International Dynamic Asset Allocation and the Effect of the Exchange Rate

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Abstract

This paper analyzes a stylized theoretical framework to examine optimal portfolio selection in an international context with an explicit focus on the effect of the exchange rate. More specifically, we study how the elimination of the exchange rate induces shifts in the optimal international portfolio. We show that the effect of the elimination of the exchange rate on the optimal portfolio is twofold. First, the volatility of the international portfolio changes (volatility effect of the exchange rate), and second, the national market prices of risk converge to common international market prices of risk (price effect of the exchange rate). This induces important shifts in the optimal international portfolio.

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1 Introduction

At the beginning of 1999 the euro was introduced creating the European Monetary Union (EMU). With the introduction, exchange rates, as separators of markets, disappeared. This was an important step in the ongoing process of economic and financial integration in the EMU member states. The current removal of financial, legal and information barriers implies that an EMU-wide economic and financial market is being created. Despite the large degree of integration, the goods markets are still segmented across countries. Business cycles and inflation rates in different countries are still highly dispersed, creating a wedge between the different economies (see Remsperger (2001)). Local economic factors might therefore still be an important driven force for an economy. This contrasts with the integration process in the financial markets. Money market rates across EMU members are equalized and bond market yields are converging. Furthermore, stock exchanges have set up initiatives to create a financial market with pan-European dimensions. The European financial market is thus, for a large extent, integrated (see Danthine et.al. (2000)). This also implies that the different national financial markets share a common pricing structure.

The increased integration and the elimination of exchange rate variability surely has an impact on the investor's optimal investment strategies. This paper analyzes a stylized theoretical framework to examine optimal portfolio selection in an international context with an explicit focus on the exchange rate. More specifically, we study the effect of the elimination of the exchange rate on the optimal international portfolio. The international portfolio problem is studied in a stochastic environment[1]. We define separately a home and a foreign stochastic environment by introducing independent predictor variables in the home and foreign asset returns. In each market (country) the state variable is the driving force for the asset prices as well as for inflation. National markets are incomplete. This modelling assumption implies that the pricing structure in each market includes both the home and the foreign risk factors, but each asset market only spans the national risk factor. Using standard international no-arbitrage arguments, the home and foreign investment environments are linked. This creates the international stochastic environment. The international market is complete. Both the home and foreign risk factors can be fully hedged in the international financial market.

We analyze the international portfolio allocation of an investor, consuming only in his own country. The investor maximizes utility over real final wealth, defined as nominal wealth deflated by national inflation[2]. We show that the international portfolio allocation

[1] A number of recent papers address the portfolio choice problem of a multi-period investor. Among others, Brennan et al. (1997), Kim and Omberg (1996), Wachter (2002), Campbell and Viceira (2002), and Lynch (2001) analyze the consequences of predictability in asset returns for financial decisions.

[2] Brennan and Xia (2001) also account for inflation risk in a (national) dynamic portfolio problem.

is composed of three components: the nominal myopic demand, the inflation demand (real myopic demand), and the intertemporal hedging demand. The exchange rate dynamics affect all of the three portfolio components in terms of differences in market prices of risk assigned by the home and the foreign market. We distinguish two different effects of the exchange rate. The first effect is the volatility effect: exchange rate volatility is added to the portfolio. The second effect is the price effect: the national market prices of risk converge to common international market prices of risk. This framework is then used to examine explicitly the impact of the elimination of the exchange rate on the portfolio decision.

The contribution of this paper concerns the inclusion of an international investment dimension in the portfolio choice problem of a multi-period investor. Ang and Bekaert (1999) also solve the dynamic international portfolio choice, but their main focus is on the diversification benefits in a market characterized by time-varying correlations and volatilities. They do not look explicitly at the portfolio shifts induced by the elimination of the exchange rate.

The organization of the paper is as follows. In Section 2, we present the set-up of the international stochastic environment. Section 3 defines the intertemporal portfolio choice and solves the dynamic problem using the Bellman principle of optimality. An economic interpretation and the impact of the exchange rate is presented in Section 4 and in Section 5 we perform a calibration exercise. Finally, Section 6 concludes.

2 The International Stochastic Environment

We derive the optimal portfolio allocation for a two-country model, denoted as home and foreign.[3] The state of each market (country) is described by an independent state variable. Assume that these state variables are perfectly captured by a two-dimensional vector of predictor variables $X(t) = \begin{bmatrix} X_1(t), & X_2(t) \end{bmatrix}'$. This vector of predictor variables (state variables) is driven by an Ornstein-Uhlenbeck process:

$$dX(t) = (k - KX(t))\,dt + \sigma_X dZ(t) \tag{1}$$

where k, K and σ_X are positive constants and K and σ_X are diagonal, implying independence of the processes X_1 and X_2; $Z(t) = \begin{bmatrix} Z_1(t), & Z_2(t) \end{bmatrix}'$ is a two-dimensional vector of independent Brownian motions. In this set-up, $Z_1(t)$ is the risk associated with the home state variable X_1 and Z_2 the risk associated with the foreign state variable X_2. For reasons that become clear later, the state variables are the dividend price ratios.

In each market, the investor can trade two assets, a risk-free asset and a risky asset. The

[3] Foreign variables are denoted with a superscript *.

return of the risk-free assets, $r(t)$ and $r^*(t)$, is an affine function of the state variables:

$$r(t)\,dt \equiv (\delta_{0,r} + \delta_{1,r} X_1(t))\,dt \tag{2}$$

$$r^*(t)\,dt \equiv \left(\delta^*_{0,r} + \delta^*_{1,r} X_2(t)\right) dt. \tag{3}$$

The home and foreign risk-free assets are driven by the home and foreign state variable, respectively. Define the home risky asset price as $P(t)$ and the foreign risky asset price as $P^*(t)$. The risky assets follow the diffusion process:

$$\frac{dP(t)}{P(t)} = (\delta_{0,P} + \delta_{1,P} X_1(t))\,dt - \sigma dZ(t) \tag{4}$$

$$\frac{dP^*(t)}{P^*(t)} = \left(\delta^*_{0,P} + \delta^*_{1,P} X_2(t)\right) dt - \sigma^* dZ(t) \tag{5}$$

where $\sigma = \left[\begin{array}{cc} \sigma_1, & 0 \end{array}\right]'$ and $\sigma^* = \left[\begin{array}{cc} 0, & \sigma^*_2 \end{array}\right]'$ are the positive volatilities of the risky assets. The home risky asset is determined by the home state variable and the foreign risky asset is determined by the foreign state variable. Given the independence of the two state variables, the home and foreign risky assets are then completely independent. Moreover, note that the home and foreign assets are perfectly negatively correlated with the home and foreign state variable, repectively. This perfect negative correlation is realistic. Empirical evidence suggests that asset returns exhibit mean reversion, and that asset returns are predictable by the dividend price ratio.[4] Perfect negative correlation between the shocks to the asset returns and the shocks to the expected excess returns then implies that stock returns are mean reverting. (see Wachter (2002), Campbell and Viceira (1999)).

The above set-up implies that in each market a single state variable, or single risk factor, is present in each risky asset. However, we assume that both state variables are priced in both the home and the foreign market. No-arbitrage assumptions now imply the following dynamics for the stochastic discount factors[5]:

$$\frac{dM(t)}{M(t)} = -r dt - \Lambda(X(t))\,dZ \tag{6}$$

$$\frac{dM^*(t)}{M^*(t)} = -r^* dt - \Lambda^*(X(t))\,dZ \tag{7}$$

with $\Lambda(X(t))$ and $\Lambda^*(X(t))$ the market prices of risk. The market prices of risk in both

[4] Empirical evidence of mean reversion in asset returns is given by Poterba and Summers (1988). Campbell and Shiller (1988), and Fama and French (1989) show that the dividend yield predicts excess returns.

[5] Campbell and Viceira (2002) give a simple derivation of the stochastic discount factor in continuous time.

countries price both the home and the foreign risk factor:

$$\Lambda(X(t)) = \left[\ \Lambda_1(X(t)),\quad \Lambda_2(X(t))\ \right] \tag{8}$$

$$\Lambda^*(X(t)) = \left[\ \Lambda_1^*(X(t)),\quad \Lambda_2^*(X(t))\ \right] \tag{9}$$

This modelling of the stochastic discount factors implies complete international markets, while the national markets are incomplete.

The home and the foreign stochastic environment taken together create the international stochastic environment. A home investor investing in an international environment is only interested in home currency payoffs. By means of the exchange rate he converts the foreign currency payoffs to home currency units. The assumption of international market integration determines the exchange rate[6]. Denote $S(t)$ the unit price of foreign currency in terms of domestic currency, international no-arbitrage implies:

$$E_t\left[\frac{M^*(t+\tau)}{M^*(t)}\frac{P^*(t+\tau)}{P^*(t)}\right] = E_t\left[\frac{M(t+\tau)}{M(t)}\frac{P^*(t+\tau)}{P^*(t)}\frac{S(t+\tau)}{S(t)}\right] \tag{10}$$

which is certainly satisfied for:

$$\frac{M^*(t+\tau)}{M^*(t)} = \frac{M(t+\tau)}{M(t)}\frac{S(t+\tau)}{S(t)} \tag{11}$$

The exchange rate thus ties the home and foreign stochastic discount factors. Applying Ito's lemma to $S(t) = \frac{M^*(t)}{M(t)}$ yields:

$$\frac{dS(t)}{S(t)} = \frac{dM^*(t)}{M^*(t)} - \frac{dM(t)}{M(t)} + \left(\frac{dM(t)}{M(t)} - \frac{dM^*(t)}{M^*(t)}\right)\left(\frac{dM(t)}{M(t)}\right) \tag{12}$$

Using (6) and (7) the dynamics of the exchange rate return can be rewritten as:

$$\frac{dS(t)}{S(t)} = (r(t) - r^*(t))\,dt + \sigma_S\Lambda'(X(t))\,dt - \sigma_S dZ \tag{13}$$

with

$$\begin{aligned} \sigma_S &= \left[\ \sigma_{S,1},\quad \sigma_{S,2}\ \right] \\ &= \left[\ \Lambda_1(X(t)) - \Lambda_1^*(X(t)),\quad \Lambda_2(X(t)) - \Lambda_2^*(X(t))\ \right] \end{aligned} \tag{14}$$

The exchange rate is fully determined by the home and foreign risk factors.

A home investor investing in the foreign risky asset passes through the exchange market. A cross-border investment is represented as $V(t) \equiv S(t)\,P^*(t)$. By Ito's lemma and using

[6] Among others, Brandt et al. (2001) and Backus et al. (2001) exploit this no-arbitrage equation to relate the exchange rate dynamics with the stochastic discount factor dynamics.

(5) and (13) yields:

$$\frac{dV(t)}{V(t)} = \frac{dP^*(t)}{P^*(t)} + \frac{dS(t)}{S(t)} + \frac{dS(t)}{S(t)}\frac{dP^*(t)}{P^*(t)} \tag{15}$$

$$\frac{dV(t)}{V(t)} = r(t)\,dt + (\sigma_S + \sigma^*)\,\Lambda'(X(t))\,dt - (\sigma_S + \sigma^*)\,dZ \tag{16}$$

$$\frac{dV(t)}{V(t)} = E_t\left(\frac{dV(t)}{V(t)}\right) - \sigma_V(X(t))\,dZ \tag{17}$$

The return of the foreign asset, converted to home currency units, equals the return on the foreign asset plus the return on the exchange rate plus the covariance between the exchange rate return and the foreign asset return (see (15)). Equation (16) shows that the expected return on a foreign asset, converted to home currency units, equals the home risk-free rate augmented with a risk-premium. The risk-premium is defined in a standard way as the amount of risk $(\sigma_S + \sigma^*)$ multiplied by the (home) price of risk $(\Lambda(X(t)))$. This shows the effect of the exchange rate as a converter of market prices of risk. That is, a home investor always prices based on his home market prices of risk, $\Lambda(X(t))$, independent of whether it concerns a home or a foreign investment opportunity. The above dynamics also show that investing in the foreign asset, an investor is able to hedge against the foreign risk. The crucial underlying assumption is that the foreign risk is priced in the home market. If the foreign risk factor is not priced in the home market, the home investor has no incentive to hold the foreign converted asset. He is not rewarded for taking the foreign risk, while he is faced with increased volatility.

The key property of an attractive international investment environment is the investor's ability to diversify away the home as well as the foreign risks. This implies non-zero market prices of risk. As is well understood, the market prices of risk are determined as the Sharpe ratio of the risky asset:

$$\Lambda_1(X(t)) = \frac{\delta_{0,P} + \delta_{1,P}X_1(t) - r(t)}{\sigma_1} \tag{18}$$

$$\Lambda_2^*(X(t)) = \frac{\delta_{0,P}^* + \delta_{1,P}^*X_2(t) - r^*(t)}{\sigma^*} \tag{19}$$

However, due to the modelling assumption that the national markets are incomplete, $\Lambda_2(X(t))$ and $\Lambda_1^*(X(t))$ cannot be derived from the observable security prices. International no-arbitrage conditions require that the exchange rate volatilities, $\sigma_{S,1}$ and $\sigma_{S,2}$, and the observable market prices of risk, $\Lambda_1(X(t))$ and $\Lambda_2^*(X(t))$, uniquely determine the market prices of risk that are unobservable from the national asset markets (see (14)):

$$\Lambda_1^*(X(t)) = \Lambda_1(X(t)) - \sigma_{S,1} \tag{20}$$

$$\Lambda_2(X(t)) = \Lambda_2^*(X(t)) + \sigma_{S,2} \tag{21}$$

Volatility of the exchange rate thus allows market prices of risk to differ across countries. It immediately follows that zero exchange rate volatility equates market prices of risk across countries. This determines the international stochastic environment.

3 Intertemporal Portfolio Choice

In this section we determine the investor's dynamic portfolio problem. Following Merton (1990), we assume no transaction costs, no short-sale constraints and the investor is a price-taker. We take the viewpoint of a home investor who has the choice of investing in three assets: (1) the home risky asset, (2) the foreign risky asset (converted to home currency units), and (3) the home risk-free bond. He can invest a fraction $\alpha(t)$ in the home risky asset, a fraction $\alpha^*(t)$ in the converted foreign risky asset, and a fraction $(1-\alpha(t)-\alpha^*(t))$ in the home risk-free bond.

We solve the portfolio choice problem for an investor with power utility over real wealth at some finite horizon T. To derive the budget constraint, denote $W(t)$ the nominal wealth and $\overline{P}(t)$ the price level in the home economy. The dynamics of the nominal budget constraint and inflation are given by:

$$\frac{dW(t)}{W(t)} = \left\{A(t)'(\mu(X(t)) - r(t)\iota) + r(t)\right\}dt - A(t)'\Sigma dZ(t) \tag{22}$$

$$\frac{d\overline{P}(t)}{\overline{P}(t)} = \pi(X(t))\,dt + \sigma_{\overline{P}}dZ(t) \tag{23}$$

with $A(t) = \begin{bmatrix} \alpha(t), & \alpha^*(t) \end{bmatrix}'$ the risky portfolio weights, $\Sigma = \begin{bmatrix} \sigma & 0 \\ \sigma_{S,1} & \sigma^* + \sigma_{S,2} \end{bmatrix}$ the portfolio's volatility, $\mu(X(t))\,dt = \begin{bmatrix} E_t\left(\frac{dP(t)}{P(t)}\right), & E_t\left(\frac{dV(t)}{V(t)}\right) \end{bmatrix}'$ the mean portfolio return and ι a two-dimensional unit vector. Expected inflation is denoted by $\pi(X(t))$ and we assume that it is an affine function of the state variables:

$$\pi(X(t)) = \delta_{0,\pi} + \delta_{1,\pi}X(t) \tag{24}$$

Moreover, the instantaneous unexpected inflation $\sigma_{\overline{P}}dZ(t)$ is perfectly correlated with the home risk factor and uncorrelated with the foreign risk factor. Formally, this implies $\sigma_{\overline{P}} = \begin{bmatrix} \sigma_{\overline{P},1}, & 0 \end{bmatrix}$. Using the nominal wealth dynamics (22) and the inflation dynamics (23), real wealth $Y(t) = \frac{W(t)}{\overline{P}(t)}$ can be derived by Ito's lemma:

$$dY(t) = Y(t)\left[\frac{dW(t)}{W(t)} - \frac{d\overline{P}(t)}{\overline{P}(t)} + \frac{d\overline{P}(t)}{\overline{P}(t)}\left(\frac{d\overline{P}(t)}{\overline{P}(t)} - \frac{dW(t)}{W(t)}\right)\right] \tag{25}$$

$$\begin{aligned} dY(t) &= Y(t)\left\{A(t)'(\mu(X(t)) - r(t)\iota) + r(t) - \pi(X(t)) + \sigma_{\overline{P}}\sigma_{\overline{P}}' + A(t)'\Sigma\sigma_{\overline{P}}'\right\}dt \\ &\quad -Y(t)\left(A(t)'\Sigma + \sigma_{\overline{P}}\right)dZ(t) \end{aligned} \tag{26}$$

We assume that the nationality of an investor is determined by the country he lives in, and thus where he consumes his wealth. This imlies that the investor takes into account the national inflation rate when maximizing his wealth. The investor solves:

$$\begin{aligned} &\max_{A(t)} E\left[U(Y(T))\right] \\ &s.t. \quad \text{equation (26)} \end{aligned}$$

where $U(Y(T)) = \frac{1}{1-\gamma}(Y(T))^{1-\gamma}$ is defined as power utility over real wealth, with γ the coefficient of relative risk aversion[7].

We use stochastic control to solve this problem[8]. Define $J(t, Y, X)$ as the maximized utility function (or value function). The Bellman principle of optimality implies the following Hamilton-Jacobi-Bellman equation (HJB herein)[9]:

$$0 = \max_{A(t)}\left\{\mathcal{D}J(t, Y, X)\right\} \tag{27}$$

where $\mathcal{D}$ is the Dynkin operator, with $\mathcal{D}J(t, Y, X) = \frac{1}{dt}E_t\left[dJ(t, Y, X)\right]$. The value function is the utility obtained by the investor if he has followed the optimal portfolio policies. That is, the investor has allocated wealth optimally among the available assets, to achieve the highest real wealth at the end of the investment horizon T. Standard dynamic programming conditions result in the optimal portfolio allocation $A(t)^{opt}$:

$$\begin{aligned} A(t)^{opt} &= \frac{-J_Y}{Y(t)J_{YY}}(\Sigma\Sigma')^{-1}(\mu(X,t) - r(t)\iota) - \frac{J_Y}{Y(t)J_{YY}}(\Sigma\Sigma')^{-1}\Sigma\sigma_{\overline{P}}' \\ &\quad - (\Sigma\Sigma')^{-1}\Sigma\sigma_{\overline{P}}' + \frac{1}{Y(t)J_{YY}}(\Sigma\Sigma')^{-1}\Sigma\sigma_X' J_{XY}, \end{aligned} \tag{28}$$

[7] We focus on the case $\gamma > 1$. Kim and Omberg (1996) discuss the solutions for γ unrestricted.

[8] Kamien and Schwartz (1995) give a clear treatment of stochastic optimal control.

[9] For the complete derivation of the dynamic programming solution, see Appendix A.

and the partial differential equation (PDE):

$$\begin{aligned}0 \quad = \quad & J_t + J_X' \left(k - KX(t)\right) + J_Y Y(t) \left(r(t) - \pi(X(t))\right) \\ & -\frac{1}{2} J_Y \left(\mu(X(t)) - r(t)\iota\right)' \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X(t)) - r(t)\iota\right) \frac{J_Y}{J_{YY}} \\ & -J_Y \sigma_{\overline{P}} \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X,t) - r(t)\iota\right) \frac{J_Y}{J_{YY}} \\ & -J_Y Y(t) \sigma_{\overline{P}} \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X(t)) - r(t)\iota\right) \\ & +J_{XY}' \sigma_X \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X(t)) - r(t)\iota\right) \frac{J_Y}{J_{YY}} - \frac{1}{2} J_Y \sigma_{\overline{P}} \sigma_{\overline{P}}' \frac{J_Y}{J_{YY}} \\ & -\frac{1}{2} J_{XY}' \sigma_X \sigma_X' \frac{J_{XY}}{J_{YY}} + \frac{1}{2} Tr\left(J_{XX'} \sigma_X \sigma_X'\right) + J_{XY}' \sigma_X \sigma_{\overline{P}}' \frac{J_Y}{J_{YY}},\end{aligned} \tag{29}$$

with terminal condition:

$$J(T, Y, X) = \frac{1}{1-\gamma} \left(Y(T)\right)^{1-\gamma}. \tag{30}$$

To solve for the optimal investment allocation, the PDE (29) needs to be solved. Assume a trial solution of the form:

$$J(t, X, Y) = \frac{Y^{1-\gamma}}{1-\gamma} \Phi\left(X(t,T)\right) \tag{31}$$

$$\Phi\left(X(t,T)\right) = exp\left(\mathcal{A}(t,T) + \mathcal{B}(t,T) X(t) + \frac{1}{2} X(t)' \mathcal{C}(t,T) X(t)\right) \tag{32}$$

$$\mathcal{A}(T) = \mathcal{B}(T) = \mathcal{C}(T) = 0 \tag{33}$$

Substitution of the trial solutions (31), (32) and (33) in the optimal investment allocation (28) yields:

$$\begin{aligned}A(t)^{opt} \quad = \quad & \frac{1}{\gamma} \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X(t)) - r(t)\iota\right) + \left(\frac{1-\gamma}{\gamma}\right) \left(\Sigma\Sigma'\right)^{-1} \Sigma \sigma_{\overline{P}}' \\ & -\frac{1}{\gamma} \left(\Sigma\Sigma'\right)^{-1} \Sigma \sigma_X' \left(\mathcal{B}(t,T)' + \mathcal{C}(t,T) X(t)\right)\end{aligned} \tag{34}$$

where $\mathcal{A}(t,T)$, $\mathcal{B}(t,T)$ and $\mathcal{C}(t,T)$ are the solutions to a system of ordinary differential equations (ODE's). This system of ODE's results from substituting the trial solution in the PDE (29). The system of ODE's can be solved numerically with the Runge Kutta simulation technique[10].

The optimal portfolio allocation consists of three components: (1) the nominal mean-variance component, (2) the inflation component (real mean-variance component), and (3) the intertemporal hedging component. The first two portfolio components are the so-called myopic demand. The first component is the standard nominal mean-variance portfolio weight. This is the portfolio that achieves the maximum Sharpe ratio. The higher the coefficient of relative risk aversion, the lower the asset demand induced by mean-variance trade-off. The second component is the inflation portfolio (real mean-variance component),

[10] An analytical solution can be found for the case where no cross-terms of the state variable $X(t)$ appear. The more general set-up has to be solved by simulation. See Judd (1998).

which adjusts for the fact that the mean-variance portfolio is in nominal rather than real terms. $(\Sigma\Sigma')^{-1}\Sigma\sigma_P'$ selects the portfolio that has maximum correlation with inflation. Given the negative correlation between inflation and asset returns, inflation decreases contemporaneous asset returns and wealth. This induces a negative inflation demand. The higher γ, the larger the negative inflation demand. To summarize, the two mean-variance components constitute the optimal allocation when the investor ignores changes about the future investment set.

The third component is the intertemporal hedging portfolio. $(\Sigma\Sigma')^{-1}\Sigma\sigma_X'$ selects the portfolio that has maximum correlation with the state variables and $\left(\mathcal{B}(t)' + \mathcal{C}(t)X(t)\right)$ measures the sensitivity of wealth towards changes in the stochastic environment.[11] Based on the dynamics of the asset returns and the negative correlation between the shocks to the asset returns and the shocks to the state variables, an increase in the state variables decreases current asset returns, but increases future wealth, and thus a positive amount is allocated to the hedge portfolio. Put differently, when investment opportunities are poor, the risky assets pay off well. This provides a good hedge, and thus the investor increases his risky portfolio holdings relative to the myopic asset demand. This is a well-known result that investors want more wealth when marginal utility is higher. Finally, a more risk averse investor allocates less to the hedging component, reducing the total risky asset holdings.

The result in (34) is similar to the standard optimal dynamic portfolio weight. The exchange rate enters through the portfolio's mean and volatility and through the sensitivity of wealth towards changes in the stochastic environment. This effect is discussed in a next section, where we isolate the exchange rate effect by comparing the optimal portfolio in the presence of exchange rate to that in the absence of exchange rate variability.

4 Exchange Rate Effects on the Optimal Portfolio

Having defined the optimal portfolio, we analyze the impact of the exchange rate on the allocation. The exchange rate effect is isolated, comparing the optimal allocation with and without exchange rates. This sheds light on how the portfolio allocation changes by the removal of exchange rate variability. We start by deriving the market prices of risk and the foreign asset return in the absence of exchange rates. For ease of notation, denote the variables in the absence of exchange rate variability with a tilde. Elimination of exchange rate variability implies that the rate of return of the exchange rate becomes zero:

$$\frac{d\widetilde{S}(t)}{\widetilde{S}(t)} = 0 \tag{35}$$

[11] This term measures the sensitivity of the logarithm of marginal utility of wealth to a stochastic opportunity set.

For the exchange rate dynamics in (13), this implies that the drift and the diffusion of the exchange rate return should be zero:

$$0 = \left(\widetilde{r}(t) - \widetilde{r}^*(t)\right) + \sigma_S \widetilde{\Lambda}'(X(t)) \tag{36}$$

$$0 = \sigma_S \tag{37}$$

Using the no-arbitrage relation $\sigma_S = \widetilde{\Lambda}(X(t)) - \widetilde{\Lambda}^*(X(t))$, this can be further simplified to:

$$\widetilde{r}(t) = \widetilde{r}^*(t) \tag{38}$$

$$\widetilde{\Lambda}(X(t)) = \widetilde{\Lambda}^*(X(t)) \tag{39}$$

In the case of (irrevocably) fixed exchange rates, the risk-free rates of return and the market prices of risk of the home and the foreign country converge to each other.

Starting from the dynamics of the foreign asset return, converted to home currency units (16), and using the above restrictions on the risk-free rates of return (38) and on the market prices of risk (39), we derive the return of the foreign asset when exchange rate variability is eliminated:

$$\frac{d\widetilde{V}(t)}{\widetilde{V}(t)} = \left(\delta_0^* + \delta_1^* X_2(t)\right) dt - \sigma^* dZ(t) = \frac{dP^*(t)}{P^*(t)} \tag{40}$$

In the absence of exchange rates the price of the risky foreign asset should be the same in both countries. This clearly shows that elimination of exchange rate variability, together with standard no-arbitrage assumptions on an international level, imply that assets are priced as if priced in a single market.

In the remainder of this section, the optimal portfolio allocation for two different scenarios are derived. A first scenario analyzes the optimal portfolio when exchange rate variability is eliminated. A second scenario derives the portfolio allocation in the presence of exchange rates. These two scenarios then allow us to assess the effect of the exchange rate.

Scenario 1: No exchange rate variability In the case of fixed exchange rates, the portfolio weight for the home asset is not affected by the portfolio weight of the foreign asset and vice versa. That is, when there are two independent classes of assets and two independent classes of risks (rows of σ_X are independent), the portfolio weights $\widetilde{\alpha}(t)$ and $\widetilde{\alpha}^*(t)$ are independent. Formally, when the exchange rate is eliminated the volatlity of the portfolio, Σ, becomes diagonal, and market prices of risks across countries converge:

$$\widetilde{\Sigma} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2^* \end{bmatrix} \tag{41}$$

$$\widetilde{\Lambda}(X(t)) = \begin{bmatrix} \widetilde{\Lambda}_1(X(t)), & \widetilde{\Lambda}_2(X(t)) \end{bmatrix} \tag{42}$$

with:

$$\widetilde{\Lambda}_1(X(t)) = \widetilde{\Lambda}_1^*(X(t)) \tag{43}$$

$$\widetilde{\Lambda}_2(X(t)) = \widetilde{\Lambda}_2^*(X(t)) \tag{44}$$

and the optimal portfolio weights reduce to:

$$\begin{aligned}\widetilde{\alpha}(t) = & \left(\frac{1}{\gamma}\right)\left[\frac{\widetilde{\Lambda}_1(X(t))}{\sigma_1}\right] + \left(\frac{\gamma-1}{\gamma}\right)\left[\frac{\sigma_{\overline{P}}}{\sigma_1}\right] \\ & - \left(\frac{1}{\gamma}\right)\left[\frac{\sigma_{X_1}}{\sigma_1}\left(\mathcal{B}_1(t,T) + \mathcal{C}_1(t,T)X_1(t)\right)\right]\end{aligned} \tag{45}$$

$$\widetilde{\alpha}^*(t) = \left(\frac{1}{\gamma}\right)\left[\frac{\widetilde{\Lambda}_2^*(X(t))}{\sigma_2^*}\right] - \left(\frac{1}{\gamma}\right)\left[\frac{\sigma_{X_2}}{\sigma_2^*}\left(\mathcal{B}_2(t,T) + \mathcal{C}_2(t,T)X_2(t)\right)\right] \tag{46}$$

with $\widetilde{\Lambda}_1(X(t))$ the market price of risk of the home risk factor in the home country, and $\widetilde{\Lambda}_2^*(X(t))$ the market price of risk of the foreign risk factor in the foreign country. Equations (45) and (46) present the standard myopic, inflation and hedging demand for two independent classes of assets spanning the complete set of risk factors. The portfolio weights of the home and the foreign risky asset are completely independent from each other. The decision to invest in the home risky asset is taken separately from the decision to invest in the foreign risky asset. If, for example, the investor only has the choice to invest in the home asset or in the foreign asset, the portfolio weights would be exactly the same. The optimal portfolio weights also show the diversification effect of investing internationally. An international portfolio allows the investor to hedge all the risk that is priced in his national market. By investing in the home asset, the investor hedges X_1 risk and by investing in the foreign asset, the investor hedges X_2 risk.

Scenario 2: Exchange rate variability In the presence of exchange rate variability, the volatility of the portfolio, Σ, is no longer diagonal, but lower triangular, and market prices of risk can differ across countries:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_{S,1} & \sigma_2^* + \sigma_{S,2} \end{bmatrix} \tag{47}$$

$$\Lambda(X(t)) = \begin{bmatrix} \Lambda_1(X(t)), & \Lambda_2(X(t)) \end{bmatrix} \tag{48}$$

with:

$$\Lambda_1(X(t)) = \Lambda_1^*(X(t)) + \sigma_{S,1} \quad (49)$$

$$\Lambda_2(X(t)) = \Lambda_2^*(X(t)) + \sigma_{S,2} \quad (50)$$

The home asset and the foreign asset converted to home currency units are no longer independent. The form of Σ and $\Lambda(X(t))$ means that the characteristics of the portfolio changed. Two effects play a role: (1) there is the additional exchange rate volatility, $\sigma_{S,1}$ and $\sigma_{S,2}$, (the volatility effect) and (2) the market prices of risk changed and are different across countries (the price effect). It will become clear that the corresponding portfolio shifts depend on the size of the (differences in) market prices of risk.

In this setting, the optimal portfolio weights reduce to:

$$\alpha(t) = \widetilde{\alpha}(t) - \frac{\sigma_{S,1}}{\sigma_1}\alpha^*(t) + \frac{1}{\gamma}\frac{\Delta\Lambda_1}{\sigma_1} - \frac{1}{\gamma}\frac{\sigma_{X_1}}{\sigma_1}(\Delta\mathcal{B}_1 + (\Delta\mathcal{C}_1)X_2) \quad (51)$$

$$\alpha^*(t) = \widetilde{\alpha}^*(t) - \left(\frac{\sigma_{S,2}}{\sigma_2^* + \sigma_{S,2}}\right)\widetilde{\alpha}^*(t) + \frac{1}{\gamma}\left(\frac{\Delta\Lambda_2}{\sigma_2^* + \sigma_{S,2}}\right) - \frac{1}{\gamma}\left(\frac{\sigma_{X_2}}{\sigma_2^* + \sigma_{S,2}}\right)(\Delta\mathcal{B}_2 + (\Delta\mathcal{C}_2)X_2) \quad (52)$$

with $\Delta\Lambda_i = \Lambda_i(X(t)) - \widetilde{\Lambda}_i(X(t))$, $\Delta\mathcal{B}_i = \mathcal{B}_i(t,T) - \widetilde{\mathcal{B}}_i(t,T)$ and $\Delta\mathcal{C}_i = \mathcal{C}_i(t,T) - \widetilde{\mathcal{C}}_i(t,T)$ the change in the market prices of risk and the sensitivity parameters between the scenario with and without exchange rate variability, respectively. Equation (51) expresses the optimal portfolio holding of the home risky asset as the sum of four components. The first component equals the allocation to the home risky asset in the absence of exchange rate variability. The second component represents the shift induced by the change in the loading to X_1 risk of the portfolio, given equal pricing characteristics. In the presence of exchange rate variability, the total portfolio volatility due to X_1 risk changed by $\sigma_{S,1}$. If the total loading on X_1 risk of the portfolio in the presence of exchange rate variability is higher compared to the loading in the absence of exchange rate variability, $\sigma_1 + \sigma_{S,1} > \sigma_1$, then, the investor holds less of the home asset. To achieve the optimal loading on X_1 risk, these additional risks are hedged away by the home asset holding. The opposite effect is true when $\sigma_1 + \sigma_{S,1} < \sigma_1$. In this case, the portfolio loads less on X_1 risk, and thus the investor holds more of the home risky asset. The exchange rate acts as an insurance. Put differently, when $\sigma_{S,1} < 0$, the exchange rate and the home asset have an opposite sensitivity to the risk factor X_1, and the investor increases his holdings of the home risky asset to obtain an optimal risk profile. This effect is called the volatility effect of the exchange rate.

The third and the fourth components arise from a different cause, namely from the change in the market prices of risk. Put differently, the market price of X_1 risk that the investor faces, Λ_1, is not identical to the market price of risk he faces in the absence of exchange rate variability, $\widetilde{\Lambda}_1$. This change in the pricing characteristics alters the investment opportunities. The new risk-return relationship induces the investor to adjust his optimal home asset holdings. The third component represents the portfolio shift induced by myopic demand considerations. When the market price of risk in the presence of exchange rate variability is higher compared to the case without exchange rate variability, the investor increases his myopic demand. This is reasonable, since the more the investor is rewarded for taking a certain amount of risk, the larger the amount of risky assets the investor wants to hold. Alternatively, when the investor is rewarded less in the presence of exchange rate variability, the investor reduces his myopic demand. Finally, the fourth component is the portfolio shift induced by hedging demand considerations. This change in the hedging demand origins from a change in the sensitivity of wealth of the investor towards changes in the investment environment. If we recall that the system of ODE's (see Appendix A) depends on the market prices of risk, the sensitivity of wealth of the investor towards changes in the investment environment alters. For example, when the sensitivity of the investors in the presence of exchange rates is larger compared to the case with no exchange rate variability, the investor wants to hedge more. Consequently, the hedging demand increases. However, when the presence of exchange rate variability makes the investors' wealth less sensitive to changes in the investment set, he decreases the hedging demand. This third and fourth components are the so-called price effects of the exchange rate. To summarize, the effect of the exchange rate on the optimal home asset holdings is thus twofold. First, it alters the total loading of the portfolio on X_1 risk by $\sigma_{S,1}$ and second, it changes the investment opportunities by altering the market price of risk from $\widetilde{\Lambda}_1$ to Λ_1.

Equation (52) expresses the optimal portfolio holding of the foreign risky asset. The shifts in the foreign portfolio holdings are similar to the shifts in the home portfolio holdings. We see four different components. The first component is the allocation to the risky foreign asset in the absence of exchange rate variability. The second component is the volatility effect of the exchange rate. It represents the shift in the portfolio induced by the altered loading on X_2 risk. In the presence of exchange rate variability, the portfolio loads an additional $\sigma_{S,2}$ on X_2 risk. When the total portfolio loading on X_2 risk is increased (decreased) the investor decreases (increases) his holdings of the foreign risky asset. Finally, the third and the fourth components are the price effects of the exchange rate. They arise due to a change in the market price of X_2 risk, Λ_2, whereby the investment opportunities change. The third component changes the optimal myopic demand and the fourth component changes the optimal hedging demand.

The removal of exchange rate variability thus induces shifts in the optimal allocation of both the home and the foreign risky asset. These shifts origin from two effects. First, the portfolio loadings change due to additional loadings on X_1 risk and X_2 risk of the exchange rate. International no-arbitrage conditions imply that these exchange rate risk loadings are in fact determined as the differential in the home and foreign market prices of risk. Secondly, the market prices of risk the investor faces have changed. This creates new investment opportunities. The importance of the portfolio shifts thus crucially depend on the (differentials in) market prices of risk. When market prices of risks are (almost) equal in both the home and foreign country, the effect of the exchange rate is negligible and the optimal portfolio holdings reduce to the optimal portfolio holdings in a fixed exchange rate regime.

5 Calibration Exercise

To provide illustrative calculations of the exchange rate effects on the optimal international portfolio, we calibrate the above international portfolio model. Table 1 reports the parameter estimates, which we assume identical for the home and foreign dynamics.

Table 1: **Model Parameters**

Parameter	Estimate
constant state variable, k	7.3E-05
mean reversion parameter, K	0.0194
volatility of state variables, σ_X	0.0002
constant asset return, $\delta_{0,P}$	0.0031
time-varying asset return, $\delta_{1,P}$	1.2061
volatility of asset returns, σ	0.0581
constant risk-free rate, $\delta_{0,r}$	0.0027
time-varying risk-free rate, $\delta_{1,r}$	0.4648
constant inflation, $\delta_{0,\pi}$	-5.4E-05
time-varying inflation, $\delta_{1,\pi}$	0.0201
volatility of inflation, $\sigma_{\overline{P}}$	8.3E-05
Sharpe ratio, $\Lambda_1(X(t))$, $\Lambda_2^*(X(t))$	0.0597

All estimates are in monthly units. Parameters are based on estimations for Germany, using Morgan Stanley Capital International total market indices (net dividends included) and the one-month LIBOR rates. The state variable is the dividend yield. Price level data are used to calculate monthly inflation rates. The correlation between the state variable and the asset returns is -0.83. In the portfolio optimization, this is set to -1.

We estimate the optimal portfolio holdings for a home and foreign dividend yield of 0.4%. We take a coefficient of risk aversion of 2, 6 and 10 with a fixed investment horizon of 10 years. Furthermore, we assume a risk loading of the exchange rate ranging from -0.02 up to 0.02.[12] For simplicity, we assume that the market prices of risk as determined by the Sharpe ratios, $\Lambda_1(X(t))$ and $\Lambda_2^*(X(t))$ are left unchanged in the cases with and without exchange rate variability. With this assumption, a negative exchange rate risk sensitivity, $\sigma_{S,2} < 0$, implies a market price of risk Λ_2 lower compared to the no exchange rate volatility market price of risk, $\widetilde{\Lambda}_2$. On the contrary, when the exchange rate volatility is postive, $\sigma_{S,2} > 0$, then the market price of risk Λ_2 is higher compared to the market price of risk in the absence of exchange rate voltility, $\widetilde{\Lambda}_2$. The portfolio shifts, thus, origin from a change in the home market price of risk of the foreign risk factor, $\Lambda_2(X(t))$, and from the elimination of exchange rate variability, $\sigma_{S,1}$ and $\sigma_{S,2}$. With these simplifying assumptions, the exchange rate effect in the home risky asset (51) reduces to $ER = -\frac{\sigma_{S,1}}{\sigma_1}\alpha^*(t)$. This is the volatility effect. For the foreign risky asset, (52) shows that the total exchange rate effect is given by $ER^* = -\left(\frac{\sigma_{S,2}}{\sigma_2^*+\sigma_{S,2}}\right)\widetilde{\alpha}^*(t) + \frac{1}{\gamma}\left(\frac{\Delta\Lambda_2}{\sigma_2^*+\sigma_{S,2}}\right) - \frac{1}{\gamma}\left(\frac{\sigma_{X_2}}{\sigma_2^*+\sigma_{S,2}}\right)(\Delta\mathcal{B}_2 + (\Delta\mathcal{C}_2)X_2)$. The first component is the volatility effect, and the next two components are the price effects of the exchange rate.

Table 2 summarizes the optimal investment strategies for different levels of exchange rate volatilities and coefficients of relative risk aversion. The investment horizon is fixed at $T = 10$ years. The portfolio weights add up to 100%. The middle column, under the exchange rate loading of zero, reports the optimal portfolio weights when exchange rate variability is eliminated. Given the symmetrical structure for the home and foreign stochatsic environment, the portfolio weight of the risky home asset and the risky foreign asset are almost identical. The small difference origins from the inflation demand, inducing the investor to hold less of the home risky asset.

The exchange rate component in the optimal international portfolio is significant, as high as 22%. Elimination of exchange rate variability, thus, induces large shifts in the optimal international portfolio. The exchange rate effect is decreasing over the coefficient of relative risk aversion. For exchange rate sensitivities $\sigma_{S,1}$ and $\sigma_{S,2}$ smaller (larger) than zero, the optimal portfolio holdings are larger (smaller) compared to the optimal portfolio holdings in the absence of exchange rate variability. This shows the insurance role of the exchange rate when the exchange rate sensitivities are negative (and thus have the opposite sign as the asset return sensitivities). For the home risky asset, this result is clear. Given the equal market price of risk $\Lambda_1(X(t))$ with and without exchange rate variability, there is only the volatility effect of the exchange rate. When the total portfolio volatility in the presence of exchange rate variability is lower (higher), the investor hedges this additional volatility

[12] This sensitivity of the exchange rate is reasonable; De Santis and Gerard (1998), and De Santis, Gerard and Hillion (1999) estimate exchange rate volatilities up to 0.03.

away by increasing (reducing) the optimal allocation to the home risky asset. For the foreign risky asset, the exchange rate has a double effect. When the total portfolio volatility in the presence of exchange rate variability is lower (higher), the investor hedges this additional volatility away by increasing (reducing) the optimal allocation to the foreign risky asset. Furthermore, when the exchange rate volatility is negative (positive), the market price of risk in the presence of exchange rate variability is lower (higher) compared to the market prices of risk in the absence of exchange rate variability. Faced with a lower (higher) market price of risk, the investor decreases (increases) his myopic and hedging demand. For the calibration reported in Table (2), we see that the volatility exchange rate effect dominates in the foreign allocation. That is, when the loading of the exchange rate is negative, $\sigma_{S,2} < 0$, the investor increases the total allocation to the foreign risky asset, while he decreases the total allocation to the foreign risky asset for a positive exchange rate volatility. Table (3) reinforces these results. This table reports the magnitude of the different exchange rate components.

Table 2: **Optimal Portfolio Strategy (%)**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
γ		-0.02	-0.002	-0.002	0	0.0002	0.002	0.02
		$\Delta\Lambda_2 < 0$				$\Delta\Lambda_2 > 0$		
2	$\alpha(t)$	83.55	63.24	61.34	61.13	60.92	59.03	40.72
	$\alpha^*(t)$	65.13	61.47	61.23	61.21	61.18	60.69	59.29
	$100 - \alpha(t) - \alpha^*(t)$	-48.68	-24.71	-22.57	-22.34	-22.10	-19.72	-0.01
6	$\alpha(t)$	38.94	28.70	27.82	27.73	27.63	26.78	18.93
	$\alpha^*(t)$	32.58	28.18	27.89	27.86	27.83	27.56	25.56
	$100 - \alpha(t) - \alpha^*(t)$	28.48	43.12	44.29	44.41	44.54	45.66	55.51
10	$\alpha(t)$	29.13	21.12	20.48	20.41	20.34	19.71	14.13
	$\alpha^*(t)$	25.33	20.88	20.58	20.55	20.52	20.25	18.22
	$100 - \alpha(t) - \alpha^*(t)$	45.54	58.00	58.94	59.04	59.14	60.04	67.65

This table reports the optimal strategies for an investor with different values of exchange rate volatility and risk aversion, for a fixed investment horizon of 10 years; $\alpha(t)$ and $\alpha^*(t)$ are the proportional allocation to the home and foreign risky assets, respectively; $100 - \alpha(t) - \alpha^*(t)$ is the proportion allocated to the riskless asset. The middle column reports the optimal allocation when exchange rate variability is eliminated.

Given the assumption that the market price of risk Λ_1 is unchanged by the elimination of exchange rate variability, the allocation to the home risky asset only has a single exchange rate component. A negative exchange rate volatility leads to a higher weight of the home risky asset. A positive exchange rate volatility leads to a lower home portfolio weight. The exchange rate component of the foreign asset allocation is more interesting. We clearly see two opposing effects of the exchange rate: when the volatility effect is positive ($ER^*_{\sigma_{S,2}} > 0$), the price effects are negative ($ER^*_{\Delta\Lambda_2,M} < 0; ER^*_{\Delta\Lambda_2,H} < 0$), and vice versa. In this example, the volatility effect dominates the price effect. An investor faced with a negative exchange rate loading, increases the holdings of the foreign risky asset. A positive exchange rate effect induces the investor to decrease the amount of foreign risky assets in his optimal portfolio. Finally, note that the price effect of the exchange rate induced by hedging considerations is small relative to the volatility effect and the price effect of the myopic demand. The volatility effect is thus mainly offset by the myopic demand price effect.

Table 3: **Exchange Rate Components (as % of total portfolio weights)**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$					
γ		-0.02	-0.002	-0.0002	0.0002	0.002	0.02
		$\Delta\Lambda_2 < 0$			$\Delta\Lambda_2 > 0$		
2	$ER_{\sigma_{S,1}}$	22.42	2.12	0.21	-0.21	-2.10	-20.41
	$ER^*_{\sigma_{S,2}}$	32.13	2.18	0.21	-0.21	-2.04	-15.67
	$ER^*_{\Delta\Lambda_2,M}$	-26.25	-1.78	-0.17	0.17	1.66	12.80
	$ER^*_{\Delta\Lambda_2,H}$	-1.96	-0.13	-0.01	0.01	0.12	0.96
6	$ER_{\sigma_{S,1}}$	11.21	0.97	0.10	-0.10	-0.95	-8.80
	$ER^*_{\sigma_{S,2}}$	14.62	0.99	0.10	-0.10	-0.93	-7.13
	$ER^*_{\Delta\Lambda_2,M}$	-8.75	-0.59	-0.06	0.06	0.55	4.27
	$ER^*_{\Delta\Lambda_2,H}$	-1.16	-0.08	-0.01	0.01	0.07	0.57
10	$ER_{\sigma_{S,1}}$	8.72	0.72	0.07	-0.07	-0.70	-6.27
	$ER^*_{\sigma_{S,2}}$	10.79	0.73	0.07	-0.07	-0.68	-5.26
	$ER^*_{\Delta\Lambda_2,M}$	-5.25	-0.36	-0.03	0.03	0.33	2.56
	$ER^*_{\Delta\Lambda_2,H}$	-0.76	-0.05	-0.01	0.00	0.05	0.37

The above calibration is repeated for the optimal investment policies for a fixed coefficient of relative risk aversion ($\gamma = 6$), and for different investment horizons. The results of this calibration are similar as the previous results presented above. They can be found in Appendix B.

To see the importance of the exchange rate on the optimal allocation in terms of wealth, we compare the certainty equivalent of wealth for the optimal portfolio in the presence of exchange rates to that of the optimal portfolio when exchange rate variability is eliminated[13]:

$$ERG = \frac{\widetilde{CE}}{CE}$$

This ratio of certainty equivalents reflects the effect of exchange rate variability on the investor's welfare. If the ratio is larger than one, the investor will be made worse off by the elimination of exchange rate variability. Adversely, if the ratio is smaller than one, the investor will be made better off by the elimination of exchange rate variability. Table 4 reports these results. The variable ERG is the ratio of the certainty equivalent of the

[13] The certainty equivalent is the amount of wealth such that the investor is indifferent between receiving it for sure at the horizon, and having his current wealth today invested optimally up to the horizon.

oprimal portfolio in the presence of exchange rate variability to the certainty equivalent of the optimal portfolio when exchange rate variability is eliminated.

Table 4: **Ratio of Certainty Equivalents of Wealth**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
γ	*ERG*	-0.02	-0.002	-0.002	0	0.0002	0.002	0.02
			$\Delta\Lambda_2 < 0$				$\Delta\Lambda_2 > 0$	
2		0.94	0.99	1.00	1.00	1.00	1.01	1.10
6		0.97	1.00	1.00	1.00	1.00	1.00	1.04
10		0.98	1.00	1.00	1.00	1.00	1.00	1.03

The ratio of the certainty equivalents, when exchange rate risk sensitivity is negative, is smaller than one, while it is larger than one when exchange rate volatility is positive. From the previous discussion, we know that a negative exchange rate sensitivity implies that the price of risk the investor faces is smaller compared to the price of risk he faces when there is no exchange rate volatility. For a given amount of risk, the investor is rewared less in the presence of exchange rate variability compared to a situation without exchange rate variability. The investors is then better off when exchange rate variability is eliminated. On the contrary, a positive exchange rate volatility implies that the price of risk the investor faces is larger compared to the price of risk he faces when there is no exchange rate volatility. The investors is then worse off when exchange rate variability is eliminated.

The gain or loss over the elimination of exchange rate variability is substantial when the exchange rate volatility is large: when the exchange rate volatility is -0.02, the gain ranges from 2 percent to 6 percent, and the loss ranges from 3 percent to 10 percent when the exchange rate volatility is 0.02.

6 Conclusion

This paper solves the dynamic portfolio allocation for an international market where asset prices are characterized by mean reversion. The model derives an analytical solution for the optimal portfolio in a two-country, two-state variable model, in which an investor can allocate his wealth between a home and a foreign asset. Moreover, inflation dynamics are included to determine the nationality of an investor. Our model is derived under the assumption that the home and foreign assets are influenced by independent sources of risk, but that both markets do price both risks. We then have incomplete national markets, but a complete international market. Once converted to home currency units, the foreign asset depends on both the home and the foreign risk factors. This setting allows us to study the effect of the exchange rate in isolation.

Using international no-arbitrage conditions, we link the home and foreign stochastic environment, and we show that the optimal international portfolio consists of three components: a nominal myopic demand, an inflation demand (real myopic demand) and a hedging demand. We show that in the absence of exchange rate variability, the optimal portfolio weights are fully independent of each other and market prices of risks across countries converge. However, when both home and foreign risk factors are present in the foreign asset price due to the conversion by the exchange rate, the portfolio's volatilities change and the market prices of risk across countries differ. The additional exchange rate volatilities are hedged by the investor. This is the volatility effect of the exchange rate. Furthermore, the investor faces new investment opportunities through the change in the market prices of risk. Accordingly, he adjusts his myopic and hedging demand optimally. This is the price effect of the exchange rate.

To determine the importance of the elimination of the exchange rate in an international portfolio, we calibrate our model for an international portfolio using the dividend yield as the state variable. We compare the optimal allocation in the presence of exchange rate variability, with the optimal allocation when exchange rates are artificially eliminated. We find that the exchange rate effect on the portfolio allocation is significant. In response to the volatility and price effect, the investor optimally reallocates his international portfolio. We show that the volatility effect of the exchange rate and the myopic demand price effect constitute the main driving forces of these portfolio shifts.

This model allows us to assess the portfolio shifts following the introduction of the Euro. Given the small difference in market prices of risk across the different EMU members, even before the monetary unification (see Dewachter et.al. (2003)), it can be expected that the portfolio shifts are limited. Furthermore, in this paper we derived the effect of the elimination of exchange rate variability on the optimal international portfolio assuming that international markets are complete, and that there are no exchange rate specific diversification benefits. The next challenge is to extend the international portfolio model to take account of an additional, priced exchange rate specific stochastic component. To fully hedge all the risks away, the optimal portfolio then consists of the home risky asset (hedging the home risk factor), the foreign converted risky asset (hedging the foreign risk factor) and the currency (hedging the exchange rate risk factor). In this case, the removal of exchange rate variability has an additional effect of removing the specific exchange rate risk factor.

A Stochastic Control Problem

We use stochastic control to solve the investors' portfolio problem. Define $J(t, Y, X)$ as the maximized utility function (or value function). The Bellman principle of optimality implies the following Hamilton-Jacobi-Bellman (HJB herein):

$$0 = \max_{A(t)} \{\mathcal{D}J(t, Y, X)\} \tag{53}$$

where $\mathcal{D}$ is the Dynkin operator, with $\mathcal{D}J(t, Y, X) = \frac{1}{dt}E_t[dJ(t, Y, X)]$. Using Ito's lemma to compute $dJ(t, Y, X)$, the control problem becomes:

$$0 = \max_{A(t)} \frac{1}{dt} E_t \left[\begin{array}{c} J_t dt + J_Y dY + J_X' dX + \frac{1}{2} J_{YY} dY dY' \\ + \frac{1}{2} Tr(J_{XX'} dX dX') + J_{XY}' dX dY \end{array} \right] \tag{54}$$

with the terminal condition:

$$J(T, Y, X) = \frac{1}{1-\gamma} (Y(T))^{1-\gamma} \tag{55}$$

where J_d is the partial derivative of J with respect to d. Using the real wealth dynamics (26), the state variable dynamics (1) and taking expectations yields:

$$0 = \max_{A(t)} \left[\begin{array}{c} J_t + J_X'(k - KX(t)) + J_Y Y(t)(r(t) - \pi(X(t))) \\ + J_Y Y(t) A(t)' (\mu(X(t)) - r(t)\iota) + J_Y Y(t) \sigma_{\overline{P}} \sigma_{\overline{P}}' \\ + J_Y Y(t) A(t)' \Sigma \sigma_{\overline{P}}' + \frac{1}{2} J_{YY} Y(t) A(t)' \Sigma\Sigma' A(t) Y(t)' \\ + J_{YY} Y(t) A(t)' \Sigma \sigma_{\overline{P}}' Y(t)' + \frac{1}{2} J_{YY} Y(t) \sigma_{\overline{P}} \sigma_{\overline{P}}' Y(t)' \\ + \frac{1}{2} Tr(J_{XX'} \sigma_X \sigma_X') - J_{YX}' \sigma_X \Sigma' A(t) Y(t)' - J_{XY}' \sigma_X \sigma_{\overline{P}}' Y(t)' \end{array} \right] \tag{56}$$

From the HJB equation, we can now compute the first-order optimality condition (FOC) for the portfolio allocation:

$$\begin{aligned} A(t)^{opt} &= \frac{-J_Y}{Y(t) J_{YY}} (\Sigma\Sigma')^{-1} (\mu(X, t) - r(t)\iota) - \frac{J_Y}{Y(t) J_{YY}} (\Sigma\Sigma')^{-1} \Sigma \sigma_{\overline{P}}' \\ &\quad - (\Sigma\Sigma')^{-1} \Sigma \sigma_{\overline{P}}' + \frac{1}{Y(t) J_{YY}} (\Sigma\Sigma')^{-1} \Sigma \sigma_X' J_{XY} \end{aligned} \tag{57}$$

Substitution of the first order condition (57) back into the HJB (56) equation yields the following partial differential equation:

$$\begin{aligned} 0 = \; & J_t + J_X' \left(k - KX(t)\right) + J_Y Y(t) \left(r(t) - \pi(X(t))\right) \\ & -\frac{1}{2} J_Y \left(\mu(X(t)) - r(t)\iota\right)' \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X(t)) - r(t)\iota\right) \frac{J_Y}{J_{YY}} \\ & -J_Y \sigma_{\overline{P}} \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X,t) - r(t)\iota\right) \frac{J_Y}{J_{YY}} \\ & -J_Y Y(t) \sigma_{\overline{P}} \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X(t)) - r(t)\iota\right) + J_{XY}' \sigma_X \sigma_{\overline{P}}' \frac{J_Y}{J_{YY}} \\ & +J_{XY}' \sigma_X \Sigma' \left(\Sigma\Sigma'\right)^{-1} \left(\mu(X(t)) - r(t)\iota\right) \frac{J_Y}{J_{YY}} \\ & -\frac{1}{2} J_Y \sigma_{\overline{P}} \sigma_{\overline{P}}' \frac{J_Y}{J_{YY}} - \frac{1}{2} J_{XY}' \sigma_X \sigma_X' \frac{J_{XY}}{J_{YY}} + \frac{1}{2} Tr\left(J_{XX'} \sigma_X \sigma_X'\right) \end{aligned} \tag{58}$$

The next step in solving this PDE is to guess a trial solution for the value function $J(t, X, Y)$. We conjecture that:

$$J(t, X, Y) = \frac{Y^{1-\gamma}}{1-\gamma} \Phi(X(t,T)) \tag{59}$$

where $\Phi(X(t,T)) = \Phi(X(T-t))$ represents the contribution to the investor's expected utility of the remaining investment opportunities up to the horizon. From (18)-(21) in the main text, we know that the market prices of risk are affine functions of the state variables. Therefore, define:

$$\Lambda(X(t)) = \lambda_0' + X(t)' \lambda_1' \tag{60}$$

$$\Lambda^*(X(t)) = \lambda_0^{*\prime} + X(t)' \lambda_1^{*\prime} \tag{61}$$

Furthermore, using the fact that $\left(\mu(X(t)) - r(t)\iota\right) = \Sigma\Lambda' = \Sigma\left(\lambda_0' + X'(t)\lambda_1'\right)$ and that $\pi(X(t)) = \delta_{0,\pi} + \delta_{1,\pi} X(t)$, and substituting the trial solution (59) in the PDE (58) yields after simplification:

$$\begin{aligned} 0 = \; & -\Phi_t + \Phi_X' \left(k - KX(t)\right) + \left(\frac{1-\gamma}{2\gamma}\right) \Phi(X(t,T)) \lambda_0' \lambda_0 \\ & + \left(\frac{1-\gamma}{\gamma}\right) \Phi(X(t,T)) X(t)' \lambda_1' \lambda_0 + \left(\frac{1-\gamma}{2\gamma}\right) \Phi(X(t,T)) X(t)' \lambda_1' \lambda_1 X(t) \\ & + (1-\gamma) \Phi(X(t,T)) \left(\delta_{0,r} + \delta_{1,r} X(t) - \delta_{0,\pi} - \delta_{1,\pi} X(t)\right) \\ & + \frac{(1-\gamma)^2}{\gamma} \Phi(X(t,T)) \sigma_{\overline{P}} \lambda_0 + \frac{(1-\gamma)^2}{\gamma} \Phi(X(t,T)) \sigma_{\overline{P}} \lambda_1 X(t) \\ & + \left(\frac{1-\gamma}{2\gamma}\right) \Phi(X(t,T)) \sigma_{\overline{P}} \sigma_{\overline{P}}' + \left(\frac{1-\gamma}{2\gamma}\right) \Phi_X' \sigma_X \sigma_X{}' \left(\frac{\Phi_X}{\Phi(X(t,T))}\right) \\ & + \frac{1}{2} Tr\left(\Phi_{XX'} \sigma_X \sigma_X'\right) - \left(\frac{1-\gamma}{\gamma}\right) \Phi_X' \sigma_X \lambda_0 \\ & - \left(\frac{1-\gamma}{\gamma}\right) \Phi_X' \sigma_X \lambda_1 X(t) - \left(\frac{1-\gamma}{\gamma}\right) \Phi_X' \sigma_X \sigma_{\overline{P}}' \end{aligned} \tag{62}$$

This equation is a linear second-order PDE, which is analytically hard to solve. Therefore, we follow the approach of Liu (2001), and rewrite the above PDE into a system of ODE's. This is done by guessing a functional form for $\Phi(X(t,T))$. Assume the trial solution:

$$\Phi(X(t,T)) = exp\left(\mathcal{A}(t,T) + \mathcal{B}(t,T)X(t) + \frac{1}{2}X(t)'\mathcal{C}(t,T)X(t)\right) \tag{63}$$

and we impose that $\Phi(X(T)) = 1$, which implies boundary conditions:

$$\mathcal{A}(T) = \mathcal{B}(T) = \mathcal{C}(T) = 0 \tag{64}$$

Substitution of this trial solution in the HJB (62) results in a quadratic equation in X. Its three coefficients need to be zero. Therefore, collecting terms in the constant, X and $X'X$ results in the following system of first-order nonlinear equations:

$$\begin{aligned} 0 \;=\;& -\mathcal{A}_t + \mathcal{B}(t,T)k + \left(\frac{1-\gamma}{2\gamma}\right)\lambda_0'\lambda_0 + \frac{(1-\gamma)^2}{\gamma}\sigma_{\overline{P}}\lambda_0 - \left(\frac{1-\gamma}{\gamma}\right)\mathcal{B}(t,T)\sigma_X\lambda_0 \\ & + (1-\gamma)(\delta_{0,r} - \delta_{0,\pi}) + \left(\frac{1-\gamma}{2\gamma}\right)\sigma_{\overline{P}}\sigma_{\overline{P}}' + \left(\frac{1}{2\gamma}\right)\mathcal{B}(t,T)\sigma_X\sigma_X'\mathcal{B}(t,T)' \\ & + \frac{1}{2}Tr\left(\mathcal{C}(t,T)\sigma_X\sigma_X'\right) - \left(\frac{1-\gamma}{\gamma}\right)\mathcal{B}(t,T)\sigma_X\sigma_{\overline{P}}' \end{aligned} \tag{65}$$

$$\begin{aligned} 0 \;=\;& -\mathcal{B}_t - \mathcal{B}(t,T)K + k'\mathcal{C}(t,T) + \left(\frac{1-\gamma}{\gamma}\right)\lambda_0'\lambda_1 + \frac{(1-\gamma)^2}{\gamma}\sigma_{\overline{P}}\lambda_1 \\ & - \left(\frac{1-\gamma}{\gamma}\right)\lambda_0'\sigma_X\mathcal{C}(t,T) - \left(\frac{1-\gamma}{\gamma}\right)\mathcal{B}(t,T)\sigma_X\lambda_1 + (1-\gamma)(\delta_{1,r} - \delta_{1,\pi}) \\ & - \left(\frac{1}{\gamma}\right)\mathcal{B}(t,T)\sigma_X\sigma_X'\mathcal{C}(t,T) - \left(\frac{1-\gamma}{\gamma}\right)\sigma_{\overline{P}}\sigma_X'\mathcal{C}(t,T) \end{aligned} \tag{66}$$

$$\begin{aligned} 0 \;=\;& -\mathcal{C}_t - 2\mathcal{C}(t,T)'K + \left(\frac{1-\gamma}{\gamma}\right)\lambda_1'\lambda_1 + \left(\frac{1}{\gamma}\right)\mathcal{C}(t,T)'\sigma_X\sigma_X'\mathcal{C}(t,T) \\ & - 2\left(\frac{1-\gamma}{\gamma}\right)\mathcal{C}(t,T)'\sigma_X\lambda_1 \end{aligned} \tag{67}$$

To solve for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ we start from the ODE (67), then solve for (66), and finally for (65). We use the fourth order Runge Kutta simulation technique. Finally, substitution of the trial solutions (59) and (63) into the portfolio first-order condition (57) determines the optimal portfolio allocation:

$$\begin{aligned} A(t)^{opt} \;=\;& \frac{1}{\gamma}(\Sigma\Sigma')^{-1}(\mu(X(t)) - r(t)\iota) + \left(\frac{1-\gamma}{\gamma}\right)(\Sigma\Sigma')^{-1}\Sigma\sigma_{\overline{P}}' \\ & - \frac{1}{\gamma}(\Sigma\Sigma')^{-1}\Sigma\sigma_X'\left(\mathcal{B}(t,T)' + \mathcal{C}(t,T)X(t)\right) \end{aligned} \tag{68}$$

where $\mathcal{B}$ and $\mathcal{C}$ are the solutions to the system of ODE's.

B Calibration

Tables 5, 6 and 7 summarize the results of the optimal portfolio policies for an investment horizon of 1, 15 and 20 years and a fixed coefficient of relative risk aversion of 6.

Table 5: **Optimal Portfolio Strategy (%)**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
T		-0.02	-0.002	0.0002	0	0.0002	0.002	0.02
			$\Delta\Lambda_2 < 0$				$\Delta\Lambda_2 > 0$	
1	$\alpha(t)$	26.57	20.09	19.48	19.41	19.34	18.74	12.90
	$\alpha^*(t)$	20.80	19.62	19.54	19.53	19.52	19.45	18.91
	$100 - \alpha(t) - \alpha^*(t)$	52.63	60.29	60.98	61.06	61.14	61.81	68.19
15	$\alpha(t)$	40.34	29.68	28.77	28.67	28.57	27.69	19.61
	$\alpha^*(t)$	33.90	29.15	28.84	28.81	28.77	28.48	26.32
	$100 - \alpha(t) - \alpha^*(t)$	25.76	41.17	42.39	42.52	42.56	43.83	54.07
20	$\alpha(t)$	40.84	30.03	29.11	29.01	28.91	28.02	19.86
	$\alpha^*(t)$	34.38	29.50	29.18	29.15	29.11	28.82	26.60
	$100 - \alpha(t) - \alpha^*(t)$	24.78	40.47	41.71	41.84	41.98	43.16	53.54

This table reports the optimal strategies for an investor with different values of exchange rate volatility and investment horizons, for a fixed coefficeint of relative risk aversion of 6; $\alpha(t)$ and $\alpha^*(t)$ are the proportional allocation to the home and foreign risky assets, respectively. The middle column reports the optimal allocation when exchange rate variability is eliminated.

Table 6: **Exchange Rate Components (as % of total portfolio weights)**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$					
T		-0.02	-0.002	-0.0002	0.0002	0.002	0.02
1			$\Delta\Lambda_2 < 0$			$\Delta\Lambda_2 > 0$	
	$ER_{\sigma_{S,1}}$	7.16	0.68	0.07	-0.07	-0.67	-6.51
	$ER^*_{\sigma_{S,2}}$	10.25	0.70	0.07	-0.07	-0.65	-5.00
	$ER^*_{\Delta\Lambda_2,M}$	-8.25	-0.59	-0.06	0.06	0.55	4.27
	$ER^*_{\Delta\Lambda_2,H}$	-0.23	-0.02	0.00	0.00	0.01	0.11
15							
	$ER_{\sigma_{S,1}}$	11.67	1.00	0.10	-0.10	-0.98	-9.06
	$ER^*_{\sigma_{S,2}}$	15.12	1.03	0.10	-0.10	-0.96	-7.38
	$ER^*_{\Delta\Lambda_2,M}$	-8.75	-0.59	-0.06	0.06	0.55	4.27
	$ER^*_{\Delta\Lambda_2,H}$	-1.28	-0.09	-0.01	0.01	0.08	0.62
20							
	$ER_{\sigma_{S,1}}$	11.83	1.02	0.10	-0.10	-0.99	-9.16
	$ER^*_{\sigma_{S,2}}$	15.30	1.04	0.10	-0.10	-0.97	-7.46
	$ER^*_{\Delta\Lambda_2,M}$	-8.75	-0.59	-0.06	0.06	0.55	4.27
	$ER^*_{\Delta\Lambda_2,H}$	-1.32	-0.09	-0.01	0.01	0.08	0.64

Table 7: **Ratio of Certainty Equivalents of Wealth**

		Exchange rate loadings, $\sigma_{S,1}$ and $\sigma_{S,2}$						
T	ERG	-0.02	-0.002	-0.002	0	0.0002	0.002	0.02
		$\Delta\Lambda_2 < 0$				$\Delta\Lambda_2 > 0$		
1		1.00	1.00	1.00	1.00	1.00	1.00	1.00
15		0.96	0.99	1.00	1.00	1.00	1.01	1.06
20		0.94	0.99	1.00	1.00	1.00	1.01	1.09

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