nu W_{t-}) - H(W_{t-}))dN_t \\ &\quad + \frac{u_{CC}u_H}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}} \frac{u_H(C(e^\nu W_t), H(e^\nu W_t))}{u_H(C(W_t), H(W_t))} \frac{Y_H(W_t, H_t)}{Y_H(e^\nu W_t, H(e^\nu W_t))} e^\nu \lambda dt, \\ dW_t &= ((r_t - \delta)W_t + H_t w_t^H - C_t)dt - (1 - e^\nu)W_{t-}dN_t. \end{aligned}$$

For the specific case of Cobb-Douglas production, $Y_t = AK_t^\alpha H_t^{1-\alpha}$, and CRRA utility as from (57), optimal behavior from the first-order conditions (52) and (53) implies

$$\begin{aligned} -\frac{u_H(C_t, H_t)}{u_C(C_t, H_t)} = Y_H(K_t, H_t) \;\Leftrightarrow\; 1 - H(W_t) &= \frac{\psi}{(1-\alpha)A} C(W_t) W_t^{-\alpha} H(W_t)^\alpha \qquad (59) \\ \Leftrightarrow\; 1 - H(e^\nu W_t) &= \frac{\psi}{(1-\alpha)A} C(e^\nu W_t) e^{-\alpha\nu} W_t^{-\alpha} H(e^\nu W_t)^\alpha. \end{aligned}$$

This pins down the jump in consumption to

$$\begin{aligned} 1 - H(e^\nu W_t) &= \frac{C(e^\nu W_t)}{C(W_t)} e^{-\alpha\nu} \left(\frac{H(e^\nu W_t)}{H(W_t)}\right)^\alpha (1 - H(W_t)) \\ \Rightarrow \quad \tilde{C}(W_t) &= \frac{1 - \tilde{H}(W_t)H(W_t)}{1 - H(W_t)} \tilde{H}(W_t)^{-\alpha} e^{\alpha\nu}, \qquad (60) \end{aligned}$$

where we define

$$\tilde{C}(W_t) \equiv \frac{C(e^{\nu}W_t)}{C(W_t)}, \quad \tilde{H}(W_t) \equiv \frac{H(e^{\nu}W_t)}{H(W_t)}.$$

We can neglect the SDE for consumption as from (59) and (60), $C(W_t) = C(H(W_t), W_t)$ and $\tilde{C}(W_t) = \tilde{C}(H(W_t))$. Economically, optimal behavior of consumption is completely described by optimal hours and financial wealth. Thus, a reduced form description is

$$\begin{aligned} dH_t &= \frac{\rho - (1-\theta)r_t + (1-\alpha\theta)\delta + \lambda - \alpha\theta C_t/W_t - \tilde{C}(W_t)^{-\theta+(1-\theta)\psi}\tilde{H}(W_t)^{(1-\theta)\psi\alpha}e^{\nu-(1-\theta)\psi\alpha\nu}\lambda}{\alpha\theta H_t^{-1} - (\psi - \theta\psi - \theta)(1-H_t)^{-1}}dt \\ &\quad +(H(e^{\nu}W_{t-}) - H(W_{t-}))dN_t, \\ dW_t &= ((r_t - \delta)W_t + H_t w_t^H - C_t)dt - (1-e^{\nu})W_{t-}dN_t. \end{aligned}$$

where for CRRA preferences, the risk premium (56) becomes

$$RP_t = \tilde{C}(W_t)^{(1-\theta)\psi-\theta}\tilde{H}(W_t)^{(1-\theta)\psi\alpha}e^{-(1-\theta)\psi\alpha\nu}(1-e^{\nu})\lambda. \tag{61}$$

In the general case, the premium depends on the optimal jumps in consumption and hours.

### 3.2.5 Results

In what follows, we restrict our discussion to the empirically most relevant case where $\theta \geq 1$. The key result is that effective risk aversion, except for the singular case $\rho = \bar{\rho}$, is still a function of financial wealth. This in turn implies a time-varying risk premium as wealth is changing stochastically over time. As shown in the appendix, elastic labor supply, $\psi \neq 0$, primarily has an effect on the optimal hours supplied, but does not substantially affect the shape and properties of the risk premium (cf. Figure A.1).

This knife-edge value $\rho = \bar{\rho}$ ensures that the individual's optimal choice of leisure is constant (cf. Bodie et al., 1992). Then, the expected proportional decline of marginal utility with respect to consumption matches the expected rate of return apart from a constant. Moreover, we obtain that the marginal propensity to save (to consume), $s(W_t) = s$, the supplied hours, $H(W_t) = H$, and the risk premium are all constant measures over time (consumption becomes a homogeneous function of degree $\alpha$). For $\rho < \bar{\rho}$ the individual prefers a higher saving rate, $s(W_t) > s$, and supplies more hours, $H(W_t) > H$. Because both optimal policy functions for consumption and hours are concave, the effective risk aversion of the value function is lower than for $\psi = 0$. The risk premium is convex in financial wealth and has the upper bound $e^{-\alpha\theta\nu}(1-e^{\nu})\lambda$ for wealth approaching zero. For $\rho > \bar{\rho}$ the marginal propensity to save is smaller, $s(W_t) < s$, and the individual supplies less hours, $H(W_t) < H$, which in fact raises the effective risk aversion of the value function. The marginal propensity to consume out of wealth decreases less rapidly than it would if the saving rate were constant, while the optimal policy functions for hours is convex, which in fact raises the effective risk aversion of the value function. Since the saving rate is decreasing in wealth, the risk premium is concave with lower bound $e^{-\theta\alpha\nu}(1 - e^{\nu})\lambda$.

A empirically testable implication is the correlation between hours and consumption. In the data, hours and consumption are positively correlated, which in turn implies a negative correlation between consumption and leisure. We may infer this property directly from the policy functions. For $\rho = \bar{\rho}$ there is zero correlation, while for $\rho < \bar{\rho}$ consumption and hours are concave functions of financial wealth (which has the usual interpretation of the capital stock per effective worker), and we obtain a positive correlation. It is only for $\rho > \bar{\rho}$ that the optimal policy function for hours is convex, which in turn would imply a counterfactual negative correlation as long as the consumption function is concave.

# 4 Conclusion

In this paper we study how non-linearities in a neoclassical production economy affect asset pricing implications. For this purpose, we choose to formulate our DSGE models in continuous-time because we believe that a clear understanding of these effects can best be achieved by working out analytical solutions. We derive closed-form solutions of the Lucas' fruit-tree model and compare the resulting risk premium to those obtained from models which account for non-linearities in the form of a neoclassical production function. Our key result is that these non-linearities can generate time-varying and asymmetric risk premia over the business cycle. The economic intuition is that individual's effective risk aversion, except for singular cases, is no longer constant in a neoclassical production economy. We show that non-normalities in the form of rare disasters substantially increases the economic relevance of these (empirical) key features.

From a methodological point of view, this paper shows that formulating the endowment economy or non-trivial production models in continuous-time gives analytical solutions for reasonable parametric restrictions or functional forms. Analytical solutions are useful for macro-finance models for at least two reasons. First, they are points of reference from which numerical methods can be used to explore a broader class of models. Second, they shed light on asset market implications without relying purely on numerical methods. This circumvents problems induced by approximation schemes which could be detrimental when studying the effects of uncertainty. Along these lines, we propose the continuous-time formulation of DSGE models as a workable paradigm in macro-finance.

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# A Appendix

## A.1 Lucas fruit-tree model in continuous-time

### A.1.1 Deriving the budget constraint

Consider a risky asset and a government bill with default risk. Suppose the price of the risky asset follows

$$dp_t = \mu p_t dt + \sigma p_t dB_t + J_t p_{t-} dN_t,$$

where $\mu$ denotes the instantaneous conditional expected percentage change in the price of asset $i$, $\sigma^2$ the instantaneous conditional variance, $B_t$ is a standard Brownian motion, and $J_t$ is a random variable representing the sensitivity of the asset price with respect to a jump of the Poisson process $N_t$ at arrival rate $\lambda$. A government bill (riskless in normal times) obeys

$$dp_0(t) = p_0(t)rdt + D_t dN_t,$$

where $D_t$ is a random variable denoting a random default risk during a contraction.

Consider a portfolio strategy which holds $n_t$ units of the risky asset and $n_0(t)$ units of the riskless asset with default risk, such that

$$W_t = n_0(t)p_0(t) + p_t n_t$$

denotes the portfolio value. Using Itô's formula, it follows

$$\begin{aligned} dW_t &= p_0(t)dn_0(t) + n_0(t)p_0(t)rdt + p_t dn_t + n_t p_t \mu dt + n_t p_t \sigma dB_t \\ &\quad + \left(n_t p_{t-} J_t + n_0(t)p_0(t_-)D_t\right) dN_t \\ &= p_0(t)dn_0(t) + n_0(t)p_0(t)rdt + p_t dn_t + w_t \mu W_t dt + w_t \sigma W dB_t \\ &\quad + \left(w_{t-} J_t + (1 - w_{t-})D_t\right) W_{t-} dN_t, \end{aligned} \tag{62}$$

where $w_t W_t \equiv n_t p_t$ denotes the amount invested in the risky asset. Since investors use their savings to accumulate assets, assuming no dividend payments, $p_0(t)dn_0(t) + p_t dn_t = -C_t dt$,

$$\begin{aligned} dW_t &= \left((\mu - r)w_t W_t + rW_t - C_t\right) dt + \sigma w_t W_t dB_t \\ &\quad + \left((J_t - D_t)w_{t-} + D_t\right) W_{t-} dN_t. \end{aligned} \tag{63}$$

### A.1.2 The short-cut approach

As a necessary condition for optimality the Bellman's principle gives at time $s$

$$\rho V(W_s) = \max_{C_s} \left\{ u(C_s) + \frac{1}{dt} E_s dV(W_s) \right\}. \tag{64}$$

Using Itô's formula (see e.g. Protter, 2004; Sennewald, 2007),

$$\begin{aligned} dV(W_s) &= \left(\left(\mu_M W_s - C_s\right) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW}\right) dt + \sigma_M W_s V_W dB_t + (V(W_s) - V(W_{s-}))dN_t \\ &= \left(\left(\mu_M W_s - C_s\right) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW}\right) dt + \sigma_M W_s V_W dB_t \\ &\quad + (V((1 - \zeta_M(t_-))W_{s-}) - V(W_{s-}))dN_t, \end{aligned}$$

where $\sigma_M^2$ is the instantaneous variance of the risky asset's return from the Brownian motion increments. If we take the expectation of the integral form, and use the property of stochastic integrals, we may write using $\zeta_M \equiv E(\zeta_M(t)|D_t = \exp(\kappa) - 1) = 1 - e^{\kappa} - (e^{\nu_1} - e^{\kappa})w$ and $\zeta_M^0 \equiv E(\zeta_M(t)|D_t = 0) = (1 - e^{\nu_2})w$,

$$\begin{aligned} E_s dV(W_s) &= \big( (\mu_M W_s - C_s) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} \\ &\quad + (V((1-\zeta_M)W_s)q + V((1-\zeta_M^0)W_s)(1-q) - V(W_s))\lambda \big) dt. \end{aligned}$$

Inserting into (64) gives the Bellman equation

$$\begin{aligned} \rho V(W_s) &= \max_{C_s} \big\{ u(C_s) + (\mu_M W_s - C_s) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} \\ &\quad + (V((1-\zeta_M)W_s)q + V((1-\zeta_M^0)W_s)(1-q) - V(W_s))\lambda \big\}. \end{aligned}$$

The first-order condition (8) makes consumption a function of the state variable. Using the maximized Bellman equation for all $s = t \in [0, \infty)$,

$$\begin{aligned} \rho V(W_t) &= u(C(W_t)) + (\mu_M W_t - C(W_t))V_W + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WW} \\ &\quad + (V((1-\zeta_M)W_t)q + V((1-\zeta_M^0)W_s)(1-q) - V(W_t))\lambda. \end{aligned}$$

Use the envelope theorem to compute the costate

$$\begin{aligned} \rho V_W &= (\mu_M V_W + (\mu_M W_t - C(W_t))V_{WW} + \sigma_M^2 W_t V_{WW} + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW} \\ &\quad + (V_W((1-\zeta_M)W_t)(1-\zeta_M)q + V_W((1-\zeta_M^0)W_s)(1-\zeta_M^0)(1-q) - V_W(W_t))\lambda. \end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned} (\rho - \mu_M + \lambda)V_W &= (\mu_M W_t - C(W_t))V_{WW} + \sigma_M^2 W_t V_{WW} + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW} \\ &\quad + (V_W((1-\zeta_M)W_t)(1-\zeta_M)q + V_W((1-\zeta_M^0)W_s)(1-\zeta_M^0)(1-q))\lambda. \quad (65) \end{aligned}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W(W_t) &= (\mu_M W_t - C_t) V_{WW} dt + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW} dt + \sigma_M W_t V_{WW} dB_t \\ &\quad + (V_W((1-\zeta_M(t_-))W_{t-}) - V_W(W_{t-}))dN_t \\ &= \big((\rho - \mu_M + \lambda)V_W - \sigma_M^2 W_t V_{WW} - V_W((1-\zeta_M)W_t)(1-\zeta_M)q\lambda \\ &\quad - V_W((1-\zeta_M^0)W_s)(1-\zeta_M^0)(1-q)\lambda\big) dt \\ &\quad + \sigma_M W_t V_{WW} dB_t + (V_W((1-\zeta_M(t_-))W_{t-}) - V_W(W_{t-}))dN_t, \end{aligned}$$

where we inserted the costate from (65). As a final step we insert the first-order condition and obtain the Euler equation (9).

### A.1.3 A more comprehensive approach

As a necessary condition for optimality the Bellman's principle gives at time $s$

$$\rho V(W_s) = \max_{(w_s,C_s)} \left\{ u(C_s) + \frac{1}{dt} E_s dV(W_s) \right\}. \tag{66}$$

Using Itô's formula,

$$\begin{aligned} dV(W_s) &= \Big(\left((w_s(\mu - r) + r)W_s - C_s\right) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW}\Big) dt + w_s \sigma W_s V_W dB_t \\ &\quad + (V(W_s) - V(W_{s-}))dN_t \\ &= \Big(\left((w_s(\mu - r) + r)W_s - C_s\right) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW}\Big) dt + w_s \sigma W_s V_W dB_s \\ &\quad + (V((1 + (J_s - D_s)w_{t_-} + D_s)W_{s_-}) - V(W_{s-}))dN_s. \end{aligned}$$

Take the expectation of the integral form, and use the property of stochastic integrals,

$$\begin{aligned} E_s dV(W_s) &= \Big( \left((w_s(\mu - r) + r)W_s - C_s\right) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW} \\ &\quad + (E[V((1 + (J_s - D_s)w_s + D_s)W_s)] - V(W_s))\lambda \Big) dt \\ &= \Big( \left((w_s(\mu - r) + r)W_s - C_s\right) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW} + (V((e^\kappa + (e^{\nu_1} - e^\kappa)w_s)W_s)q \\ &\quad + V((1 + (e^{\nu_2} - 1)w_s)W_s)(1 - q) - V(W_s))\lambda \Big) dt. \end{aligned}$$

The first-order conditions (14) and (15) make the controls a function of the state variable. Using the maximized Bellman equation,

$$\begin{aligned} \rho V(W_t) &= u(C(W_t)) + ((\mu - r)w(W_t)W_t + rW_t - C(W_t)) V_W + \tfrac{1}{2} w(W_t)^2 \sigma^2 W_t^2 V_{WW} \\ &\quad + \big(V((e^\kappa + (e^{\nu_1} - e^\kappa)w(W_t))W_t)q + V((1 + (e^{\nu_2} - 1)w(W_t))W_t)(1 - q) \\ &\quad - V(W_t)\big)\lambda. \end{aligned} \tag{67}$$

Use the envelope theorem to compute the costate

$$\begin{aligned} \rho V_W &= ((\mu - r)w(W_t) + r)V_W + ((\mu - r)w(W_t)W_t + rW_t - C(W_t))V_{WW} \\ &\quad + w(W_t)^2 \sigma^2 W_t V_{WW} + \tfrac{1}{2} w(W_t)^2 \sigma^2 W_t^2 V_{WWW} \\ &\quad + V_W((e^\kappa + (e^{\nu_1} - e^\kappa)w_t)W_t)(e^\kappa + (e^{\nu_1} - e^\kappa)w_t)q\lambda \\ &\quad + V_W((1 + (e^{\nu_2} - 1)w_t)W_t)(1 + (e^{\nu_2} - 1)w_t)(1 - q)\lambda - V_W(W_t)\lambda. \end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned} (\rho - ((\mu - r)w_t + r) + \lambda)V_W &= ((\mu - r)w_t W_t + rW_t - C_t)V_{WW} \\ &\quad + w_t^2 \sigma^2 W_t V_{WW} + \tfrac{1}{2} w_t^2 \sigma^2 W_t^2 V_{WWW} \\ &\quad + V_W((e^\kappa + (e^{\nu_1} - e^\kappa)w_t)W_t)(e^\kappa + (e^{\nu_1} - e^\kappa)w_t)q\lambda \\ &\quad + V_W((1 + (e^{\nu_2} - 1)w_t)W_t)(1 + (e^{\nu_2} - 1)w_t)(1 - q)\lambda. \end{aligned}$$

Using Itô's formula, the costate obeys

$$
\begin{aligned}
dV_W(W_t) &= ((\mu - r)w_t W_t + rW_t - C_t)\, V_{WW} dt + \tfrac{1}{2} w_t^2 \sigma^2 W_t^2 V_{WWW} dt + w_t \sigma W_t V_{WW} dB_t \\
&\quad + (V_W((1 + (J_t - D_t)w_{t\_} + D_t)W_{t-}) - V_W(W_{t-}))dN_t \\
&= \big((\rho - ((\mu - r)w_t + r) + \lambda)V_W - w_t^2 \sigma^2 W_t V_{WW} \\
&\quad - V_W((e^\kappa + (e^{\nu_1} - e^\kappa)w_t)W_t)(e^\kappa + (e^{\nu_1} - e^\kappa)w_t)q\lambda \\
&\quad - V_W((1 + (e^{\nu_2} - 1)w_t)W_t)(1 + (e^{\nu_2} - 1)w_t)(1 - q)\lambda\big) dt \\
&\quad + w_t \sigma W_t V_{WW} dB_t + (V_W((1 + (J_t - D_t)w_{t\_} + D_t)W_{t-}) - V_W(W_{t-}))dN_t,
\end{aligned}
$$

where we inserted the costate from above. As a final step, we insert the first-order condition for consumption to obtain the Euler equation (16).

### A.1.4 Proof of Proposition 2.1

For constant relative risk aversion, $\theta$, the utility function reads

$$
u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \quad \theta > 0. \tag{68}
$$

From (67) we have the maximized Bellman equation where we use functional equations from first-order conditions (14) and (15),

$$
\begin{aligned}
C(W_t) &= V_W^{-\frac{1}{\theta}} \\
w(W_t) &= \frac{C(W_t)^{-\theta}}{\theta C(W_t)^{-\theta-1} C_W W_t} \frac{\mu - r}{\sigma^2} + \frac{C((e^\kappa + (e^{\nu_1} - e^\kappa)w_t)W_t)^{-\theta}}{\theta C(W_t)^{-\theta-1} C_W W_t} \frac{e^{\nu_1} - e^\kappa}{\sigma^2} q\lambda \\
&\quad + \frac{C((1 + (e^{\nu_2} - 1)w_t)W_t)^{-\theta}}{\theta C(W_t)^{-\theta-1} C_W W_t} \frac{e^{\nu_2} - 1}{\sigma^2} (1 - q)\lambda.
\end{aligned}
$$

We use an *educated guess*,

$$
\bar{V} = \mathbb{C}_0 \frac{W_t^{1-\theta}}{1-\theta}, \tag{69}
$$

where $\bar{V}_W = \mathbb{C}_0 W_t^{-\theta}$, and $\bar{V}_{WW} = -\theta \mathbb{C}_0 W_t^{-\theta-1}$ to solve the resulting equation. Note that optimal consumption is linear in wealth, $C(W_t) = \mathbb{C}_0^{-1/\theta} W_t$, which implies that the optimal portfolio weight is constant and implicitly given by

$$
w = \frac{\mu - r}{\theta\sigma^2} + (e^\kappa + (e^{\nu_1} - e^\kappa)w)^{-\theta} \frac{e^{\nu_1} - e^\kappa}{\theta\sigma^2} q\lambda + (1 + (e^{\nu_2} - 1)w)^{-\theta} \frac{e^{\nu_2} - 1}{\theta\sigma^2} (1 - q)\lambda.
$$

Using the result that $w(W_t) = w$ is constant, and inserting the candidate policy function for consumption into the maximized Bellman equation (67), we arrive at

$$
\begin{aligned}
\rho \mathbb{C}_0 \frac{W_t^{1-\theta}}{1-\theta} &= \frac{\mathbb{C}_0^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + ((\mu - r)wW_t + rW_t - \mathbb{C}_0^{-\frac{1}{\theta}} W_t)\mathbb{C}_0 W_t^{-\theta} - \theta \tfrac{1}{2} w^2 \sigma^2 \mathbb{C}_0 W_t^{1-\theta} \\
&\quad + \big((e^\kappa + (e^{\nu_1} - e^\kappa)w)^{1-\theta} q + (1 + (e^{\nu_2} - 1)w)^{1-\theta}(1 - q) - 1\big)\mathbb{C}_0 \frac{W_t^{1-\theta}}{1-\theta} \lambda.
\end{aligned}
$$

Defining $\mu_M \equiv (\mu - r)w + r$, $\sigma_M \equiv w\sigma$, $\zeta_M \equiv 1 - e^{\kappa} - (e^{\nu_1} - e^{\kappa})w_t$, and collecting terms,

$$\begin{aligned} \rho &= \mathbb{C}_0^{-\frac{1}{\theta}} + (1-\theta)((\mu - r)w + r - \mathbb{C}_0^{-\frac{1}{\theta}}) - (1-\theta)\theta\tfrac{1}{2}w^2\sigma^2 \\ &\quad + \big((e^{\kappa} + (e^{\nu_1} - e^{\kappa})w)^{1-\theta}q + (1 + (e^{\nu_2} - 1)w)^{1-\theta}(1-q) - 1\big)\lambda \\ \Rightarrow \quad \mathbb{C}_0 &= \left(\frac{\rho + \lambda - (1-\theta)\mu_M - (1-\zeta_M)^{1-\theta}\lambda}{\theta} + (1-\theta)\tfrac{1}{2}\sigma_M^2\right)^{-\theta}. \end{aligned}$$

This proofs that the guess (69) indeed is a solution, and by inserting the guess together with the constant, we obtain the policy functions for the portfolio weights and consumption.

Table A.1: Calibrated model and the risk premium (endowment economy)

| | | (1) | (2) | (3) | (4) | (5) | (6) | (7) |
|---|---|---|---|---|---|---|---|---|
| | | | | | | Parameters | | |
| | | No disasters | Baseline | Low $\theta$ | High $\lambda$ | Low $q$ | Low $\bar{\mu}$ | Low $\rho$ |
| $\theta$ | (coef. of relative risk aversion) | 4 | **4** | **3** | 4 | 4 | 4 | 4 |
| $\bar{\sigma}$ | (s.d. of growth rate, no disasters) | 0.02 | **0.02** | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| $\rho$ | (rate of time preference) | 0.03 | **0.03** | 0.03 | 0.03 | 0.03 | 0.03 | **0.02** |
| $\bar{\mu}$ | (growth rate, deterministic part) | 0.025 | **0.025** | 0.025 | 0.025 | 0.025 | **0.020** | 0.025 |
| $\lambda$ | (disaster probability) | 0 | **0.017** | 0.017 | **0.025** | 0.017 | 0.017 | 0.017 |
| $q$ | (default probability in disaster) | 0 | **0.4** | 0.4 | 0.4 | **0.3** | 0.4 | 0.4 |
| $1-e^{\bar{\nu}}$ | (size of disaster) | 0 | **0.4** | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 |
| $1-e^{\kappa}$ | (size of default) | 0 | **0.4** | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 |
| | | | | | | Variables | | |
| Default risk | | 0 | **0.021** | 0.012 | 0.03 | 0.016 | 0.021 | 0.021 |
| Disaster risk | | 0 | **0.031** | 0.019 | 0.046 | 0.036 | 0.031 | 0.031 |
| Residual risk | | 0.002 | **0.002** | 0.001 | 0.002 | 0.002 | 0.002 | 0.002 |
| Implicit risk premium | | 0.002 | **0.054** | 0.032 | 0.078 | 0.054 | 0.054 | 0.054 |
| Expected market rate | | 0.128 | **0.06** | 0.067 | 0.028 | 0.06 | 0.04 | 0.05 |
| Expected bill rate | | 0.126 | **0.031** | 0.051 | -0.013 | 0.026 | 0.011 | 0.021 |
| Market premium | | 0.002 | **0.029** | 0.016 | 0.041 | 0.033 | 0.029 | 0.029 |
| Expected market rate, conditional | | 0.128 | **0.066** | 0.074 | 0.038 | 0.066 | 0.046 | 0.056 |
| Face bill rate | | 0.126 | **0.034** | 0.054 | -0.009 | 0.028 | 0.014 | 0.024 |
| Market premium, conditional | | 0.002 | **0.033** | 0.02 | 0.047 | 0.038 | 0.033 | 0.033 |
| Sharpe ratio, conditional | | 0.08 | **1.641** | 0.996 | 2.366 | 1.901 | 1.641 | 1.641 |
| Expected growth rate | | 0.025 | **0.016** | 0.016 | 0.012 | 0.016 | 0.011 | 0.016 |
| Expected growth rate, conditional | | 0.025 | **0.025** | 0.025 | 0.025 | 0.025 | 0.02 | 0.025 |

Table A.2: Calibrated model and the risk premium (endowment economy)

| | | (1) | (2) | (3) | (4) | (5) | (6) | (7) |
|---|---|---|---|---|---|---|---|---|
| | | | | | | Parameters | | |
| | | No default | Baseline | High $\bar{\sigma}$ | High $\lambda$ | High $q$ | Low $1-e^{\bar{\nu}}$ | Low $1-e^{\kappa}$ |
| $\theta$ | (coef. of relative risk aversion) | 4 | **4** | 4 | 4 | 4 | 4 | 4 |
| $\bar{\sigma}$ | (s.d. of growth rate, no disasters) | 0.02 | **0.02** | **0.05** | 0.02 | 0.02 | 0.02 | 0.02 |
| $\rho$ | (rate of time preference) | 0.03 | **0.03** | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 |
| $\bar{\mu}$ | (growth rate, deterministic part) | 0.025 | **0.025** | 0.025 | 0.025 | 0.025 | 0.025 | 0.025 |
| $\lambda$ | (disaster probability) | 0.017 | **0.017** | 0.017 | **0.2** | 0.017 | 0.017 | 0.017 |
| $q$ | (default probability in disaster) | **0** | **0.4** | 0.4 | 0.4 | **1** | 0.4 | 0.4 |
| $1-e^{\bar{\nu}}$ | (size of disaster) | 0.4 | **0.4** | 0.4 | **0.034** | 0.4 | **0.2** | 0.4 |
| $1-e^{\kappa}$ | (size of default) | 0.4 | **0.4** | 0.4 | **0.034** | 0.4 | 0.4 | **0.2** |
| | | | | | | Variables | | |
| Default risk | | 0 | **0.021** | 0.021 | 0.003 | 0.052 | 0.007 | 0.01 |
| Disaster risk | | 0.052 | **0.031** | 0.031 | 0.004 | 0 | 0.002 | 0.042 |
| Residual risk | | 0.002 | **0.002** | 0.01 | 0.002 | 0.002 | 0.002 | 0.002 |
| Implicit risk premium | | 0.054 | **0.054** | 0.062 | 0.009 | 0.054 | 0.01 | 0.054 |
| Expected market rate | | 0.06 | **0.06** | 0.047 | 0.102 | 0.06 | 0.108 | 0.06 |
| Expected bill rate | | 0.013 | **0.031** | 0.01 | 0.099 | 0.058 | 0.106 | 0.022 |
| Market premium | | 0.047 | **0.029** | 0.037 | 0.002 | 0.002 | 0.003 | 0.038 |
| Expected market rate, conditional | | 0.066 | **0.066** | 0.054 | 0.108 | 0.066 | 0.112 | 0.066 |
| Face bill rate | | 0.013 | **0.034** | 0.013 | 0.102 | 0.065 | 0.108 | 0.023 |
| Market premium, conditional | | 0.054 | **0.033** | 0.041 | 0.006 | 0.002 | 0.003 | 0.043 |
| Sharpe ratio, conditional | | 2.681 | **1.641** | 0.824 | 0.292 | 0.08 | 0.162 | 2.161 |
| Expected growth rate | | 0.016 | **0.016** | 0.015 | 0.019 | 0.016 | 0.021 | 0.016 |
| Expected growth rate, conditional | | 0.025 | **0.025** | 0.024 | 0.025 | 0.025 | 0.025 | 0.025 |

Table A.3: Calibrated model and the risk premium (production economy)

| | | (1) | (2) | (3) | (4) | (5) | (6) | (7) |
|---|---|---|---|---|---|---|---|---|
| | | | | | | Parameters | | |
| | | No disasters | Baseline | High $\theta$ | Low $\alpha$ | Low $\delta$ | High $\lambda$ | High $\|\nu\|$ |
| $\theta$ | (coef. of relative risk aversion) | 4 | **4** | **6** | 4 | 4 | 4 | 4 |
| $\alpha$ | (output elasticity of capital) | 0.75 | **0.75** | 0.75 | **0.33** | 0.75 | 0.75 | 0.75 |
| $\delta$ | (capital depreciation, deterministic part) | 0.1 | **0.1** | 0.1 | 0.1 | **0.05** | 0.1 | 0.1 |
| $\rho$ | (rate of time preference) | 0.05 | **0.05** | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| $\sigma$ | (s.d. of stochastic depreciation, no disasters) | 0 | **0** | 0 | 0 | 0 | 0 | 0 |
| $\bar{\sigma}$ | (s.d. of TFP growth) | 0 | **0** | 0 | 0 | 0 | 0 | 0 |
| $\bar{\mu}$ | (growth rate TFP, deterministic part) | 0 | **0** | 0 | 0 | 0 | 0 | 0 |
| $\lambda$ | (disaster probability) | **0** | **0.017** | 0.017 | 0.017 | 0.017 | **0.02** | 0.017 |
| $1 - e^{\nu}$ | (size of disaster) | 0 | **0.4** | 0.4 | 0.4 | 0.4 | 0.4 | **0.5** |
| | | | | | | Variables | | |
| Implied knife-edge value $\bar{\rho}$ | | 0.200 | **0.230** | 0.435 | 0.035 | 0.130 | 0.236 | 0.251 |
| Implicit risk premium | | | | | | | | |
| steady state, conditional | | 0 | **0.024** | 0.034 | 0.014 | 0.027 | 0.028 | 0.045 |
| zero wealth (left limit) | | 0 | **0.032** | 0.068 | 0.013 | 0.032 | 0.037 | 0.068 |
| Market rate, steady state (gross) | | 0.150 | **0.131** | 0.116 | 0.147 | 0.077 | 0.128 | 0.122 |
| Bill rate, steady state (gross) | | 0.150 | **0.107** | 0.081 | 0.133 | 0.051 | 0.101 | 0.078 |

## A.2 A model of growth under uncertainty

### A.2.1 The Bellman equation and the Euler equation

As a necessary condition for optimality the Bellman's principle gives at time $s$

$$\rho V(W_s, A_s) = \max_{C_s} \left\{ u(C_s) + \frac{1}{dt} E_s dV(W_s, A_s) \right\}.$$

Using Itô's formula yields

$$\begin{aligned} dV &= V_W(dW_s - J_s W_{s-} dN_t) + V_A dA_s + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_s^2 + V_{WW}\sigma^2 W_s^2\right) dt \\ &\quad + [V(W_s, A_s) - V(W_{s-}, A_{s-})] dN_t \\ &= ((r_s - \delta)W_s + w_s^L - C_s)V_W dt + V_W \sigma W_s dZ_s + V_A \bar{\mu} A_s dt + V_A \bar{\sigma} A_s dB_s \\ &\quad + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_s^2 + V_{WW}\sigma^2 W_s^2\right) dt + [V(e^{\nu} W_{s-}, A_{s-}) - V(W_{s-}, A_{s-})] dN_t. \end{aligned}$$

Using the property of stochastic integrals, we may write

$$\begin{aligned} \rho V(W_s, A_s) = \max_{C_s} \{ & u(c_s) + ((r_s - \delta)W_s + w_s^L - C_s)V_W + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_s^2 + V_{WW}\sigma^2 W_s^2\right) \\ & + V_A \bar{\mu} A_s + [V(e^{\nu} W_s, A_s) - V(W_s, A_s)]\lambda \} \end{aligned}$$

for any $s \in [0, \infty)$. Because it is a necessary condition for optimality, we obtain the first-order condition (39) which makes optimal consumption a function of the state variables.

For the *evolution of the costate* we use the maximized Bellman equation

$$\begin{aligned} \rho V(W_t, A_t) &= u(C(W_t, A_t)) + ((r_t - \delta)W_t + w_t^L - C(W_t, A_t))V_W + V_A \bar{\mu} A_t \\ &\quad + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_t^2 + V_{WW}\sigma^2 W_t^2\right) + [V(e^{\nu} W_t, A_t) - V(W_t, A_t)]\lambda, \end{aligned} \tag{70}$$

where $r_t = r(W_t, A_t)$ and $w_t^L = w(W_t, A_t)$ follow from the firm's optimization problem, and the envelope theorem (also for the factor rewards) to compute the costate,

$$\begin{aligned} \rho V_W &= \bar{\mu} A_t V_{AW} + ((r_t - \delta)W_t + w_t^L - C_t)V_{WW} + (r_t - \delta)V_W + \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) \\ &\quad + V_{WW}\sigma^2 W_t + [V_W(e^{\nu} W_t, A_t)e^{\nu} - V_W(W_t, A_t)]\lambda. \end{aligned}$$

Collecting terms we obtain

$$\begin{aligned} (\rho - (r_t - \delta) + \lambda)V_W &= V_{AW}\bar{\mu} A_t + ((r_t - \delta)W_t + w_t^L - C_t)V_{WW} + \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) \\ &\quad + \sigma^2 V_{WW} W_t + V_W(e^{\nu} W_t, A_t)e^{\nu}\lambda. \end{aligned}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W &= V_{AW}\bar{\mu} A_t dt + V_{AW}\bar{\sigma} A_t dB_t + \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) dt \\ &\quad + ((r_t - \delta)W_t + w_t^L - C_t)V_{WW} dt + V_{WW}\sigma W_t dZ_t + [V_W(W_t, A_t) - V_W(W_{t-}, A_{t-})] dN_t, \end{aligned}$$

where inserting yields

$$
\begin{aligned}
dV_W &= (\rho - (r_t - \delta) + \lambda) V_W dt - V_W(e^\nu W_t, A_t) e^\nu \lambda - \sigma^2 V_{WW} W_t dt + V_{AW} A_t \bar{\sigma} dB_t + V_{WW} W_t \sigma dZ_t \\
&\quad + [V_W(e^\nu W_{t-}, A_{t-}) - V_W(W_{t-}, A_{t-})] dN_t,
\end{aligned}
$$

which describes the evolution of the costate variable. As a final step, we insert the first-order condition (39) to obtain the Euler equation (40).

### A.2.2 Proof of Proposition 3.1

The idea of this proof is to show that an educated guess of the value function, the maximized Bellman equation (70) and the first-order condition (39) are both fulfilled. We guess that the value function reads

$$
V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} + f(A_t). \tag{71}
$$

From (39), optimal consumption is a constant fraction of wealth,

$$
C_t^{-\theta} = \mathbb{C}_1 W_t^{-\theta} \quad \Leftrightarrow \quad C_t = \mathbb{C}_1^{-1/\theta} W_t.
$$

Now use the maximized Bellman equation (70), the property of the Cobb-Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^\alpha L_t^{-\alpha}$, together with the transformation $K_t \equiv L W_t$, and insert the solution candidate,

$$
\begin{aligned}
\rho V(W_t, A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + ((r_t - \delta) W_t + w_t^L - C(W_t, A_t)) V_W + V_A \bar{\mu} A_t \\
&\quad + \tfrac{1}{2}\left(V_{AA} \bar{\sigma}^2 A_t^2 + V_{WW} \sigma^2 W_t^2\right) + [V(e^\nu W_t, A_t) - V(W_t, A_t)]\lambda \\
\Leftrightarrow \ \rho \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + (\alpha A_t W_t^{\alpha-1} W_t - \delta W_t + (1-\alpha) A_t W_t^\alpha - \mathbb{C}_1^{-1/\theta} W_t) \mathbb{C}_1 W_t^{-\theta} \\
&\quad - \tfrac{1}{2}\theta \mathbb{C}_1 W_t^{1-\theta} \sigma^2 - g(A_t) + (e^{(1-\theta)\nu} - 1) \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} \lambda,
\end{aligned}
$$

where we defined $g(A_t) \equiv \rho f(A_t) - f_A \bar{\mu} A_t - \frac{1}{2} f_{AA} \bar{\sigma}^2 A_t^2$. When imposing the condition $\alpha = \theta$ and $g(A_t) = \mathbb{C}_1 A_t$ it can be simplified to

$$
\begin{aligned}
(\rho - (e^{(1-\theta)\nu} - 1)\lambda) \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} + g(A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + (A_t W_t^{\alpha-\theta} - \delta W_t^{1-\theta} - \mathbb{C}_1^{-1/\theta} W_t^{1-\theta}) \mathbb{C}_1 \\
&\quad - \tfrac{1}{2}\theta \mathbb{C}_1 W_t^{1-\theta} \sigma^2 \\
\Leftrightarrow \ (\rho - (e^{(1-\theta)\nu} - 1)\lambda) W_t^{1-\theta} &= \theta \mathbb{C}_1^{-1/\theta} W_t^{1-\theta} - (1-\theta)\delta W_t^{1-\theta} - \tfrac{1}{2}\theta(1-\theta) W_t^{1-\theta} \sigma^2,
\end{aligned}
$$

which implies that

$$
\mathbb{C}_1^{-1/\theta} = \frac{\rho - (e^{(1-\theta)\nu} - 1)\lambda + (1-\theta)\delta + \frac{1}{2}\theta(1-\theta)\sigma^2}{\theta}.
$$

This proofs that the guess (71) indeed is a solution, and by inserting the guess together with the constant, we obtain the optimal policy function for consumption.

### A.2.3 Proof of Proposition 3.5

The idea of this proof follows Section A.2.2. An educated guess of the value function is

$$V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta}. \tag{72}$$

From (39), optimal consumption is a constant fraction of income,

$$C_t^{-\theta} = \mathbb{C}_1 W_t^{-\alpha\theta} A_t^{-\theta} \quad \Leftrightarrow \quad C_t = \mathbb{C}_1^{-1/\theta} W_t^{\alpha} A_t.$$

Now use the maximized Bellman equation (70), the property of the Cobb-Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^{\alpha} L^{-\alpha}$, together with the transformation $K_t \equiv L W_t$, and insert the solution candidate,

$$\begin{aligned} \rho V(W_t, A_t) \quad = \quad & \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{\alpha-\alpha\theta} A_t^{1-\theta}}{1-\theta} + ((r_t - \delta) W_t + w_t^L - C(W_t, A_t)) V_W + V_A \bar{\mu} A_t \\ & + \tfrac{1}{2} \left( V_{AA} \bar{\sigma}^2 A_t^2 + V_{WW} \sigma^2 W_t^2 \right) + [V(e^{\nu} W_t, A_t) - V(W_t, A_t)]\lambda, \end{aligned}$$

which is equivalent to

$$\begin{aligned} (\rho - (e^{(1-\alpha\theta)\nu} - 1)\lambda) \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta} \quad = \quad & \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{\alpha-\alpha\theta} A_t^{1-\theta}}{1-\theta} - \theta \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} \bar{\mu} A_t^{-\theta} \\ & + \left( \alpha A_t W_t^{\alpha} - \delta W_t + (1-\alpha) A_t W_t^{\alpha} - \mathbb{C}_1^{-1/\theta} W_t^{\alpha} A_t \right) \mathbb{C}_1 W_t^{-\alpha\theta} A_t^{-\theta} \\ & + \tfrac{1}{2} \left( \theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2 \right) \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta}. \end{aligned}$$

Collecting terms gives

$$\begin{aligned} (\rho - (e^{(1-\alpha\theta)\nu} - 1)\lambda) = (1-\alpha\theta) & \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}-1} W_t^{\alpha-1} A_t}{1-\theta} - \theta\bar{\mu} + (1-\alpha\theta) A_t W_t^{\alpha-1} - (1-\alpha\theta)\delta \\ & - (1-\alpha\theta) \mathbb{C}_1^{-1/\theta} W_t^{\alpha-1} A_t + \tfrac{1}{2} \left( \theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2 \right) \\ \Leftrightarrow \; \rho + \theta\bar{\mu} - \tfrac{1}{2} & \left( \theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2 \right) + (1-\alpha\theta)\delta = \\ & \left( \frac{\theta}{1-\theta} \mathbb{C}_1^{-1/\theta} + 1 \right) (1-\alpha\theta) A_t W_t^{\alpha-1}, \end{aligned}$$

which has a solution for $\mathbb{C}_1^{-1/\theta} = (\theta - 1)/\theta$ and

$$\rho = (e^{(1-\alpha\theta)\nu} - 1)\lambda - \theta\bar{\mu} + \tfrac{1}{2} \left( \theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2 \right) - (1-\alpha\theta)\delta.$$

This proofs that the guess (72) indeed is a solution, and by inserting the guess together with the constant, we obtain the optimal policy function for consumption.

## A.3 A model of growth under uncertainty with leisure

### A.3.1 The Bellman equation and the Euler equation

As a necessary condition for optimality the Bellman's principle gives at time $s$

$$\rho V(W_s, A_s) = \max_{C_s, H_s} \left\{ u(C_s, H_s) + \frac{1}{dt} E_s dV(W_s, A_s) \right\}.$$

Using Itô's formula yields

$$\begin{aligned} dV &= V_W(dW_s - J_s W_{s-} dN_t) + V_A dA_s + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_s^2 + V_{WW}\sigma^2 W_s^2\right) dt \\ &\quad + [V(W_s, A_s) - V(W_{s-}, A_{s-})] dN_t \\ &= ((r_s - \delta)W_s + H_s w_s^H - C_s) V_W dt + V_W \sigma W_s dZ_s + V_A \bar{\mu} A_s dt + V_A \bar{\sigma} A_s dB_s \\ &\quad + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_s^2 + V_{WW}\sigma^2 W_s^2\right) dt + [V(e^\nu W_{s-}, A_{s-}) - V(W_{s-}, A_{s-})] dN_t. \end{aligned}$$

Using the property of stochastic integrals, we may write

$$\begin{aligned} \rho V(W_s, A_s) = \max_{C_s, H_s} \Big\{ & u(C_s, H_s) + ((r_s - \delta)W_s + H_s w_s^H - C_s) V_W \\ & + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_s^2 + V_{WW}\sigma^2 W_s^2\right) + V_A \bar{\mu} A_s + [V(e^\nu W_s, A_s) - V(W_s, A_s)]\lambda \Big\} \end{aligned}$$

for any $s \in [0, \infty)$. Because it is a necessary condition for optimality, we obtain the first-order conditions (52) and (53) which make optimal consumption and hours functions of the state variables, $C_t = C(W_t, A_t)$ and $H_t = H(W_t, A_t)$, respectively.

For the *evolution of the costate* we use the maximized Bellman equation

$$\begin{aligned} \rho V(W_t, A_t) &= u(C(W_t, A_t), H(W_t, A_t)) + ((r_t - \delta)W_t + H(W_t, A_t) w_t^H - C(W_t, A_t)) V_W \\ &\quad + V_A \bar{\mu} A_t + \tfrac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_t^2 + V_{WW}\sigma^2 W_t^2\right) + [V(e^\nu W_t, A_t) - V(W_t, A_t)]\lambda, \quad (73) \end{aligned}$$

where $r_t = r(W_t, A_t)$ and $w_t^L = w(W_t, A_t)$ follow from the firm's optimization problem, and the envelope theorem (also for the factor rewards) to compute the costate,

$$\begin{aligned} \rho V_W &= \bar{\mu} A_t V_{AW} + ((r_t - \delta)W_t + H_t w_t^H - C_t) V_{WW} + (r_t - \delta) V_W + \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) \\ &\quad + V_{WW}\sigma^2 W_t + [V_W(e^\nu W_t, A_t) e^\nu - V_W(W_t, A_t)]\lambda. \end{aligned}$$

Collecting terms we obtain

$$\begin{aligned} (\rho - (r_t - \delta) + \lambda) V_W &= V_{AW} \bar{\mu} A_t + ((r_t - \delta)W_t + H_t w_t^H - C_t) V_{WW} \\ &\quad + \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) + \sigma^2 V_{WW} W_t + V_W(e^\nu W_t, A_t) e^\nu \lambda. \end{aligned}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W &= V_{AW} \bar{\mu} A_t dt + V_{AW} \bar{\sigma} A_t dB_t + \tfrac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) dt \\ &\quad + ((r_t - \delta)W_t + H_t w_t^H - C_t) V_{WW} dt + V_{WW}\sigma W_t dZ_t + [V_W(W_t, A_t) - V_W(W_{t-}, A_{t-})] dN_t, \end{aligned}$$

where inserting yields

$$\begin{aligned} dV_W &= (\rho - (r_t - \delta) + \lambda)V_W dt - V_W(e^\nu W_t, A_t)e^\nu \lambda - \sigma^2 V_{WW} W_t dt + V_{AW} A_t \bar{\sigma} dB_t + V_{WW} W_t \sigma dZ_t \\ &\quad + [V_W(e^\nu W_{t-}, A_{t-}) - V_W(W_{t-}, A_{t-})] dN_t, \end{aligned}$$

which describes the evolution of the costate variable. As a final step, we insert the first-order condition (52) to obtain the Euler equation (55).

### A.3.2 Proof of Proposition 3.5

The idea of this proof follows Section A.2.2. An educated guess of the value function is

$$V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta}. \tag{74}$$

From the first-order conditions (52) and (53), we obtain

$$\begin{aligned} C_t^{-\theta}(1-H_t)^{(1-\theta)\psi} &= \mathbb{C}_1 W_t^{-\alpha\theta} A_t^{-\theta}, \\ \psi C_t^{1-\theta}(1-H_t)^{(1-\theta)\psi-1} &= w_t^H \mathbb{C}_1 W_t^{-\alpha\theta} A_t^{-\theta} \quad \Rightarrow \quad \psi C_t/(1-H_t) = (1-\alpha) A_t W_t^\alpha H_t^{-\alpha}. \end{aligned}$$

Suppose that optimal hours are constant, $H_t = H$, then optimal consumption becomes a constant fraction of income,

$$C_t = (1-s) A_t W_t^\alpha H^{1-\alpha}, \quad 1-s \equiv (1-\alpha)\frac{1-H}{\psi H}, \quad \psi \neq 0.$$

Inserting everything into (73) gives

$$\begin{aligned} \rho \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta} &= \frac{C_t^{1-\theta}(1-H)^{(1-\theta)\psi}}{1-\theta} + \mathbb{C}_1 W_t^{-\alpha\theta} A_t^{-\theta} \left\{ A_t W_t^\alpha H^{1-\alpha} - \delta W_t - C_t \right\} \\ &\quad + V_A \bar{\mu} A_t + \tfrac{1}{2}\left( V_{AA} \bar{\sigma}^2 A_t^2 + V_{WW} \sigma^2 W_t^2 \right) + (V(e^\nu W_t, A_t) - V(W_t, A_t))\lambda \\ &= \frac{(1-s)^{1-\theta} H^{(1-\theta)(1-\alpha)}(1-H)^{(1-\theta)\psi}}{1-\theta} A_t^{1-\theta} W_t^{\alpha-\alpha\theta} \\ &\quad + \mathbb{C}_1 W_t^{\alpha-\alpha\theta} A_t^{1-\theta} H^{1-\alpha} - \delta \mathbb{C}_1 W_t^{1-\alpha\theta} A_t^{-\theta} - (1-s)\mathbb{C}_1 W_t^{\alpha-\alpha\theta} H^{1-\alpha} A_t^{1-\theta} \\ &\quad + \left( -\theta\bar{\mu} + \tfrac{1}{2}\left( \theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\sigma)\sigma^2 \right) \right) \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta} \\ &\quad + (e^{\nu(1-\alpha\theta)} - 1)\lambda \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta}. \end{aligned}$$

Collecting terms, we may write

$$\begin{aligned} &\left( \rho + (1-\alpha\theta)\delta + \left( \theta\bar{\mu} - \tfrac{1}{2}\left( \theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\sigma)\sigma^2 \right) \right) - (e^{\nu(1-\alpha\theta)} - 1)\lambda \right) \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta} = \\ &\quad \left( (1-s)^{1-\theta} H^{(1-\theta)(1-\alpha)}(1-H)^{(1-\theta)\psi} + \left( H^{1-\alpha} - (1-s)H^{1-\alpha} \right)(1-\theta)\mathbb{C}_1 \right) \frac{A_t^{1-\theta} W_t^{\alpha-\alpha\theta}}{1-\theta}. \end{aligned}$$

Hence, for $\rho = \bar{\rho}$ and

$$\mathbb{C}_1 = -\frac{(1-s)^{1-\theta} H^{(1-\alpha)(1-\theta)} (1-H)^{(1-\theta)\psi}}{(1-\theta)H^{1-\alpha} - (1-\theta)(1-s)H^{1-\alpha}},$$

the constant saving rate is indeed the optimal solution. The optimal hours can be obtained from the first-order condition for consumption

$$\begin{aligned} C_t(1-H)^{-\frac{1-\theta}{\theta}\psi} &= \mathbb{C}_1^{-1/\theta} W_t^{\alpha} A_t \\ \Leftrightarrow \frac{1-\alpha}{\psi} H^{-\alpha}(1-H)^{1-\frac{1-\theta}{\theta}\psi} &= \mathbb{C}_1^{-1/\theta}. \end{aligned}$$

Inserting the condition for $\mathbb{C}_1$, we obtain

$$\begin{aligned} \left(\frac{1-\alpha}{\psi}\right)^{-\theta} H^{\alpha\theta}(1-H)^{-\theta+(1-\theta)\psi} &= -\frac{(1-s)^{1-\theta} H^{(1-\alpha)(1-\theta)} (1-H)^{(1-\theta)\psi}}{(1-\theta)H^{1-\alpha} - (1-\theta)(1-s)H^{1-\alpha}} \\ \Leftrightarrow \frac{\psi}{1-\alpha} &= -\frac{1-H}{(1-\theta)H - (1-\theta)(1-\alpha)(1-H)/\psi}. \end{aligned}$$

Collecting terms yields

$$\begin{aligned} \psi &= -\frac{(1-\alpha)(1-H)}{(1-\theta)H - (1-\theta)(1-\alpha)(1-H)/\psi} \\ \Leftrightarrow -\psi(1-\theta)H &= \theta(1-\alpha)(1-H) \\ \Leftrightarrow H &= \frac{\theta(1-\alpha)}{\theta(1-\alpha) - \psi(1-\theta)} \end{aligned}$$

which are admissible solutions if and only if $0 < H < 1$, which holds for $\theta > 1$.

### A.3.3 Obtaining the reduced form

In order to keep notation simple, this section provides the full derivation for a deterministic system. The complete derivation for the stochastic system is available on request from the author. Observe that the system of ODEs reads

$$\begin{aligned} dC_t &= -\frac{u_C}{u_{CC}}(r_t - \rho - \delta)\,dt - \frac{u_{CH}}{u_{CC}} dH_t, \\ dH_t &= \frac{u_{HC}u_C - u_{CC}u_H}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}}(\rho - (r_t - \delta))dt \\ &\quad - \frac{u_{CC}u_H}{Y_{HH}/Y_H u_H u_{CC} + \bar{u}} \frac{Y_{HK}}{Y_H}((r_t - \delta)W_t + H_t w_t^H - C_t)dt, \\ dW_t &= ((r_t - \delta)W_t + H_t w_t^H - C_t)dt, \end{aligned}$$

where we can neglect the first ODE because in equilibrium $C_t = C(H(W_t))$. We find that

$$\begin{aligned} dH_t &= \frac{-(1-\theta)\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1} - \theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}}{Y_{HH}/Y_H \theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1} + \bar{u}}(\rho - (r_t - \delta))dt \\ &\quad - \frac{\theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1}}{Y_{HH}/Y_H \theta\psi C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-1} + \bar{u}} \frac{Y_{HK}}{Y_H}((r_t - \delta)W_t + H_t w_t^H - C_t)dt, \end{aligned}$$

where

$$\begin{aligned}\bar{u} &= (1-\theta)^2\psi^2 C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-2} + (\theta\psi^2 - \theta^2\psi^2 - \psi\theta)C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-2} \\ &= \left((1-\theta)^2\psi^2 + \theta\psi^2 - \theta^2\psi^2 - \psi\theta\right)C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-2} \\ &= \psi\left(\psi - \theta\psi - \theta\right)C_t^{-2\theta}(1-H_t)^{2(1-\theta)\psi-2}.\end{aligned}$$

Hence, inserting $\bar{u}$ and collecting terms yields

$$\begin{aligned}dH_t &= \frac{-1}{Y_{HH}/Y_H\theta + \left((1-\theta)\psi - \theta\right)(1-H_t)^{-1}}(\rho - (r_t - \delta))dt \\ &\quad - \frac{\theta}{Y_{HH}/Y_H\theta + \left((1-\theta)\psi - \theta\right)(1-H_t)^{-1}}\frac{Y_{HK}}{Y_H}((r_t-\delta)W_t + H_t w_t^H - C_t)dt.\end{aligned}$$

Inserting remaining partial derivatives yields,

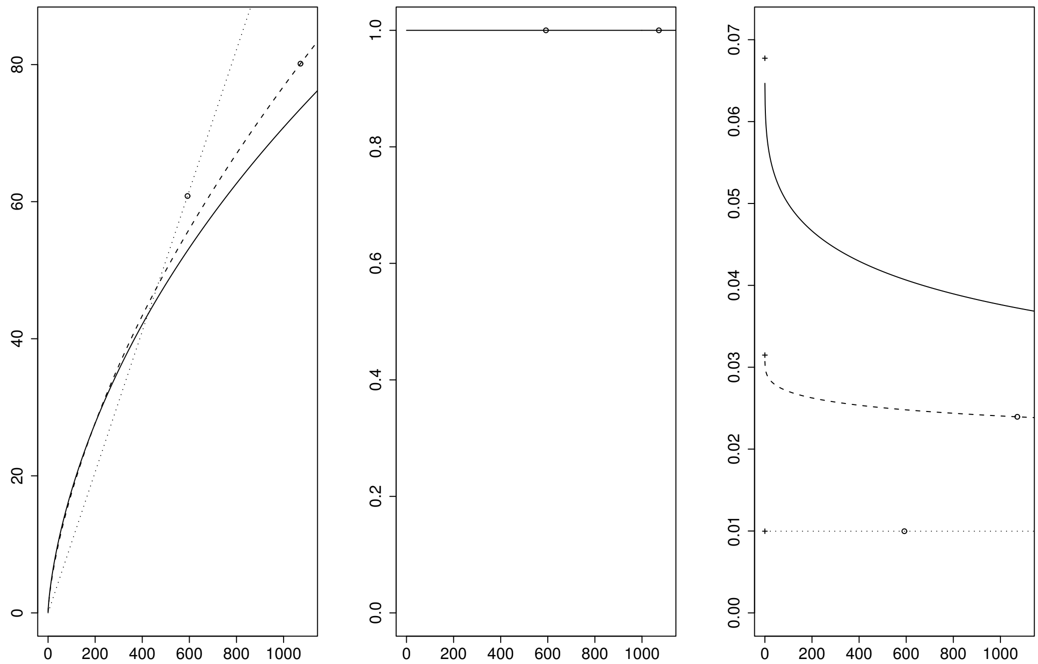
$$\begin{aligned}dH_t &= \frac{-\rho + r_t - \delta}{-\alpha\theta H_t^{-1} + \left((1-\theta)\psi - \theta\right)(1-H_t)^{-1}}dt + \frac{-\theta(r_t - \alpha\delta - \alpha C_t/W_t)}{-\alpha\theta H_t^{-1} + \left((1-\theta)\psi - \theta\right)(1-H_t)^{-1}}dt \\ &= \frac{\rho - r_t + \delta + \theta(r_t - \alpha\delta - \alpha C_t/W_t)}{\alpha\theta H_t^{-1} + \left(\theta - (1-\theta)\psi\right)(1-H_t)^{-1}}dt.\end{aligned}$$

To summarize, the reduced form description of the deterministic model can be written as

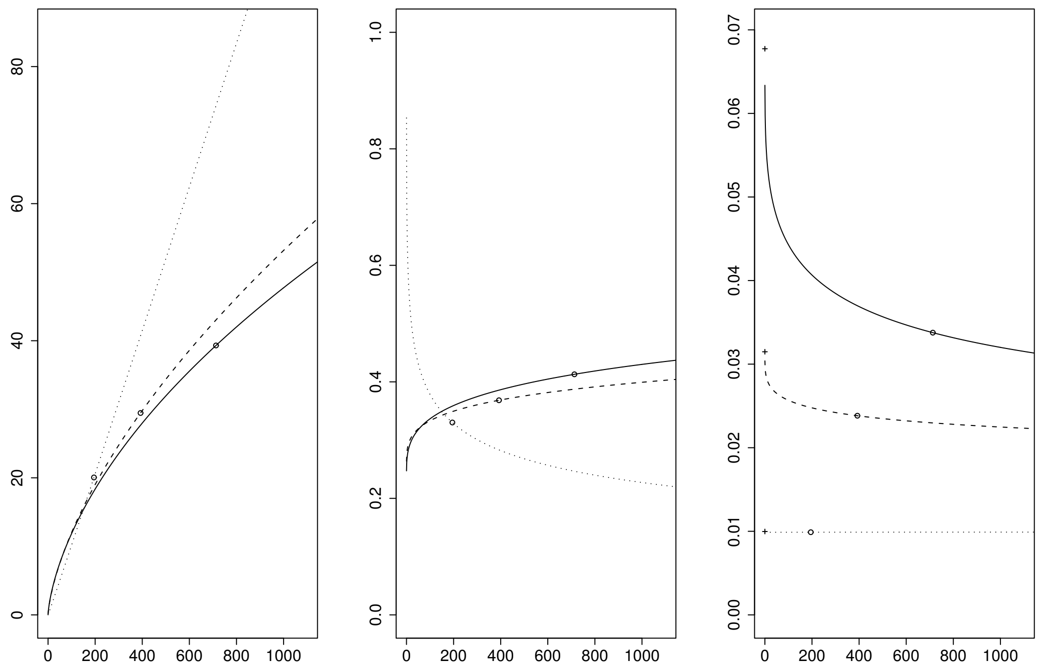
$$\begin{aligned}dH_t &= \frac{\rho + (1-\alpha\theta)\delta - (1-\theta)r_t - \alpha\theta C_t/W_t}{\alpha\theta H_t^{-1} + \left(\theta - (1-\theta)\psi\right)(1-H_t)^{-1}}dt \\ dW_t &= ((r_t - \delta)W_t + H_t w_t^H - C_t)dt,\end{aligned}$$

where $C_t = C(H(W_t), W_t)$.

Figure A.1: Risk premia in a production economy

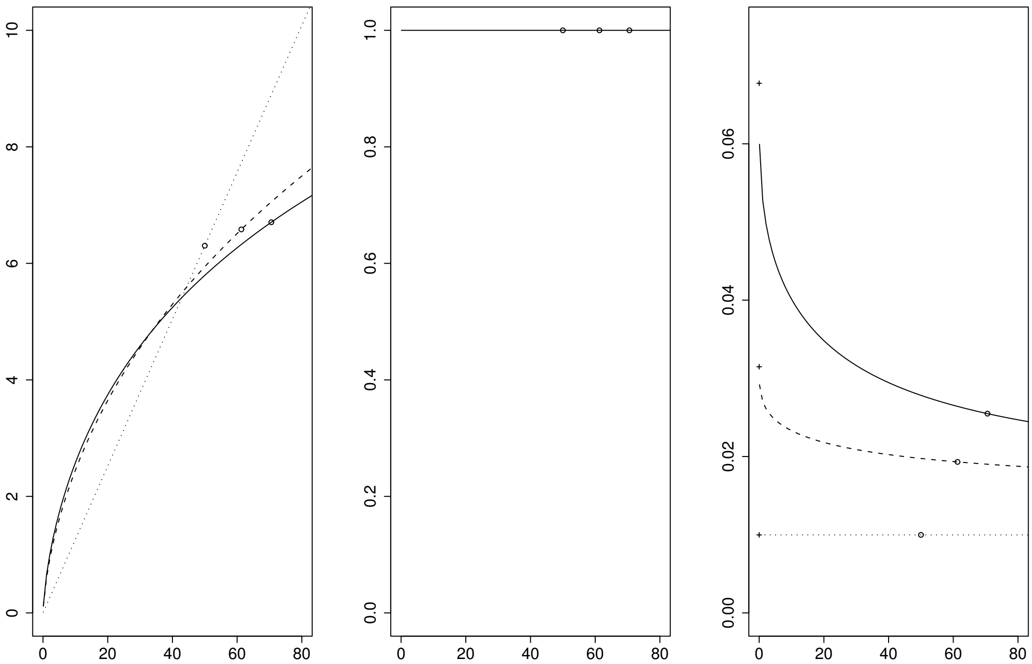


Notes: These figures illustrate the optimal policy functions for consumption (left panel), for hours (middle panel) and the risk premium (right panel) as a function of individual wealth for different levels of relative risk aversion for the case of $\sigma = \bar{\sigma} = \bar{\mu} = 0$, for calibrations $(\rho, \alpha, \theta, \delta, \lambda, 1 - e^{\nu}, \psi) = (.05, .75, \cdot, .1, .017, .4, 0)$ where $\theta = .75$ (dotted), $\theta = 4$ (dashed), and $\theta = 6$ (solid).

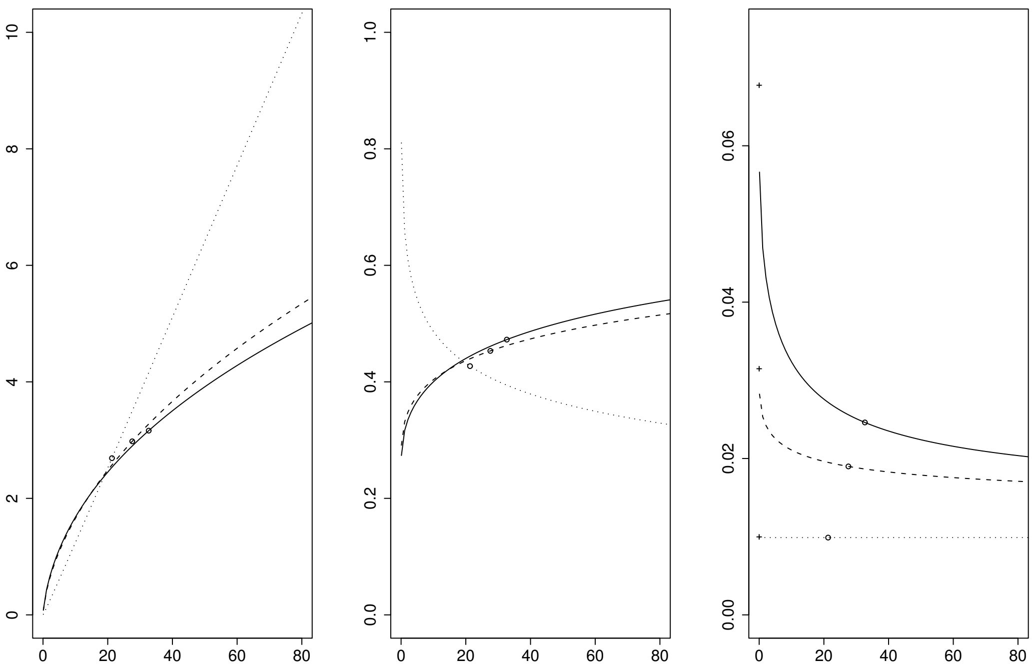


Notes: These figures illustrate the optimal policy functions for consumption (left panel), for hours (middle panel) and the risk premium (right panel) as a function of individual wealth for different levels of relative risk aversion for the case of $\sigma = \bar{\sigma} = \bar{\mu} = 0$, for calibrations $(\rho, \alpha, \theta, \delta, \lambda, 1 - e^{\nu}, \psi) = (.05, .75, \cdot, .1, .017, .4, 1)$ where $\theta = .75$ (dotted), $\theta = 4$ (dashed), and $\theta = 6$ (solid).

Figure A.2: Risk premia in a production economy




Notes: These figures illustrate the optimal policy functions for consumption (left panel), for hours (middle panel) and the risk premium (right panel) as a function of individual wealth for different levels of relative risk aversion for the case of $\sigma = \bar{\sigma} = \bar{\mu} = 0$, for calibrations $(\rho, \alpha, \theta, \delta, \lambda, 1 - e^{\nu}, \psi) = (.03, .75, \cdot, .25, .017, .4, 0)$ where $\theta = .75$ (dotted), $\theta = 4$ (dashed), and $\theta = 6$ (solid).




Notes: These figures illustrate the optimal policy functions for consumption (left panel), for hours (middle panel) and the risk premium (right panel) as a function of individual wealth for different levels of relative risk aversion for the case of $\sigma = \bar{\sigma} = \bar{\mu} = 0$, for calibrations $(\rho, \alpha, \theta, \delta, \lambda, 1 - e^{\nu}, \psi) = (.03, .75, \cdot, .25, .017, .4, 1)$ where $\theta = .75$ (dotted), $\theta = 4$ (dashed), and $\theta = 6$ (solid).