On Rational Exuberance*

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Abstract

The main aim of this paper is to show that expectation-driven fluctuations of a rational bubble may exist. We consider a simple overlapping generations model where money is used to finance a share of consumption of second period of life. The rest of this consumption is financed by credit, using remunerated non-monetary savings coming from the holdings of capital and bonds. Because they have no fundamental value, bonds represent a pure bubble. In this economy, collateral matters because a larger level of non-monetary savings increases the share of consumption financed by credit. We show that the bubbly steady state can be locally indeterminate under arbitrarily small market distortions. Hence, persistent fluctuations of equilibria with (rational) bubbles are explained by self-fulfilling expectations. Finally, we prove that a not too expansive monetary policy can not only rule out sunspot fluctuations, but also improves welfare at the steady state.

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1 Introduction

" [...] Clearly, sustained low inflation implies less uncertainty about the future, and lower risk premiums imply higher prices of stocks and other earning assets. We can see that in the inverse relationship exhibited by price/earnings ratios and the rate of inflation in the past. But how do we know when *irrational*

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exuberance has unduly escalated asset values, which then become subject to unexpected and prolonged contractions as they have in Japan over the past decade? [...] "

In Remarks by Chairman Alan Greenspan. At the Annual Dinner and Francis Boyer Lecture of The American Enterprise Institute for Public Policy Research, Washington, D.C. December 5, 1996.

Starting from these two controversial words of Alan Greenspan "irrational exuberance", we are interested in the existence and persistence of rational exuberance rather than irrational exuberance. We address this issue by showing the existence of persistent expectation-driven fluctuations of a rational bubble.

As explained in Tirole (1982, 1985), the overlapping generations model is the useful framework to show the existence of a rational bubble in a dynamic general equilibrium model, because agents have short lifes and horizons.[1] As it is well-known, Tirole (1985) shows the existence of bubbly steady state, which requires that the steady state without bubble is dynamically inefficient. Moreover, there is a unique dynamic path which monotonically converges to this steady state.

Being also interested in rational bubbles within a dynamic general equilibrium model, we want to show that these rational bubbles can experience persistent fluctuations, that are not explained by shocks on fundamentals, but are rather driven by the volatility of expectations, which corresponds to the main idea behind exuberance. Our explanation is mainly based on the features of the credit market.

We extend the overlapping generations model proposed by Tirole (1985) where consumers can save through two assets, productive capital and a pure bubble (bond), introducing money as a third asset. As in Hahn and Solow (1995), money is needed, because of a binding cash-in-advance constraint:[2] a share of consumption of the second period of life is paid using money balances,[3] the rest being financed by non-monetary savings or credit (capital and bonds). A novel feature of this model consists in assuming that the share of consumption financed by credit is increasing with the level of non-monetary savings. This corresponds to the simple idea that a consumer owning more collateral (capital and bonds) can increase his share of consumption financed by credit. Moreover, we note that the higher is the credit share, the lower the market distortions on the credit market.

In this framework, we show that there exists a steady state with a positive bubble and we focus, throughout the paper, on its properties. We explore the effect of a modification of the constant money growth on the stationary

[1] See also Tirole (1990) for an introductory survey.

[2] In contrast to several contributions (Michel and Wigniolle (2003, 2005), Weil (1987)), the real value of money does not correspond to the bubble in our framework. Money is valued because we focus on equilibria where the cash-in-advance constraint is always binding. Instead, a bubble can exist because there is a bond without fundamental value.

[3] See Crettez et al. (1999) for a presentation of different types of cash-in-advance constraints in the Diamond overlapping generations model.

allocation. We show in particular that, when it is not too large, a decrease of the money growth rate improves welfare.

Dynamics are studied through a local analysis. We start by showing that under a constant credit share, the bubbly steady state can never be indeterminate. Only endogenous cycles of period two can emerge. On the contrary, when collateral matters, i.e. the credit share is increasing in non-monetary savings, endogenous cycles cannot only occur, but the bubbly steady state can be indeterminate. Therefore, persistent expectation-driven fluctuations of the rational bubble can emerge, providing a foundation for rational exuberance. It is also worthwhile to notice that these fluctuations appear for arbitrarily small distortions on the credit market[4] and are mainly explained by the opposite dynamic patterns of real money balances and non-monetary savings.

This issue of fluctuations of a rational bubble has been addressed in some previous papers. Weil (1987) shows the existence of a sunspot equilibrium, where the bubble can crash with a positive probability. However, his analysis is based on markovian and exogenous probabilities of change and is not able to explain persistent fluctuations of the bubble. In Azariadis and Reichlin (1996), endogenous fluctuations of the bubble (debt) may occur through a Hopf bifurcation. However, in contrast to our result, their analysis requires sufficiently large increasing returns,[5] i.e. strong market imperfections. Finally, Michel and Wigniolle (2003, 2005) provide an alternative explanation of the fluctuations of a bubble. Cycles between a regime with a bubble (real money balances) and a regime where a cash-in-advance constraint is binding are exhibited. Hence, fluctuations occurs, but in contrast to our analysis, the bubble does not persist along all the dynamic path.

The remainder of the paper is organized as follows. In Section 2 we present the model, while we define the intertemporal equilibrium in Section 3. Section 4 is devoted to study the bubbly steady state. In Section 5, we show the indeterminacy of the bubbly steady state. Section 6 concludes the paper, while many technical details are gathered in the Appendix.

2 The model

We consider an overlapping generations model with two-period lived households and discrete time, $t = 0, 1, ..., +\infty$.

2.1 Households

At period t, N_t individuals are born. Every one consumes a quantity c_{1t} of the final good and supplies inelastically one unit of labor when young, and consumes c_{2t+1} when old. Population growth is constant, $n \equiv N_{t+1}/N_t > 0$.

[4] This means a small elasticity of the credit share with respect to non-monetary savings and a credit share close to one.

[5] Indeed, the real interest rate has to be increasing in capital.

In order to ensure the consumption during the retirement age, people save through a diversified portfolio of nominal balances M_{t+1}, bonds B_{t+1} and productive capital K_{t+1}.[6] Bonds are remunerated at an interest rate, capital is used by firms to produce the consumption good, and money is needed because of a cash-in-advance constraint in the second period of life. We further note p_t the price of consumption good, i_{t+1} and r_{t+1} the rental rates on bonds and capital, respectively, and w_t the real wage.

Preferences are summarized by a Cobb-Douglas utility function in consumption of both periods:

$$U\left(c_{1t}, c_{2t+1}\right) \equiv {c_{1t}}^{a} c_{2t+1}^{1-a} \tag{1}$$

The representative household of a generation born at time t derives consumption and assets demands (money, bonds and capital), by maximizing the utility function (1) under the first and second-period budget constraints:

$$\frac{M_{t+1}}{p_t N_t} + \frac{B_{t+1}}{p_t N_t} + \frac{K_{t+1}}{N_t} + c_{1t} \leq \tau_t + w_t \tag{2}$$

$$c_{2t+1} \leq \frac{M_{t+1}}{p_{t+1} N_t} + i_{t+1} \frac{B_{t+1}}{p_{t+1} N_t} + r_{t+1} \frac{K_{t+1}}{N_t} \tag{3}$$

where $\tau_t = \left(M_{t+1} - M_t\right) / \left(p_t N_t\right)$ are the monetary transfers distributed to young households. In addition, at the second period of life, each consumer faces a cash-in-advance constraint:

$$\left[1 - \gamma\left(s_t\right)\right] p_{t+1} c_{2t+1} \leq \frac{M_{t+1}}{N_t} \tag{4}$$

where s_t represents non-monetary savings:

$$s_t \equiv \frac{B_{t+1}}{p_t N_t} + \frac{K_{t+1}}{N_t} \tag{5}$$

A share $1 - \gamma\left(s\right) \in (0,1)$ of consumption purchases has to be paid cash.[7] The remaining part $\gamma\left(s\right)$ can be paid at the end of the period and denotes the credit share, that is the fraction of consumption good bought on credit. Individual non-monetary savings s_t works as collateral in order to reduce the need of cash, i.e. the larger the collaterals, the easier the purchasing on credit.[8]

Assumption 1 *$\gamma\left(s\right) \in (0,1)$ is a continuous function defined on $[0, +\infty)$, C^2 on $(0, +\infty)$ and strictly increasing ($\gamma'\left(s\right) > 0$). In addition, we define:*

$$\eta_1\left(s\right) \equiv \frac{\gamma'\left(s\right) s}{\gamma\left(s\right)},\ \eta_2\left(s\right) \equiv \frac{\gamma''\left(s\right) s}{\gamma'\left(s\right)} \tag{6}$$

$$\eta_\eta\left(s\right) \equiv \frac{\eta_1'\left(s\right) s}{\eta_1\left(s\right)} = 1 - \eta_1\left(s\right) + \eta_2\left(s\right)$$

[6] We assume a full capital depreciation during the period.

[7] We observe that, alternatively, $1/\left[1 - \gamma\left(s\right)\right]$ can be interpreted as the endogenous velocity of money.

[8] In fact, we extend the cash-in-advance constraint proposed in Hahn and Solow (1995) to the case where the share of consumption when old paid by cash depends on non-monetary savings.

We note that when $\eta_1(s) = 0$ and $\gamma(s)$ tends to 1, money is no more need and there is no more any credit market distortion. Our framework becomes similar to the model studied in the seminal paper by Tirole (1985).

Using $\pi_{t+1} \equiv p_{t+1}/p_t$, a no-arbitrage condition follows from the choice of a non-monetary portfolio:

$$i_{t+1} = \pi_{t+1} r_{t+1} \tag{7}$$

Introducing the variables per capita $m_t \equiv M_t/(p_t N_t)$, $b_t \equiv B_t/(p_t N_t)$ and $k_t \equiv K_t/N_t$, the constraints (2)-(4) can be rewritten:

$$\begin{aligned} n\pi_{t+1} m_{t+1} + s_t + c_{1t} &\leq \tau_t + w_t & (8)\\ c_{2t+1} &\leq n m_{t+1} + r_{t+1} s_t & (9)\\ [1 - \gamma(s_t)]\, c_{2t+1} &\leq n m_{t+1} & (10) \end{aligned}$$

where now

$$s_t = n(k_{t+1} + \pi_{t+1} b_{t+1}) \tag{11}$$

Each household maximizes (1) under the reduced budget and cash-in-advance constraints (8)-(10) to determine his optimal portfolio (m_{t+1}, s_t) and his optimal consumption plan (c_{1t}, c_{2t+1}).

In order to ensure the different constraints to be binding, we assume that money is a dominated asset, that is $r_{t+1} > 1/\pi_{t+1}$ or, equivalently, $i_{t+1} > 1$. The opportunity cost of holding money, that is the nominal interest rate $i_{t+1} - 1$, is supposed to be strictly positive.

Assumption 2 *Let $\omega_{t+1} \equiv s_t/(s_t + n\pi_{t+1} m_{t+1})$. For all $t \geq 0$, we assume $i_t > 1$ and*

$$\eta_1(s_t) < \frac{1 - \gamma(s_t)}{\gamma(s_t)} \frac{\omega_{t+1}}{1 - \omega_{t+1}} \tag{12}$$

Using this assumption, we show that:

Lemma 1 *Under Assumption 2, constraints (8)-(10) are binding.*

Proof. See the Appendix.

Inequality (12) puts an upper bound to the credit-share elasticity $\eta_1(s)$. In fact, if collaterals matter too much, people no longer hold money and the cash-in-advance constraint fails to be binding.

Let $R^s_{t+1} \equiv r_{t+1} - \gamma'(s_t)\, c_{2t+1}$ and $R^m_{t+1} \equiv 1/\pi_{t+1} - \gamma'(s_t)\, c_{2t+1}$. Under Assumption 2, solving the optimal households' behavior, we get:

$$\frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} = \frac{1}{\pi_{t+1}} \frac{R^s_{t+1}}{\gamma(s_t) R^m_{t+1} + [1 - \gamma(s_t)] R^s_{t+1}} > \frac{1}{\pi_{t+1}} \tag{13}$$

where the last inequality holds because money is a dominated asset ($R^m_{t+1} < R^s_{t+1}$).[9] We further note that under a constant credit share ($\gamma(s) = \gamma$), equation

[9] Second order conditions are derived in the Appendix. We show that they are satisfied if $\eta_2(s) \leq 2(\eta_1(s) - 1)$ or $\eta_1(s)$ sufficiently low.

(13) rewrites:

$$\frac{U_1(c_{1t}, c_{2t+1})}{U_2(c_{1t}, c_{2t+1})} = \frac{r_{t+1}}{1 + (1-\gamma)(i_{t+1} - 1)}$$

While the left-hand side is a marginal rate of intertemporal substitution, the right-hand side would reduce to r_{t+1} when γ tends to 1, as in the non-monetary model by Diamond (1965). In this limit case, there is no market distortion. When $\gamma < 1$, money demand entails an opportunity cost which lowers the real return on portfolio. More precisely, the household has to pay cash $1 - \gamma$ to enjoy an additional unit of consumption when old. The interest rate $i_{t+1} - 1$ on the cash holding entails an opportunity cost $(1-\gamma)(i_{t+1} - 1)$ which reduces the purchasing power of non-monetary saving. When the credit share further depends on collaterals, we get an additional distortion due to the marginal impact of savings on the credit share ($\gamma'(s) > 0$).

2.2 Firms

A competitive representative firm produces the final good using the constant returns to scale technology $f(K/N)N$, where the intensive production function $f(k)$ satisfies:

Assumption 3 *$f(k)$ is a continuous function defined on $[0, +\infty)$ and C^2 on $(0, +\infty)$, strictly increasing ($f'(k) > 0$) and strictly concave ($f''(k) < 0$). We further assume $\lim_{k \to 0^+} f'(k) = +\infty$ and $\lim_{k \to +\infty} f'(k) = 0$.*

The firm maximizes the (real) profits $f(K_t/N_t)N_t - w_t N_t - r_t K_t$, taken the prices as given, and we obtain:

$$\begin{aligned} r_t &= f'(k_t) \equiv r(k_t) \\ w_t &= f(k_t) - k_t f'(k_t) \equiv w(k_t) \end{aligned} \tag{14}$$

For further reference, note $\alpha(k) \equiv f'(k)\, k/f(k) \in (0,1)$ the capital share in total income and $\sigma(k) \equiv [f'(k)\, k/f(k) - 1]\, f'(k)/[kf''(k)] > 0$ the elasticity of capital-labor substitution. The two following elasticities can be easily derived:

$$\begin{aligned} \varepsilon_r(k) &\equiv \frac{r'(k)\, k}{r(k)} = -\frac{1 - \alpha(k)}{\sigma(k)} \\ \varepsilon_w(k) &\equiv \frac{w'(k)\, k}{w(k)} = \frac{\alpha(k)}{\sigma(k)} \end{aligned} \tag{15}$$

2.3 Monetary policy

The monetary policy implements a constant money growth, $M_{t+1}/M_t = \mu$. Using real variables per capita, we obtain:

$$\mu = n\pi_{t+1} m_{t+1}/m_t \tag{16}$$

As seen above, we assume, for simplicity, that money is distributed by the monetary authority to young consumers through the lump-sum transfer $\tau_t = (M_{t+1} - M_t) / (p_t N_t)$, or equivalently,

$$\tau_t = n\pi_{t+1} m_{t+1} - m_t \tag{17}$$

2.4 Bonds

Bonds follow $B_{t+1} = i_t B_t$.[10] Using real variables per capita, one equivalently get:

$$i_t b_t = n\pi_{t+1} b_{t+1} \tag{18}$$

Because they have zero intrinsic (fundamental) value, these bonds are pure bubbles.

3 Equilibrium

Substituting (17) in the first-period budget constraint (8), we obtain:

$$m_t + s_t + c_{1t} = w(k_t) \tag{19}$$

where m_t represents demand for real balances per capita.[11] Using (9) and (10), we get:

$$m_{t+1} = s_t \frac{r(k_{t+1})}{n} \frac{1 - \gamma(s_t)}{\gamma(s_t)} \tag{20}$$

$$c_{2t+1} = r(k_{t+1}) \frac{s_t}{\gamma(s_t)} \tag{21}$$

Substituting (20) into (16), we deduce the inflation factor:

$$\pi_{t+1} = \frac{\mu}{n} \frac{\gamma(s_t)}{\gamma(s_{t-1})} \frac{1 - \gamma(s_{t-1})}{1 - \gamma(s_t)} \frac{r(k_t) s_{t-1}}{r(k_{t+1}) s_t} \tag{22}$$

Using these expressions, we can now derive the two equations that will determine the dynamics of this economy. First, from (13), (21) and (22), the consumers' intertemporal trade-off can be rewritten:

$$x_{t+1} = \frac{1-a}{a} \frac{[1 - \eta_1(s_t)] s_t r(k_{t+1})}{\gamma(s_t) s_t + \mu [1 - \gamma(s_t) - \eta_1(s_t)] s_{t-1} \frac{r(k_t)}{n} \frac{\gamma(s_t)}{1-\gamma(s_t)} \frac{1-\gamma(s_{t-1})}{\gamma(s_{t-1})}} \tag{23}$$

[10] For instance, one can assume that this asset is supplied by the government. b_t can be considered as a (real) engagement to repay b_t unit of consumption, whatever the price p_t. Alternatively, B_t can be interpreted as the (monetary) price of a quantity of asset normalized to one. In both the cases, B_t is a non-predetermined variable.

[11] Note that aggregating (9) and (19), and substituting (11) and (18), we find:

$$c_{1t} + c_{2t}/n + nk_{t+1} = r(k_t) k_t + w(k_t) = f(k_t)$$

where

$$x_{t+1} \equiv \frac{c_{2t+1}}{c_{1t}} = \frac{s_t r\left(k_{t+1}\right)/\gamma\left(s_t\right)}{w\left(k_t\right) - s_t - s_{t-1}\frac{r(k_t)}{n}\frac{1-\gamma(s_{t-1})}{\gamma(s_{t-1})}} \tag{24}$$

is obtained from (19), (20) and (21).[12] Second, combining (7), (11) and (18) gives:

$$r\left(k_t\right)\left(s_{t-1} - nk_t\right) = n\left(s_t - nk_{t+1}\right) \tag{25}$$

Definition 1 *An intertemporal equilibrium with perfect foresight is a sequence* $(s_{t-1}, k_t) \in \mathbb{R}^2_{++}$, $t = 0, 1, ..., +\infty$, *such that (23)-(25) are satisfied, given* $k_0 = K_0/N_0 > 0$.

Equations (23)-(25) constitute a two-dimensional dynamic system determining the sequence $(s_{t-1}, k_t)_{t\geq 0}$ where k_t is the only one predetermined variable.

Let us notice that, substituting the definition of ω_{t+1} and (20) into (12), we have $\eta_1\left(s_t\right) < 1/i_{t+1}$. Hence, at equilibrium, Assumption 2 imposes:

$$1 < i_{t+1} < 1/\eta_1\left(s_t\right) \tag{26}$$

for all $t = 0, 1, ..., +\infty$.

4 Steady state analysis

A steady state is a solution $(s, k) \in \mathbb{R}^2_{++}$ that satisfies:

$$x = \frac{1-a}{a}\frac{\left(1-\eta_1(s)\right)r(k)}{\gamma(s) + \mu\left(1-\gamma(s)-\eta_1(s)\right)r(k)/n} \tag{27}$$

with

$$x = \frac{r(k)}{\gamma(s)\left(w(k)/s - 1\right) - \left(1-\gamma(s)\right)r(k)/n}$$

and[13]

$$r(k)\left(s - nk\right) = n\left(s - nk\right) \tag{28}$$

By direct inspection of this last equation, we deduce that two steady states may exist, the one without bubble (bubbleless steady state) where $s = nk$, and the one with a bubble (bubbly steady state) where $s > nk$. Recall that in this paper, we are interested in the role of monetary policy and credit share[14] on the level of the bubble as well as on the occurrence of persistent fluctuations of the bubble. Therefore, we will focus on the bubbly steady state and we will not analyze the bubbleless steady state.[15]

[12] The positivity of the right-hand side of (23) is ensured by (12) (see the proof of Lemma 1). Hence, x_{t+1}, solution of (23), will be positive at equilibrium.

[13] Equation (28) is equivalent to $r(k)b = nb$.

[14] Notice that $\gamma(s)$ can also be seen as coming from the credit market regulation.

[15] It is in fact possible to show the existence of such a steady state and analyze extensively its properties.

Using (27) and (28), a steady state with $s > nk$ is a solution $(s,k) \in \mathbb{R}^2_{++}$ satisfying:

$$r(k) = n \tag{29}$$

$$\frac{a}{1-a}\frac{ns/\gamma(s)}{w(k)-s/\gamma(s)} = \frac{n[1-\eta_1(s)]}{\gamma(s)+\mu[1-\gamma(s)-\eta_1(s)]} \tag{30}$$

It is useful to notice that the level of the capital-labor ratio is given by the golden rule (29). Therefore, this also determines the level of wage $w(k)$. Given k, non-monetary saving s is given by equation (30).

We further notice that, at the steady state, equation (16) simplifies to $\mu = n\pi$. Using equation (18), we also have $i = \pi n$. Therefore, according to equation (26), Assumption 2 is satisfied if and only if the following inequalities hold:

$$1 < \mu < 1/\eta_1(s) \tag{31}$$

We pursue the analysis of the bubbly steady state by showing its existence. Second, we analyze how the credit share and money growth affect the stationary allocation. We end by examining the relationship between monetary policy and welfare.

4.1 Existence

To establish the existence of a steady state with a positive bubble, we further assume:

Assumption 4

$$\frac{anf'^{-1}(n)}{\gamma(nf'^{-1}(n))\,w(f'^{-1}(n)) - nf'^{-1}(n)} < \frac{(1-a)[1-\eta_1(nf'^{-1}(n))]}{\mu-\gamma(nf'^{-1}(n))(\mu-1)-\mu\eta_1(nf'^{-1}(n))}$$

When this last inequality is satisfied, we can prove the following proposition:

Proposition 1 *Let $\underline{s} \equiv nf'^{-1}(n)$ and $\overline{s}$ be defined by $w \equiv w(f'^{-1}(n)) = \overline{s}/\gamma(\overline{s})$. Under Assumptions 1-4, there exists a steady state characterized by the golden rule, $r(k) = n$, and a positive bubble, $s \in (\underline{s}, \overline{s})$. Moreover, when γ is constant, uniqueness of this steady state is ensured.*

Proof. See the Appendix. ■

We note that, by continuity, uniqueness of the steady state with bubble is still satisfied when the credit share $\gamma(s)$ is no more constant but its elasticity $\eta_1(s)$ is weak for all $s \in (\underline{s}, \overline{s})$. However, multiplicity cannot a priori be excluded if this credit share elasticity is sufficiently large for some values of s.

4.2 Comparative statics under η_1 constant

Comparative statics are analyzed assuming that the credit share elasticity $\eta_1(s)$ is constant ($\eta_\eta(s) = 0$). We start by studying the role of the credit constraint on non-monetary savings and, therefore, on the size of the bubble. To simplify the analysis, we assume that:

Assumption 5

$$\mu < 1 + \frac{(1-\eta_1)^2}{\eta_1}\frac{1-a}{a}\frac{w}{s}$$

We notice that this additional assumption is not restrictive when η_1 is low. Since the credit share elasticity η_1 is constant, we can show that:

Proposition 2 *Under Assumptions 1-5 and η_1 constant, non-monetary saving s and the bubble b are both increasing with η_1 because $\mu > 1$.*

Proof. See the Appendix. ■

Under a positive, but not too large (Assumption 5), rate of money growth $\mu - 1$, the more sensitive the credit share to collaterals, the higher the non-monetary saving. Indeed, under a higher η_1, the cash-in-advance constraint $c_2 \leq nm/[1-\gamma(s)]$ enlarges more, following an increasing of s. This reinforces the raise of c_2 and hence of s. Since η_1 affects neither the capital-labor ratio, nor the inflation, a more sensitive credit share to collaterals also increases the size of the bubble.

To examine the role of the level of the credit share on non-monetary saving and on the size of the bubble, we further assume that $\eta_1 = 0$, i.e. the credit share γ is constant. Then, it is possible to evaluate the effect of γ on s. One can show that under a positive monetary growth rate $\mu > 1$, non-monetary saving is increasing with the credit share, which is essentially due to a raise of the bubble size b.[16] Indeed, households are required to hold less cash, improving the non-monetary part in total saving. Since the capital-labor ratio is given by the golden rule, the non-monetary saving raises because the bubble becomes larger.

[16] Assume γ constant ($\eta_1 = 0$). Using the expressions of the ratio between consumptions and non-monetary saving:

$$\begin{aligned} x &= \frac{1-a}{a}\frac{n}{\gamma + (1-\gamma)\mu} \\ s &= kn\frac{1-\alpha}{\alpha}\frac{(1-a)\gamma}{1+a(1-\gamma)(\mu-1)} \end{aligned}$$

we differentiate (30) with respect s and γ, and get:

$$\frac{ds}{d\gamma}\frac{\gamma}{s} = \frac{1+a(\mu-1)}{1+a(\mu-1)(1-\gamma)}$$

which is strictly positive since $\mu > 1$. Moreover, using $b = [s - nf'^{-1}(n)]/\mu$, we have $db/d\gamma = (1/\mu)ds/d\gamma$, which is also strictly positive.

The capital-labor ratio, which is determined by the golden rule $r(k) = n$, can never be affected by the monetary policy when there is a bubble. However, non-monetary saving is modified by money growth because the size of the bubble is. When μ, η_1, a, w and s satisfy Assumption 5, we can show:

Proposition 3 *Under Assumptions 1-5 and η_1 constant, non-monetary saving s is decreasing with μ if and only if $\eta_1 < 1 - \gamma$.*

Proof. See the Appendix. ■

A higher μ increases the inflation rate and the nominal interest rate, i.e. the opportunity cost of holding money. This reduces the demand of real balances. When the credit share is little sensitive to collaterals ($\eta_1 < 1 - \gamma$), the cash-in-advance constraint lowers the future consumption which, according to the budget constraint, requires less non-monetary saving ($\varepsilon_{s\mu} < 0$). Conversely, if credit market sensitivity to collaterals becomes large enough ($\eta_1 > 1 - \gamma$), individuals can reduce the burden of cash-in-advance by purchasing collaterals ($\varepsilon_{s\mu} > 0$) so compensating the increase of nominal interest rate.[17]

It is also of interest to compute how the (real) bubble b adjusts in response to a change of money growth. We can show that:

Corollary 1 *Under Assumptions 1-5 and η_1 constant, the (real) bubble b is decreasing with μ if $\eta_1 < 1 - \gamma$.*

Proof. See the Appendix. ■

When $\eta_1 < 1 - \gamma$, a higher rate of money growth reduces the size of the (real) bubble because it lowers non-monetary saving, but also because inflation raises. If $\eta_1 > 1 - \gamma$, a more expansive monetary policy can increase the size of the (real) bubble if the increase of non-monetary saving is sufficiently large. This occurs for instance if the credit share γ is sufficiently low.

4.3 Welfare

To further analyze the role of money growth on the stationary allocation, we investigate now how μ affect consumers' welfare evaluated at the steady state. As already explained, since there is a bubble, the capital-labor ratio k given by the golden rule does not depend on the monetary policy, whereas non-monetary saving s is affected by the choice of μ. Therefore consumptions when young and old also. Indeed, c_1 and c_2 can now be written:

$$c_1 = f(k) - nk - \frac{s}{\gamma(s)} \tag{32}$$

$$c_2 = n\frac{s}{\gamma(s)} \tag{33}$$

[17] This interpretation is confirmed by the fact that the ratio between consumptions $x = c_2/c_1$ is increasing with respect to s (see the proof of Proposition 3). Therefore, this ratio increases (decreases) with μ when $\eta_1 > 1 - \gamma$ ($\eta_1 < 1 - \gamma$).

The stationary level of welfare of an household is given by $W = U(c_1, c_2)$. Let:

$$\begin{aligned} \mu_1 &\equiv \frac{\gamma}{\eta_1 - (1-\gamma)} \\ \mu_2 &\equiv 1 + \frac{1-\eta_1}{1-\eta_1+\eta_\eta} \frac{(1-\eta_1)^2}{\eta_1} \frac{1-a}{a} \frac{w}{s} \end{aligned}$$

After some computations,[18] we obtain:

$$\varepsilon_{W\mu} = \varepsilon_{Uc_2} \frac{\mu}{\gamma} \frac{1-\eta_1}{\eta_1} \frac{1-\eta_1}{1-\eta_1+\eta_\eta} \frac{\mu-1}{\mu-\mu_1} \frac{1-\gamma-\eta_1}{\mu-\mu_2} \tag{34}$$

To establish how the welfare evolves according to a variation of $\mu > 1$, we further assume[19]:

Assumption 6 $\eta_\eta > \eta_1 - 1$.

We are able to show the following proposition:

Proposition 4 *Consider that Assumptions 1-4 and 6 are satisfied.*

1. *When $\eta_1 < 1 - \gamma$: the welfare W is decreasing in μ for $1 < \mu < \mu_2$, and increasing for $\mu > \mu_2$;*
2. *When $1-\gamma < \eta_1$: the welfare W is decreasing in μ for $1 < \mu < \min\{\mu_1, \mu_2\}$, increasing for $\min\{\mu_1, \mu_2\} < \mu < \max\{\mu_1, \mu_2\}$, and decreasing again for $\mu > \max\{\mu_1, \mu_2\}$.*

In the limit case where $\mu = 1$, the welfare W reaches a local maximum.

Proof. See the Appendix. ■

Recall that a variation of μ induces a decrease or an increase of non-monetary savings s depending on the value of η_1 with respect to $1 - \gamma$ (see Proposition 3). Moreover, by direct inspection of (32) and (33), we see that consumption c_1 is decreasing in s, whereas c_2 is raising with s. Hence, when $\eta_1 < 1 - \gamma$ and μ not too large, a higher money growth rate, decreasing non-monetary savings, has a predominant effect on welfare through consumption of the second period of life. On the contrary, when $\eta_1 > 1 - \gamma$ and μ not too large, welfare decreases with the money growth rate, because the raise of non-monetary savings reduces consumption of the first period, which has the most important effect on welfare.

In any case, it is important to notice that starting with a not too large money growth rate, a decrease of μ is welfare improving.

Finally, we note that in the limit case where μ tends to 1, we have $U_1(c_1, c_2)/U_2(c_1, c_2) = r = n$ (see equation (27)), which corresponds to the usual intertemporal trade-off in the Diamond (1965) model without cash-in-advance constraint. Market distortions do no more affect consumers' choice. This also corresponds to the Friedman rule $i = n\pi = 1$.

[18] See the Appendix for more details on the derivation of equation (34).

[19] Notice that Assumption 6 includes the isoelastic case ($\eta_\eta = 0$).

5 Sunspot bubbles

We will show the existence of sunspot bubbles, that is, multiple equilibria that converge to a steady state characterized by a positive bubble. We will address this issue establishing that the steady state with a positive rational bubble can be locally indeterminate and, therefore, expectation-driven fluctuations of the bubble can arise, without any shock on the fundamentals. We will further emphasize the crucial role of collateral on the credit share. Indeed, when the credit share is constant, the steady state will never be indeterminate, whereas when it depends on non-monetary savings, indeterminacy could occur under arbitrarily weak market distortions.

We start by linearizing the dynamic system (23)-(25) around the steady state with a positive bubble[20] and obtain the following proposition:

Proposition 5 *Let*

$$Z_1 \equiv (1-\gamma-\eta_1)\left[\frac{1-a}{a}+\mu\frac{1-\gamma-\eta_1}{(1-\gamma)(1-\eta_1)}\right] \tag{35}$$

$$Z_2 \equiv \gamma\left[\frac{\mu-1}{1-\eta_1}\left(1+\eta_1+\frac{\eta_1}{1-\eta_1}\eta_2\right)-\mu\frac{1-\gamma-\eta_1^2}{(1-\gamma)(1-\eta_1)}-\frac{1-a}{a}\right] \tag{36}$$

$$Z_3 \equiv \frac{1-a}{a}\left(1+\eta_1\frac{1-y}{y}\right)+\mu\frac{1-\eta_1-\gamma}{1-\eta_1}\left(1+\frac{\eta_1}{1-\gamma}\frac{1-y}{y}\right) \tag{37}$$

where the capital share in total non-monetary saving $y \equiv rk/(rk+ib) = nk/s \in (0,1]$, $\gamma \equiv \gamma(s)$, $\eta_1 \equiv \eta_1(s)$ and $\eta_2 \equiv \eta_2(s)$ are all evaluated at the steady state.

Under Assumptions 1-4, the characteristic polynomial evaluated at a steady state with a positive bubble ($r(k) = n$, $y \in (0,1)$) is defined by $P(X) \equiv X^2 - TX + D = 0$, where:

$$D = \frac{Z_1}{Z_2}-\frac{1-\alpha}{\sigma}\frac{Z_3}{Z_2} \equiv D(\sigma) \tag{38}$$

$$T = 1+D(\sigma)-\frac{1-\alpha}{\sigma}\frac{1-y}{y}\left(\frac{Z_1}{Z_2}-1\right) \equiv T(\sigma) \tag{39}$$

Proof. See the Appendix. ■

Following Grandmont et al. (1998), the local stability properties of the steady state, that is, the location of the eigenvalues with respect to the unit circle, can be characterized in the (T,D)-plane (see Figures 1 and 2). More explicitly, we evaluate the characteristic polynomial $P(X) \equiv X^2 - TX + D = 0$ at -1, 0 and 1. Along the line (AC), one eigenvalue is equal to 1, i.e. $P(1) = 1 - T + D = 0$. Along the line (AB), one eigenvalue is equal to -1, i.e. $P(-1) = 1 + T + D = 0$. On the segment $[BC]$, the two eigenvalues are complex and conjugate

[20] For sake of conciseness, we will not analyze local dynamics in the neighborhood of the steady state without bubble. Moreover, this would not help us to prove the result we focus on.

with unit modulus, i.e. $D = 1$ and $|T| < 2$. Therefore, inside the triangle ABC, the steady state is a sink, i.e. locally indeterminate ($D < 1$ and $|T| < 1 + D$). It is a saddle point if (T, D) lies on the right or left sides of both (AB) and (AC) ($|1 + D| < |T|$). It is a source otherwise. Moreover, when a parameter varies continuously, we can examine how (T, D) moves in the (T, D)-plane. A (local) bifurcation arises when at least one eigenvalue crosses the unit circle, that is, when the pair (T, D) crosses one of the loci (AB), (AC) or $[BC]$. According to the changes of the bifurcation parameter, a transcritical bifurcation (generically) occurs when (T, D) goes through (AC), a flip bifurcation (generically) arises when (T, D) crosses (AB), whereas a Hopf bifurcation (generically) happens when (T, D) goes through the segment $[BC]$.

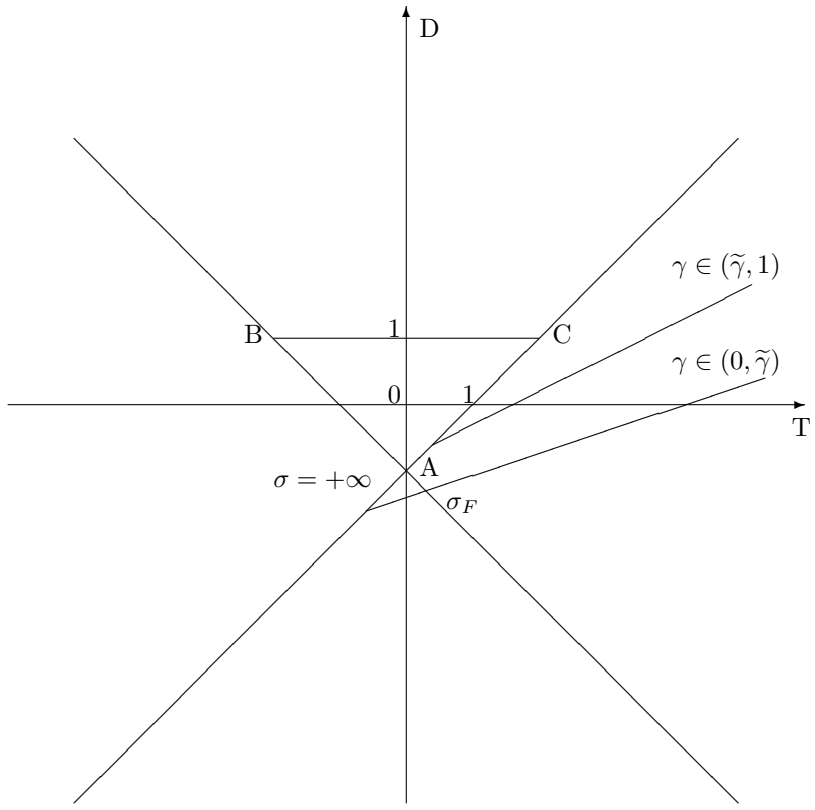

Figure 1: Local dynamics when γ is constant

A convenient parameter to discuss the stability of the steady state and the occurrence of bifurcations in the (T, D)-plane is the elasticity of capital-labor substitution $\sigma \in (0, +\infty)$. When this bifurcation parameter varies, the locus

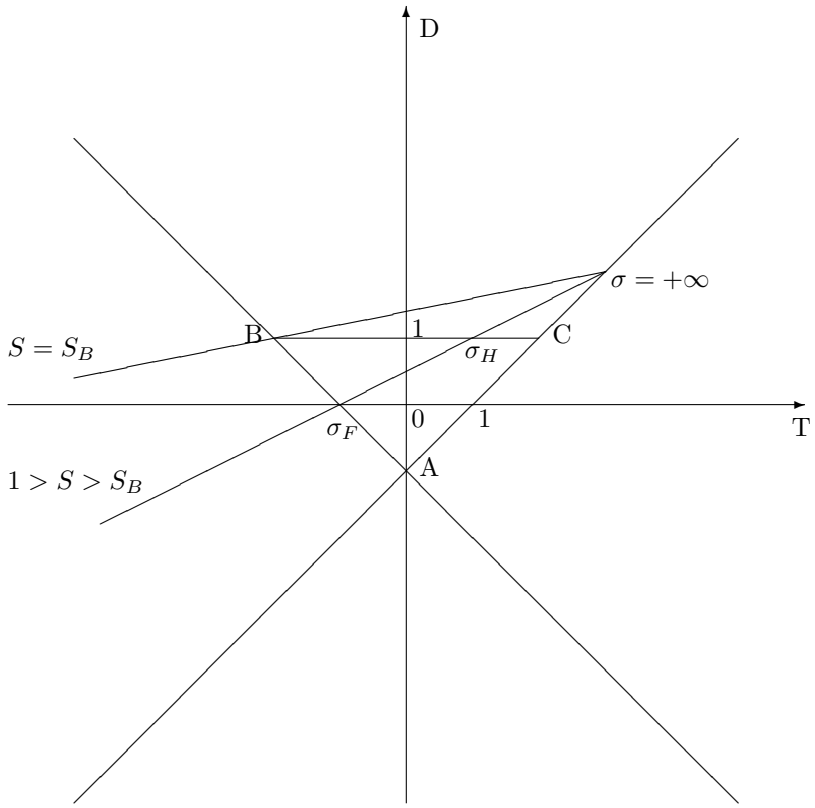

Figure 2: Indeterminate bubble

$\Sigma \equiv \{(T(\sigma), D(\sigma)) : \sigma \geq 0\}$ describes a half-line with a slope given by:

$$S = \frac{D'(\sigma)}{T'(\sigma)} = \frac{Z_3}{Z_3 + (Z_1 - Z_2)(1-y)/y} \tag{40}$$

We further note that the half-line Σ is characterized by the endpoint $(T(+\infty), D(+\infty))$ given by:

$$D(+\infty) = Z_1/Z_2 \text{ and } 1 - T(+\infty) + D(+\infty) = 0$$

which is located on the line (AC). Finally, the starting point $(T(0^+), D(0^+))$ is such that $T(0^+) = \pm\infty$ and $D(0^+) = \pm\infty$, depending on the slope S.

To clearly establish the crucial role played by collaterals on the credit share, we begin by analyzing the case where the credit share γ is constant, i.e. $\eta_1 = \eta_2 = 0$. Using equations (35)-(37), we get:

$$Z_1/Z_2 = -[1 + a(\mu - 1)](1-\gamma)/\gamma < 0 \tag{41}$$
$$Z_3/Z_2 = -[1 + a(\mu - 1)]/\gamma < 0 \tag{42}$$

Hence, the slope S belongs to $(0,1)$ and $D(\sigma)$ is decreasing. This also means that $T(0^+) = +\infty$ and $D(0^+) = +\infty$. Moreover, since $D(+\infty) = Z_1/Z_2 < 0$, $(T(+\infty), D(+\infty))$ is on the line (AC) below the horizontal axis. Let:

$$\widetilde{\gamma} \equiv \frac{1 + a(\mu - 1)}{2 + a(\mu - 1)} \in (1/2, 1) \tag{43}$$

We easily deduce that for $\gamma \in (\widetilde{\gamma}, 1)$, $D(+\infty) > -1$, whereas for $\gamma \in (0, \widetilde{\gamma})$, $D(+\infty) < -1$. Therefore, for $\gamma \in (\widetilde{\gamma}, 1)$, the half-line Σ is below (AC) and above (AB). For $\gamma \in (0, \widetilde{\gamma})$, Σ is still below (AC) but crosses (AB) when $\sigma = \sigma_F$,[21] with:

$$\sigma_F \equiv (1 - \alpha)\left[\frac{1 - y}{2y} + \frac{1 + a(\mu - 1) + \gamma(1 - y)/y}{(1 - \gamma)[1 + a(\mu - 1)] + \gamma}\right] \tag{44}$$

Using these geometrical results, we deduce the following proposition:

Proposition 6 *Let $\widetilde{\gamma}$ be defined by (43), σ_F by (44), γ be constant and $\eta_1 = \eta_2 = 0$. Under Assumptions 1-4, the following generically holds.*

(i) When $\gamma \in (\widetilde{\gamma}, 1)$, the bubbly steady state is a saddle for all $\sigma > 0$.

(ii) When $\gamma \in (0, \widetilde{\gamma})$, the bubbly steady state is a saddle for $0 < \sigma < \sigma_F$, undergoes a flip bifurcation for $\sigma = \sigma_F$, becoming a source for $\sigma > \sigma_F$.

When the credit share γ is constant, local indeterminacy can never emerge, which excludes expectation-driven fluctuations. When γ is sufficiently large, the bubbly steady state is a saddle for all degree of the capital-labor substitution. This result is similar to Tirole (1985), which corresponds to the limit case where γ tends to 1. In contrast, when γ is weaker, the bubbly steady state looses its saddle-path stability through the occurrence of a 2-cycle, becoming unstable if the capital-labor substitution is sufficiently large.

Assuming now that credit share is affected by collaterals ($\eta_1 \neq 0$, $\eta_2 \neq 0$), we will show that the steady state may be locally indeterminate, i.e. expectation-driven fluctuations of the (rational) bubble may occur. Furthermore, we will prove that such fluctuations appear under arbitrarily weak market distortions, that is, η_1 close to 0 and γ close to 1.

Hence, we focus now on the conditions to get local indeterminacy, i.e. whenever Σ goes inside the triangle ABC (see Figure 2). As it is well-known, $1 - T(\sigma) + D(\sigma) > 0$ is a necessary condition to be inside ABC. Using (38) and (39), this inequality is equivalent to $Z_1/Z_2 > 1$. This implies that $(T(+\infty), D(+\infty))$ lies on the line (AC) above the point C. Then, two requirements are needed to get indeterminacy, i.e. Σ goes through ABC: $D(\sigma)$ has to be increasing and S must belong to $(S_B, 1)$, where $S_B \equiv (Z_1 - Z_2)/(Z_1 + 3Z_2) \in (0,1)$ is the value of the slope S such that the half-line Σ goes through the point B. In such a case, we also have $T(0^+) = -\infty$ and $D(0^+) = -\infty$. At this stage, we further note that $D'(\sigma) > 0$ is equivalent to $Z_3/Z_2 > 0$, which together with $Z_1/Z_2 > 1$ ensures that $S < 1$.

These geometrical results are summarized in the following proposition:

[21] The critical value σ_F solves $1 + T(\sigma_F) + D(\sigma_F) = 0$.

Proposition 7 *Let*

$$\begin{aligned}\sigma_F &\equiv (1-\alpha)\frac{2Z_3+(Z_1-Z_2)\frac{1-y}{y}}{2(Z_1+Z_2)}\\ \sigma_H &\equiv (1-\alpha)\frac{Z_3}{Z_1-Z_2}\end{aligned}$$

be the critical values of the capital-labor substitution such that $1+T(\sigma_F)+D(\sigma_F)=0$ *and* $D(\sigma_H)=1$*, respectively.*

Under Assumptions 1-4, the steady state with a positive bubble is locally indeterminate if the conditions (i) $Z_1/Z_2>1$*, (ii)* $Z_3/Z_2>0$ *and (iii)* $(Z_1-Z_2)/(Z_1+3Z_2)<S$ *are satisfied, where* Z_1*,* Z_2*,* Z_3 *are given by (35)-(37), and* S *by (40).*

In this case, local indeterminacy occurs for $\sigma\in(\sigma_F,\sigma_H)$*. Generically, the steady state undergoes a flip bifurcation for* $\sigma=\sigma_F$ *and a Hopf bifurcation for* $\sigma=\sigma_H$*.*

We remark that, since $0<\sigma_F<\sigma_H<+\infty$, there is no room for a locally indeterminate bubble when the production factors are either too weak substitutes (σ sufficiently close to 0) or too large substitutes (σ high enough).

Conditions (i)-(iii) are not sufficiently meaningful and we need to connect them to the structural parameters of the model that have a more clear economic interpretation. In particular, money growth μ and (γ,η_1,η_2), which summarize the role of collaterals and credit share, will play key roles.

We define the following critical values:

$$\begin{aligned}\underline{\mu} &\equiv \frac{1-\gamma}{\eta_1}\frac{1+\eta_1+(1-\eta_1)\frac{1-a}{a}}{1+\eta_1-\gamma}\\ \overline{\mu} &\equiv 1+\left[a\left(\frac{\gamma}{1-\eta_1}\frac{1+\eta_1}{1-\eta_1}-1\right)\right]^{-1}\\ \theta_1 &\equiv -\frac{1-\eta_1^2}{\eta_1}\left[1-\frac{1-\eta_1}{1+\eta_1}\left(1+\frac{\frac{1}{a}+\frac{\mu\eta_1}{1-\eta_1}\frac{1-\gamma-\eta_1}{1-\gamma}}{\mu-1}\right)\right]\\ \theta_2 &\equiv -\frac{1-\eta_1^2}{\eta_1}\left[1-\frac{1-\eta_1}{1+\eta_1}\left(1+\frac{\frac{1}{a}-\frac{M}{1-\eta_1}}{\mu-1}\right)\frac{1-\eta_1}{\gamma}\right]\end{aligned}$$

where M is given by (72).[22] We further assume that:

Assumption 7 $\gamma<1-\eta_1$ *and* $\underline{\mu}<\mu<\overline{\mu}$*.*

To underline that the interval $\left(\underline{\mu},\overline{\mu}\right)$ can be nonempty, we observe that $\lim_{\gamma\to1-\eta_1}\left(\overline{\mu}-\underline{\mu}\right)=0$, while

$$\left.\frac{\partial\left(\overline{\mu}-\underline{\mu}\right)}{\partial\gamma}\right|_{\gamma=1-\eta_1}=-\frac{1-a}{2a\eta_1}<0$$

[22] See the proof of Proposition 8.

Therefore, the interval $\left(\underline{\mu}, \overline{\mu}\right)$ becomes nonempty as soon as γ slightly decreases from $1-\eta_1$. We note also that $\underline{\mu} > 1$, and $\overline{\mu} < 1/\eta_1$ for η_1 sufficiently close to 0.[23] This means that Assumption 7 is in accordance with Assumption 2 (see also inequalities (31)) and the second order condition of utility maximization when η_1 is sufficiently low and γ close to 1, i.e. for arbitrarily weak market imperfections.

Then, we are able to show the following proposition:

Proposition 8 *Under Assumption 7, the conditions (i)-(iii) of Proposition 7 are satisfied if*

$$\max\left\{\theta_1, \theta_2\right\} < \eta_2 \tag{45}$$

Moreover, we notice that $\underline{\mu} < \mu$ is equivalent to $\theta_1 < 0$ and

$$M > (\overline{\mu} - \mu)\frac{1+\eta_1}{1-\eta_1}\left[\gamma - (1-\eta_1)\frac{1-\eta_1}{1+\eta_1}\right], \tag{46}$$

satisfied for μ sufficiently close to $\overline{\mu}$, entails $\theta_2 < 0$.

Proof. See the Appendix. ■

This proposition shows that when collaterals matters $\eta_1 \neq 0$, endogenous cycles can occur not only through a flip bifurcation (cycle of period 2) but also through a Hopf bifurcation, promoting the emergence of an invariant closed curve around the steady state.

Moreover, the steady state can be locally indeterminate, meaning that expectation-driven fluctuations of the bubble can occur around the bubbly steady state. In other words, persistent fluctuations of the bubble may occur driven by the volatility of expectations, i.e. the rational exuberance of the agents.

To our knowledge, this result is new and occurs when η_1 is sufficiently low and γ close to 1, i.e. under arbitrarily small market distortions. Furthermore, it requires neither a too low not a too high elasticity of capital-labor substitution (see Proposition 7). We can even underline that indeterminacy may arise under usual specifications of the technology. For instance, in the Cobb-Douglas case, this requires $\sigma_F < 1 < \sigma_H$, which is equivalent to:

$$Z_2 > Z_1 - (1-\alpha)Z_3 \tag{47}$$

$$Z_2 > \frac{2(1-\alpha)Z_3 - \left(2 - \frac{1-\alpha}{\alpha}\frac{1-y}{y}\right)Z_1}{2 + \frac{1-\alpha}{\alpha}\frac{1-y}{y}} \tag{48}$$

Using (35) and (37), (47) is satisfied for η_1 sufficiently close to $1-\gamma$ because this ensures the right-hand side of the inequality to be negative and we have $Z_2 > 0$.[24] Using now (36), we note that with an appropriate choice of η_2, inequality (48) is satisfied.

[23] Indeed, $\overline{\mu} < 1/\eta_1$ is equivalent to $a > \eta_1(1-\eta_1)/[\gamma(1+\eta_1) - (1-\eta_1)^2]$

[24] See the proof of Proposition 8.

Finally, we notice that Proposition 8 has also some implications for the monetary policy. Indeed, indeterminacy requires $\underline{\mu} < \mu < \overline{\mu}$ (Assumption 7). Therefore, choosing a money growth factor μ higher than $\overline{\mu}$ or lower than $\underline{\mu}$ rules out expectation-driven fluctuations. We argue however, that choosing μ smaller than $\underline{\mu}$ and even sufficiently close to 1 is better. Indeed, in such a case, the monetary authority does not only stabilize fluctuations due to self-fulfilling expectations, but also improves consumers' welfare at the steady state (see Proposition 4).

Economic intuition

We give now an economic interpretation of the results obtained in this section. The intuition we provide is based on the explanation of non monotonic trajectories.

We start with the case where the credit share γ is constant ($\eta_1 = 0$), but strictly smaller than 1. Assuming a decrease of the capital stock k_t from its steady state value, the real wage w_t becomes smaller and the real interest rate r_t higher. When the elasticity of capital-labor substitution is not too weak, this induces a lower level of $r_t s_{t-1}$. Since, using equation (20), we have:

$$m_t = \frac{1}{n}\frac{1-\gamma}{\gamma} r_t s_{t-1} \tag{49}$$

real money balances m_t decreases. As a direct implication, we also get a decrease of $\pi_{t+1} m_{t+1}$ (see equation (16)).

Using now (23) and (24) with γ constant and $\eta_1 = 0$, we obtain:

$$s_t = (1-a)w_t - s_{t-1}\frac{r_t}{n}\frac{1-\gamma}{\gamma}[1 + a(\mu - 1)] \tag{50}$$

Since both w_t and $r_t s_{t-1}$ decrease, two opposite effects affect savings s_t. In particular, we note that the second effect comes from the decrease of money holding, and obviously disappears in the limit case where the credit share γ tends to 1.

Assuming that the second effect dominates, savings s_t increases. Using (22), we deduce that $i_{t+1} = \pi_{t+1} r_{t+1}$ decreases, meaning that the opportunity cost of holding money is reduced. Therefore, money balances m_{t+1} increases, which implies a decrease of inflation π_{t+1} because, as seen above, $\pi_{t+1} m_{t+1}$ reduces. From equation (49), this increase of the real money stock implies a raise of $r_{t+1} s_t$. When capital and labor are not too weak substitutes, capital k_{t+1} becomes higher. Since the bubble $\pi_{t+1} b_{t+1}$ has the same return, it increases also.

This explains that, following a decrease of capital from the steady state, future capital goes in the opposite direction, explaining oscillations. When γ is constant and not too close to 1, we have seen that instability emerges (see Proposition 6). We argue that this comes from two main effects: the strong impact of $r_t s_{t-1}$ on s_t (see (50)) and the proportional relationship between $r_t s_{t-1}$ and m_t (see (49)).

When indeterminacy (local stability) may emerge (see Proposition 8), γ cannot be constant and is closer to 1. Hence, the effect of $r_t s_{t-1}$ on s_t is lower (see (50)). Moreover, the relationship between $r_t s_{t-1}$ and m_t is no more strictly proportional, but becomes:

$$m_t = \frac{1}{n} \frac{1-\gamma(s_{t-1})}{\gamma(s_{t-1})} r_t s_{t-1} \tag{51}$$

Note that the elasticity of $(1-\gamma(s))/\gamma(s)$ with respect to s is equal to $-\eta_1/(1-\gamma)$, which belongs to $(-1,0)$ and is quite small in absolute value under Assumption 7. Therefore, when $r_t s_{t-1}$ decreases, and s_{t-1} as well, the effect on m_t is dampened. In other words, two crucial channels for the occurrence of non-monotonic dynamics are weaker when $\eta_1 > 0$, which provides the intuition for local stability or indeterminacy of the bubbly steady state when collateral matters. Finally, we notice that equation (25) can be rewritten:

$$\pi_t b_t r_t = n \pi_{t+1} b_{t+1} \tag{52}$$

The oscillations just described above can be sustained by optimistic expectations on the future value of the bubble $\pi_{t+1} b_{t+1}$, meaning that consumers born in $t-1$ will (slightly) increase their share of savings through the bubble $\pi_t b_t$, which implies an effective increase of the bubble at the next period $\pi_{t+1} b_{t+1}$, since r_t also raises.

6 Conclusion

The main contribution of this paper is to show that exuberance may be rational. To do that, we extend the overlapping generations model analyzed by Tirole (1985), where the existence of a bubbly steady state is proved, to take in account some form of credit market imperfection. A share of consumption of second period of life has to by paid using money, while savings is also used to buy productive capital and a pure bubble. Collateral matters because a higher level of non-monetary savings reduces this share of consumption financed by money balances.

In this framework, we show that the bubbly steady state can be locally indeterminate because of the role of collateral. Therefore, expectation-driven fluctuations of the bubble can prevail. We further notice that the existence of such fluctuations requires arbitrarily small market distortions. We finally emphasize that a not too expansive monetary policy may rule out endogenous fluctuations, while it improves welfare at the steady state.

We should however underline that we restrict our analysis to equilibria where the cash-in-advance constraint is always binding. In contrast to Michel and Wigniolle (2003, 2005), who use however a simpler framework, the economy cannot switch between two regimes, where the finance constraint is respectively binding or not. Analyzing such dynamic interesting patterns in our model is left for further research.

7 Appendix

Proof of Lemma 1

We maximize the Lagrangian function:

$$\begin{aligned} & U\left(c_{1t}, c_{2t+1}\right) \\ & +\lambda_{1t}\left(\tau_t + w_t - n\pi_{t+1}m_{t+1} - s_t - c_{1t}\right) \\ & +\lambda_{2t+1}\left(nm_{t+1} + r_{t+1}s_t - c_{2t+1}\right) \\ & +\nu_{t+1}\left(nm_{t+1} - \left[1-\gamma\left(s_t\right)\right]c_{2t+1}\right) \end{aligned} \tag{53}$$

with respect to $(m_{t+1}, s_t, c_{1t}, c_{2t+1}, \lambda_{1t}, \lambda_{2t+1}, \nu_{t+1})$. Since $\lambda_{1t} = U_1\left(c_{1t}, c_{2t+1}\right) > 0$, then (8) becomes binding. Because

$$\begin{aligned} \lambda_{2t+1} &= \lambda_{1t}\frac{1-\pi_{t+1}\gamma'\left(s_t\right)c_{2t+1}}{r_{t+1}-\gamma'\left(s_t\right)c_{2t+1}} \\ \nu_{t+1} &= \lambda_{1t}\left(\pi_{t+1} - \frac{1-\pi_{t+1}\gamma'\left(s_t\right)c_{2t+1}}{r_{t+1}-\gamma'\left(s_t\right)c_{2t+1}}\right) \end{aligned}$$

strict positivity of λ_{2t+1} and ν_{t+1} requires

$$\pi_{t+1} > \frac{1-\pi_{t+1}\gamma'\left(s_t\right)c_{2t+1}}{r_{t+1}-\gamma'\left(s_t\right)c_{2t+1}} > 0$$

or, equivalently,

$$i_{t+1} > \frac{r_{t+1}-i_{t+1}\gamma'\left(s_t\right)c_{2t+1}}{r_{t+1}-\gamma'\left(s_t\right)c_{2t+1}} > 0 \tag{54}$$

Inequality $r_{t+1} - i_{t+1}\gamma'\left(s_t\right)c_{2t+1} > 0$ is equivalent to (12). Moreover, $i_{t+1} > 1$ implies $r_{t+1} - \gamma'\left(s_t\right)c_{2t+1} > r_{t+1} - i_{t+1}\gamma'\left(s_t\right)c_{2t+1} > 0$, which ensures that both inequalities in (54) hold. ■

A sufficient condition for utility maximization

We compute the Hessian matrix of the Lagrangian function (53) with respect to $(\lambda_{1t}, \lambda_{2t+1}, \nu_{t+1}, c_{1t}, c_{2t+1}, s_t, m_{t+1})$:[25]

$$H \equiv \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 & -n\pi \\ 0 & 0 & 0 & 0 & -1 & r & n \\ 0 & 0 & 0 & 0 & \gamma-1 & c_2\gamma' & n \\ -1 & 0 & 0 & U_{11} & U_{12} & 0 & 0 \\ 0 & -1 & \gamma-1 & U_{12} & U_{22} & \nu\gamma' & 0 \\ -1 & r & c_2\gamma' & 0 & \nu\gamma' & \nu c_2\gamma'' & 0 \\ -n\pi & n & n & 0 & 0 & 0 & 0 \end{bmatrix}$$

In order to get a regular (i.e. strict) local maximum, we need to check the negative definition of H over the set of points satisfying the constraints. Let

[25] For simplicity, the arguments of the functions and the time subscripts are omitted.

m and n denote the numbers of constraints and variables, respectively. If the determinant of H has sign $(-1)^n$ and the last $n-m$ diagonal principal minors have alternating signs, then the optimum is a regular local maximum. In our case $n=4$ and $m=3$. Therefore, we simply require $\det H > 0$, that is,

$$\begin{aligned}\det H &= -n^2\left[\left(\gamma-\pi\left[c_2\gamma'-r\left(1-\gamma\right)\right]\right)^2 U_{11}\right. \\ &\quad +2\left(c_2\gamma'-r\right)\left(\gamma-\pi\left[c_2\gamma'-r\left(1-\gamma\right)\right]\right)U_{12} \\ &\quad +\left(c_2\gamma'-r\right)^2 U_{22} \\ &\quad \left.-\nu\gamma\left[2\gamma'\left(c_2\gamma'-r\right)-\gamma c_2\gamma''\right]\right]>0 \end{aligned} \tag{55}$$

Using (10) and (9), we find $c_{2t+1}/r_{t+1} = s_t/\gamma\left(s_t\right)$. Substituting in (55) and reducing in elasticities, in order to satisfy the SOCs (locally), we require:

$$\begin{aligned}\det H &= -\left(nr\right)^2\left[\zeta_0+\zeta_1^2 U_{11}+2\zeta_1\left(\eta_1-1\right)U_{12}+\left(\eta_1-1\right)^2 U_{22}\right] \\ &= -\left(nr\right)^2\left[\zeta_0+\left[\begin{array}{cc}\zeta_1 & \eta_1-1\end{array}\right]\left[\begin{array}{cc}U_{11} & U_{12}\\ U_{12} & U_{22}\end{array}\right]\left[\begin{array}{c}\zeta_1\\ \eta_1-1\end{array}\right]\right]>0\end{aligned} \tag{56}$$

where

$$\begin{aligned}\zeta_0 &= \zeta_0 \equiv \nu\eta_1\left[\eta_2+2\left(1-\eta_1\right)\right]\frac{\gamma}{r}\frac{\gamma}{s} \\ \zeta_1 &= \zeta_1 \equiv \pi\left(1-\gamma-\eta_1\right)+\frac{\gamma}{r}\end{aligned}$$

Condition (56) ensures the concavity in the utility maximization program under three constraints. We observe that the negative definiteness of U entails

$$\left[\begin{array}{cc}\zeta_1 & \eta_1-1\end{array}\right]\left[\begin{array}{cc}U_{11} & U_{12}\\ U_{12} & U_{22}\end{array}\right]\left[\begin{array}{c}\zeta_1\\ \eta_1-1\end{array}\right]<0 \tag{57}$$

A sufficient condition, jointly with (57), is $\zeta_0 < 0$ or, equivalently, $\eta_2 \leq 2\left(\eta_1-1\right)$, that is a sufficient degree of concavity of the credit share.[26] It is also useful to notice that the second order condition is satisfied under a sufficiently small elasticity of credit share η_1, which implies ζ_0 close to zero.

In the Cobb-Douglas case, $\zeta_0+\zeta_1^2 U_{11}+2\zeta_1\left(\eta_1-1\right)U_{12}+\left(\eta_1-1\right)^2 U_{22}<0$ becomes:

$$\nu\eta_1\left(\eta_2+2\left(1-\eta_1\right)\right)\frac{\gamma}{n}\frac{\gamma}{s} < a\left(1-a\right){c_1}^a c_2^{1-a}\left[\frac{\gamma+\mu\left(1-\gamma-\eta_1\right)}{nc_1}+\frac{1-\eta_1}{c_2}\right]^2 \tag{58}$$

■

Proof of Proposition 1

The capital-labor ratio k is determined by the golden rule $r\left(k\right) = n$ (see (29)). Using Assumption 3, there exists a unique solution to this equation,

[26] In the isoelastic case, the concavity of credit share is weak: $\eta_2 = \eta_1 - 1$ and $\zeta_0 > 0$. In order to get a local maximum, we require a sufficiently powerful concavity of utility function.

$k = f'^{-1}(n)$. This also determines the real wage $w(k) = w\left(f'^{-1}(n)\right) = w$. Then, s is a solution of $g(s) = h(s)$, with:

$$g(s) \equiv \frac{a}{1-a} x(s), \text{ where } x(s) \equiv \frac{ns/\gamma(s)}{w - s/\gamma(s)} \tag{59}$$

$$h(s) \equiv \frac{n[1-\eta_1(s)]}{\gamma(s) + \mu[1-\gamma(s)-\eta_1(s)]} \tag{60}$$

Since the steady state is characterized by a positive bubble ($b > 0$), we have $s > \underline{s}$. Moreover, because $\eta_1(s) < 1$, $s/\gamma(s)$ is increasing in s, which implies that $x(s) > 0$ requires $s < \overline{s}$. Therefore, all the stationary solutions s belong to $(\underline{s}, \overline{s})$.[27]

To prove the existence of a stationary solution s, we use the continuity of $g(s)$ and $h(s)$, ensured because $\gamma(s)$ is C^2 (see Assumption 1). Using (59) and (60), we determine the boundary values of $g(s)$ and $h(s)$:

$$\begin{aligned}
\lim_{s\to\underline{s}} g(s) &= \frac{a}{1-a}\frac{n^2 k}{w\gamma(nk) - nk} > 0, \ \lim_{s\to\overline{s}} g(s) = +\infty, \\
\lim_{s\to\underline{s}} h(s) &= \frac{n[1-\eta_1(nk)]}{\gamma(nk) + \mu[1-\gamma(nk)-\eta_1(nk)]} > 0, \\
\lim_{s\to\overline{s}} h(s) &= \frac{n[1-\eta_1(\overline{s})]}{\gamma(\overline{s}) + \mu[1-\gamma(\overline{s})-\eta_1(\overline{s})]}
\end{aligned}$$

where $k = f'^{-1}(n)$.

Assumption 4 ensures that $\lim_{s\to\underline{s}} g(s) < \lim_{s\to\underline{s}} h(s)$, while we have $\lim_{s\to\overline{s}} g(s) > \lim_{s\to\overline{s}} h(s)$. Therefore, there exists at least one value $s^* \in (\underline{s}, \overline{s})$ such that $g(s^*) = h(s^*)$.

To address the uniqueness versus the multiplicity of stationary solutions s, we compute the two following elasticities:

$$\begin{aligned}
\varepsilon_g(s) &\equiv \frac{g'(s)\,s}{g(s)} = \frac{w[1-\eta_1(s)]}{w - s/\gamma(s)} > 0 \\
\varepsilon_h(s) &\equiv \frac{h'(s)\,s}{h(s)} = \frac{\eta_1(s)[\eta_\eta(s) + 1 - \eta_1(s)]}{1-\eta_1(s)} \frac{(\mu-1)\gamma(s)}{\gamma(s) + \mu[1-\gamma(s)-\eta_1(s)]}
\end{aligned}$$

A sufficient condition for uniqueness is $\varepsilon_h(s) < \varepsilon_g(s)$ for all $s \in (\underline{s}, \overline{s})$. We deduce that when $\gamma(s)$ is constant ($\eta_1(s) = 0$), uniqueness is ensured because $\varepsilon_h(s) = 0 < \varepsilon_g(s)$. ■

Proof of Proposition 2

Assuming η_1 constant and differentiating (30) with respect s and η_1, we obtain:

$$\varepsilon_{s\eta_1} \equiv \frac{ds}{d\eta_1}\frac{\eta_1}{s} = -\left[\frac{1-\eta_1}{\eta_1}\left(\eta_1 + (1-\eta_1)\frac{w}{s}\frac{1-\eta_1}{1-\mu}\frac{1-a}{a}\right)\right]^{-1}$$

[27] We notice that $(\underline{s}, \overline{s})$ is nonempty. Using (11) and (19), we obtain $w > s \geq nf'^{-1}(n) = \underline{s}$. Because $1-\eta_1(s)$, the elasticity of $s/\gamma(s)$, belongs to $(0,1)$, $s < w$ implies $s < \overline{s}$. We deduce that $\underline{s} < \overline{s}$.

Since $\eta_1 < 1$ and Assumption 5 is satisfied, $\varepsilon_{s\eta_1} > 0$ if and only if $\mu > 1$. According to (11), (22) and (29), we have $b = [s - nf'^{-1}(n)]/\mu$. We easily conclude that b is increasing with η_1 if and only if the same condition is satisfied. ■

Proof of Proposition 3

We differentiate (30) with respect to μ and s. Using $\eta_\eta = 0$, we obtain:

$$\varepsilon_{s\mu} \equiv \frac{ds}{d\mu}\frac{\mu}{s} = \frac{\mu}{\gamma}\frac{1-\gamma-\eta_1}{\eta_1(\mu-1) - (1-\eta_1)^2 \frac{w}{s}\frac{1-a}{a}}$$

Since, under Assumption 5, the denominator of the right-hand side is strictly negative, the proposition immediately follows. ■

Proof of Corollary 1

Differentiating $b = [s - nf'^{-1}(n)]/\mu$, we get:

$$\varepsilon_{b\mu} \equiv \frac{db}{d\mu}\frac{\mu}{b} = \frac{s\varepsilon_{s\mu}}{b\mu} - 1$$

Using Proposition 3, we easily deduce that $\varepsilon_{b\mu} < 0$ if $\eta_1 < 1 - \gamma$. ■

Derivation of equation (34)

Consider the welfare function $W = U(c_1, c_2)$ and note:

$$(\varepsilon_{W\mu}, \varepsilon_{Uc_2}, \varepsilon_{c_2\mu}) \equiv \left(\frac{\partial W}{\partial \mu}\frac{\mu}{W}, \frac{\partial U}{\partial c_2}\frac{c_2}{U}, \frac{dc_2}{d\mu}\frac{\mu}{c_2}\right)$$

We can easily get:

$$\varepsilon_{W\mu} = \varepsilon_{Uc_2}\varepsilon_{c_2\mu}\left(1 + \frac{a}{1-a}x\frac{dc_1/d\mu}{dc_2/d\mu}\right) \tag{61}$$

Differentiating now (32) and (33), we obtain:

$$\frac{dc_1}{d\mu} = -(1-\eta_1)\frac{1}{\gamma}\frac{ds}{d\mu} \tag{62}$$

$$\frac{dc_2}{d\mu} = n(1-\eta_1)\frac{1}{\gamma}\frac{ds}{d\mu} \tag{63}$$

Substituting (62) and (63) in (61) and noticing that $\varepsilon_{c_2\mu} = (1-\eta_1)\varepsilon_{s\mu}$, we get:

$$\varepsilon_{W\mu} = \varepsilon_{Uc_2}\varepsilon_{s\mu}(1-\eta_1)\left(1 - \frac{a}{1-a}\frac{x}{n}\right) \tag{64}$$

Equations (30) implicitly defines s as function of μ. Applying the Implicit Function Theorem, we compute the following elasticity:

$$\varepsilon_{s\mu} = \frac{\mu}{\gamma}\frac{1-\gamma-\eta_1}{\eta_1(\mu-1)\frac{1-\eta_1+\eta_\eta}{1-\eta_1} - (1-\eta_1)^2\frac{w}{s}\frac{1-a}{a}} \tag{65}$$

Substituting (65) in (64), we have:

$$\varepsilon_{W\mu} = \varepsilon_{Uc_2} \frac{\mu}{\gamma} \left(1 - \frac{x}{n} \frac{a}{1-a}\right) \frac{1-\gamma-\eta_1}{(\mu-1)\frac{\eta_1}{1-\eta_1}\frac{1-\eta_1+\eta_\eta}{1-\eta_1} - (1-\eta_1)\frac{w}{s}\frac{1-a}{a}}$$

Using the critical values μ_1 and μ_2, we deduce equation (34). ■

Proof of Proposition 4

Using equation (34) and Assumption 6, the sign of $\varepsilon_{W\mu}$ is equivalent to the sign of:

$$\frac{\mu-1}{\mu-\mu_1}\frac{1-\gamma-\eta_1}{\mu-\mu_2} \tag{66}$$

We note first that under Assumption 6, we have $\mu_2 > 1$. By direct inspection of (66), we deduce that:

1. When $\eta_1 < 1-\gamma$, we have $\mu_1 < 0$ and $1 < \mu_2$. Then, $\varepsilon_{W\mu} > 0$ is strictly positive for $0 < \mu < 1$, $\varepsilon_{W\mu} < 0$ for $1 < \mu < \mu_2$, and $\varepsilon_{W\mu} > 0$ for $\mu > \mu_2$;
2. When $1-\gamma < \eta_1$, we have $1 < \mu_1$ and $1 < \mu_2$. Then, $\varepsilon_{W\mu} > 0$ for $0 < \mu < 1$, $\varepsilon_{W\mu} < 0$ for $1 < \mu < \min\{\mu_1, \mu_2\}$, $\varepsilon_{W\mu} > 0$ for $\min\{\mu_1, \mu_2\} < \mu < \max\{\mu_1, \mu_2\}$, and $\varepsilon_{W\mu} < 0$ for $\mu > \max\{\mu_1, \mu_2\}$.

The limit case where $\mu = 1$ corresponds to a local maximum ($\varepsilon_{W\mu} = 0$). We deduce the proposition taking in account that $\mu > 1$. ■

Proof of Proposition 5

We linearize the system (23)-(25) around a steady state (with or without bubble) with respect to $(k_t, s_{t-1}, k_{t+1}, s_t)$. We obtain:

$$Z_2 \frac{ds_t}{s} = \varepsilon_r \left(\gamma y \frac{1-a}{a} + \frac{1-\gamma}{1-\gamma-\eta_1} Z_1\right) \frac{dk_t}{k} + Z_1 \frac{ds_{t-1}}{s} \tag{67}$$

$$y\frac{n}{r}\frac{dk_{t+1}}{k} - \frac{n}{r}\frac{ds_t}{s} = \left[y - (1-y)\,\varepsilon_r\right]\frac{dk_t}{k} - \frac{ds_{t-1}}{s} \tag{68}$$

where

$$Z_1 \equiv (1-\gamma-\eta_1)\left[\frac{1-a}{a} + \mu\frac{1-\gamma-\eta_1}{(1-\gamma)(1-\eta_1)}\right]$$

$$Z_2 \equiv \left(\mu - \frac{n}{x}\frac{1-a}{a}\right)\left(1 + \frac{\eta_1\eta_\eta}{1-\eta_1}\right) - \mu\gamma\frac{1-\gamma-\eta_1^2}{(1-\gamma)(1-\eta_1)} - \gamma\frac{n}{r}\frac{1-a}{a}$$

and r, ε_r and η_η the stationary values of $r(k)$, $\varepsilon_r(k)$ and $\eta_\eta(s)$, respectively.

The characteristic polynomial is given by $P(X) \equiv X^2 - TX + D = 0$, where T and D represent the trace and the determinant of the associated Jacobian matrix, respectively. After some computations, we get:

$$D = \frac{1}{Z_2}\frac{r}{n}\left(Z_1\left[1 + \varepsilon_r\frac{y(1-\gamma)+(1-y)\eta_1}{y(1-\gamma-\eta_1)}\right] + \varepsilon_r\gamma\frac{1-a}{a}\right) \tag{69}$$

$$T = \frac{r}{n} + \frac{n}{r}D + \varepsilon_r\frac{1-y}{y}\left(\frac{Z_1}{Z_2} - \frac{r}{n}\right) \tag{70}$$

We deduce the expressions given in the proposition considering $y \in (0,1)$ and $r = n$, and using:

$$x = \frac{1-a}{a} \frac{n(1-\eta_1)}{\gamma + \mu(1-\gamma-\eta_1)}$$

■

Proof of Proposition 8

We prove that, under Assumption 7, condition (45) implies the condition (i)-(iii) of Proposition 7, i.e. is sufficient for local indeterminacy.

Assuming $Z_2 > 0$,[28] conditions (i)-(iii) for local indeterminacy in Proposition 7 are equivalent to $Z_1 > Z_2$, $Z_3 > 0$ and

$$Z_2^2 - 2\left(Z_1 + 2Z_3 \frac{y}{1-y}\right) Z_2 + Z_1^2 < 0 \tag{71}$$

that is, to $Z_3 > 0$ and $0 < Z_1 - Z_2 < M$, with:

$$M \equiv 2Z_3 \frac{y}{1-y} \left(\sqrt{1 + \frac{Z_1}{Z_3} \frac{1-y}{y}} - 1 \right) \tag{72}$$

The inequality $\theta_1 < \eta_2$ is equivalent to $Z_2 > 0$, while the assumption $\gamma < 1 - \eta_1$ implies $Z_3 > 0$. Since $\underline{\mu} > 1$, we have $\overline{\mu} > 1$, which means that:

$$(1-\eta_1) \frac{1-\eta_1}{1+\eta_1} < \gamma \tag{73}$$

According to $1 < \underline{\mu} < \mu$ and (73), $\mu < \overline{\mu}$ implies $0 < Z_1 - Z_2$, while $\theta_2 < \eta_2$ is equivalent to $Z_1 - Z_2 < M$. ■

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