# Beating the Market?

## A Mathematical Puzzle to Market Efficiency

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*Abstract* The efficient market hypothesis is highly discussed—supported and criticized—in economic literature. In its weakest form it states that there are no price trends. When weakening the no-trending assumption only a little to arbitrary short and small and fully unknown trends, by use of control techniques it is very easy to construct trading strategies with zero initial investment and positive expected gain. Since even the trend's sign may be unknown, a possible trader does not have to think about predictable patterns etc. Even if compared to buy-and-hold strategies and adjusted for risk, the control-based strategies are preferable.

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# 1 Introduction

While in the 70's the market efficiency hypothesis was highly accepted (Fama, 1965, 1970), later on, it was highly criticized—and defended (Malkiel, 1989, 2005). Much of the critics concerned so-called predictable patterns, for example the January effect, i.e., high positive returns in the first two weeks of January. The defenders of the market efficiency hypothesis have several arguments against this, e.g., that patterns will self-destroy once published or that small possible gains will vanish when trading costs have to be payed.

Additionally, there is the so-called joint hypothesis problem which states that market efficiency and the used market model have to be tested nearly always simultaneously. That means, if the test fails, no one knows whether the market is not efficient or whether the model used is not sufficient. A second point of critics on the critics is the distinction between statistical inefficiency and economical inefficiency. The first one means that one can construct a test for showing that there are, for example, predictable patterns. The second one means that a trader has to be able to exploit this. And the last point to defend the market efficiency hypothesis I want to mention is that even if one can construct a strategy with "too high" returns, e.g., by taking into account some external variables, it may be that these variables are better ratios for measuring risk. When introducing risk-adjusted returns, excess returns are no contradiction when they go hand in hand with excess risk.

In the next section I will give a very short review of market efficiency, its critics, and its defenses, i.e., the critics of the critics. But for now, I will go on with motivating this paper. Much of the discussion on market efficiency, technical trading, and beating the market follows the idea that a trader (i) has to find a predictable pattern, (ii) has to construct a trading strategy exploiting this pattern, and, (iii) has to test this new strategy against randomly selected broad index buy-and-hold strategies (Malkiel, 1973).

However, a new strand of research—mainly in engineering sciences and mathematics—goes another way: Assume task (i) can be skipped and so directly trading strategies can be constructed. These strategies usually are model-free and do neither use predictions of patterns nor estimations of parameters. In short and using the terminology of the control community: They are constructed to be robust against the price. Instead of task (iii), which relies on real market data, (performance) properties are proven mathematically. This way, the overfitting problem (cf. Bailey et al., 2014) is avoided.

While in the control literature results on control-based trading strategies attract high attention, in the economical literature they are widely unknown. The aim of the work at hand is to review known results on a particular control-based strategy, the so-called Simultaneously Long Short (SLS) Strategy, to extend the results in different directions, and, finally, to bring them into the context of market efficiency, which is not adequatelly discussed in control literature. The paper is organized as follows: After reviewing the market efficiency literature and the control-based trading literature, new results are obtained and their relation to efficient markets is discussed.

Since the aim of the work at hand is to bring together economical ideas like market efficiency and control theoretic ideas like feedback trading the one or the other part might be uncommon to the reader. After discussing the efficient market hypothesis, reviewing the feedback trading literature, constructing new trading strategies, as well as proving properties of these strategies, as this parts will be brought together. At the end, since some of these properties do not fit to efficient markets I discuss this puzzle.

## 2 Literature Review

In this section, I briefly discuss market efficiency, its critics and its defense. After that, I introduce the SLS rule as it is known from the control literature and state the most important results from this work.

## 2.1 Review of Market Efficiency

In this section, I will give a very brief overview about market efficiency. Because there is a very broad literature on this topic and there are also a lot of very good and thus very famous overviews I refer the interested reader to these overviews (e.g., Fama, 1991; Malkiel, 2003). Besides the definition and discussion of market efficiency, I will discuss some topics where definitions are not clear—focused on the discussion of the SLS strategy at the end of this work.

In its *strong* version, the market efficiency states that all information is reflected in the price. That means, no "sophisticated" trader—even no "insider," who has private information—performs on average better than a simple buy-and-hold trader. That means, when there is no change in the fundamental value, all price movements are fully random without any trend. Mathematically spoken, the price process is a random walk around its fundamental value. A little bit weaker and maybe closer to markets is the assumption that only nearly all information is incorporated in the price. But the costs for getting the missing information and for trading the asset are higher than the possible gain of exploiting these information (Fama, 1991).

The *semi-strong* version of the market efficiency hypothesis states that all public information is reflected in the price. That means "insider trading" may be profitable, which is widely accepted. For example, the findings on the effects of Value Line rank changes are a sign that insider trading may be profitable (Stickel, 1985), (summarized in Fama, 1991). However, all public information is immediately incorporated in the asset price. The word "immediately" has to be understood in an averaged sense, i.e., markets may overreact to new information or underreact and markets may reflect information too early or too late, but on average all these effects are balancing out (Fama, 1995). In other words, fundamental value analysis, i.e., trying to calculate the fundamental or intrinsic value (simplistic: the real value), is on average not profitable at all, because an asset's actual price is at any point of time the best estimate for the fundamental value (based on public information). Fundamentalists can make profit if they find relevant information faster and rate the effects to the fundamental values under analysis better. Thus all fundamentalists try to be as fast and as accurate as possible—thereby adjusting prices instantaneously to the intrinsic values. Since no one knows who is the fastest and the best, on average fundamentalists cannot expect excess gains.

Last, the *weak* version of market efficiency states that insider trading as well as fundamental analysis may be profitable but technical analysis is not. That means, no one can use past returns to predict future ones. Also in this version, chartists cannot make money on average, markets have no memory, and patterns do not exist. Or, even a little bit weaker, when there exists a dependence of past and future returns, these anomalies are so small that they are not exploitable. To sum up, in all versions of the market efficiency hypothesis, for chartists it is not possible to make money on average. Because on the other hand there is a lot of literature on the profitability of technical trading and there are numerous funds managers who rely on such strategies, the task is always considered to be empirical. That means, chartist fund managers are challenged to provide statistics that their strategies outperform random-selected buy-and-hold strategies.

Hereafter, I will summarize a selection of common critics to the market efficiency hypothesis and state some arguments of the defenders of the hypothesis against these critics. One strand of critics to the market efficiency hypothesis relies on predictable patterns. With statistical or data science methods, patterns, i.e., an on average recurring behavior of stock market prices, were found: the Monday effect (lower returns on Mondays; Cross (1973); French (1980)), the month effect (higher returns at the last day of the month; Ariel (1987)), the holiday effect (higher returns at the day before a holiday; Ariel (1990)), and the most famous January effect (higher returns in January and even higher returns in the first five days of January; Keim (1983); Roll (1983)).

But—following Malkiel (2003)—predictable patterns will self-destroy once published.

Exemplary for the January effect: If the January effect exists, traders would buy at the last days of December and sell at the very beginning of Januar. That means, the pattern would move a few days. Observing this, traders would buy and sell again a few days earlier. And so on. At the end, the January effect would be destroyed. A second attack to this strand of critics is that the effects of (predictable) patterns are too small to exploit them (Lakonishok and Smidt, 1988), especially when trading costs are considered. This last argument can be generalized: Only because there is a statistical inefficiency (i.e., predictability in returns, which are shown by use of data science methods) that does not mean that a trader can make profit of it—when the effect and the power of the statistic is small relative to additional costs. That means, economical inefficiency had to be shown by trading performance statistics.

Another strand of critics to market efficiency is that stock returns may be predictable using some external variables, for example, dividend yields (D/P; Rozeff (1984); Shiller (1984)), earning per price ratios (E/P; Campbell and Shiller (1988)), or the firms' size (Banz, 1981). But, as summarized by Fama (1991), these dependencies are either too small to exploit them (especially when trading costs are taken into account) or—like in the case of the size effect—they have another reason: Taking into account some external variables with predictive power may just mean that these variables are better ratios for measuring risk. As mentioned above, the definition of market efficiency is not clear at all. Despite the statistical inefficiency vs. economical inefficiency problem, one can find statements like "traders cannot expect excess returns" as well as "traders can only expect excess returns when they accept excess risk" in the literature. So, often the term *risk-adjusted gains* is used. Here, the next problem arises: How to measure risk? Often the Capital Asset Pricing Model's (CAPM's) $\beta$ or the standard deviation is used. I will come back to this problem in the discussion section again.

At the end of this section, I want to mention a few more problems very briefly. First, all empirical findings concerning market efficiency might be results of *data-dredging* (also known as p-hacking), i.e., the results might be found by use of data-mining techniques searching for significant p-values without causality or hypothesis. However, there are studies indicating that there are (with constant fundamentals) long term trends (possibly sinusodial) (Granger and Morgenstern, 1962; Saad et al., 1998). Second, there is the *joint hypothesis problem,* which states that market efficiency can (nearly) always only be tested when simultaneously using a market model. A consequence is that if a test fails, no one can say whether the market efficiency hypothesis is wrong or whether the used market model is insufficient. An exception are so-called *event studies* (Fama et al., 1969). Event studies analyze how fast and to which extent stock prices adjust to announcements, i.e., to new public information. So, event studies lie in the field of the semi-strong form of the market efficiency hypothesis and not of the field of the weak one.

And last, there is the *momentum effect*, which states that assets that performed well over the last few months will do so over the next few months and similar for bad assets (Jegadeesh and Titman, 1993, 2001; Fama and French, 1996, 2008). Thus, there seems to be valid critics to the weak form market efficiency based on empirical/statistical methods—maybe explainable by behavioral economics.

## 2.2 Simultaneously Long Short (SLS) Trading

There is a strand of research in the control literature that seemingly does not care about market efficiency. There, by use of feedback techniques, which are used in engineering sciences and analyzed in applied mathematics, trading strategies are constructed that are robust against noisy prices $p_t$. The control theoretic way of thinking is different from classical finance: Neither fundamentals $f_t$ are calculated nor price patterns are searched for estimating future returns $\mathbb{E}\left[\frac{p_{t+1}-p_t}{p_t}\right]$ because the strategies do not use estimations of future returns.

Traders relying on control-based trading strategies are called feedback traders. They calculate their investment, i.e., their net asset position, which is an *input* variable to the system “financial market” at every point of time, as a function of an *output* variable of the system, usually the gain. In the next section, I will extend the strategies to other output variables. But for now, I define the Simultaneous Long Short (SLS) strategy as used in the control literature and present the most important results.

As mentioned above, a feedback trader $\ell$ (in this section) computes at time $t$ the investment $I^{\ell}(t)$ as a function of the own gain $g^{\ell}(t)$ and—some would call it naively—of nothing else:

$$I^{\ell}(t) = F(g^{\ell}(t))$$

Since the results from the literature to be presented next are obtained in different market models, some in discrete (indicated by subscript $t$), some in continuous time (indicated by $t$ in brackets), I will give the definition of the strategy for a stochastic model in continuous time, which can easily be rewritten to other settings. The trader’s gain is calculated by use of the investment and the return of investment:

$$g^{\ell}(t) = \int_0^t I^{\ell}(t) \cdot \frac{dp(t)}{p(t)}$$

The big question that has to be answered is how to choose the function $F$. One possibility for $F$ is the so-called *linear long* feedback trading rule

$$I^L(t) = I_0^L + K^L g^L(t),$$

where $I_0^L > 0$ is the initial investment of the linear long rule and $K^L > 0$ is the so-called feedback parameter. It is easy to see that the linear long feedback trader is a trend following long trader in continuous time, when the price process is continuous, too. That means, this trader type makes money when prices rise and loses money when prices fall. Since the required SLS rule shall be robust against variations in prices, i.e., trend following is a non-desired property, the linear long rule has to be modified. For this, the *linear short* feedback rule is defined first:

$$I^S(t) = -I_0^S - K^S g^S(t)$$

This trader is (when time and price are continuous) an anti trend following short investor who loses money when prices rise and earns money when prices fall. The SLS rule is now simply defined as the superposition of the linear long and the linear short rule with the same parameters, i.e., $I_0^L = I_0^S =: I_0^*$ and $K^L = K^S =: K$:

$$I^{SLS}(t) = I^L(t) + I^S(t)$$

Note that the long side's gain $g^L$ and the short side's gain $g^S$ have to be calculated separately and that the initial investment of the SLS rule is always zero:

$$I^{SLS}(0) = I^L(0) + I^S(0) = I_0^* - I_0^* = 0$$

A flow diagram for the SLS rule is given in Fig. 1.

As can be seen, the short side's strategy requires for sure the possibility for *short selling*. Besides this market requirement, a few more assumptions are needed. In the analytic parts of the work at hand, *costless trading*, i.e., no additional costs related with buying or selling assets, *adequate resources*, i.e., no financial constraints, which could prohibit any desired transaction, *perfect liquidity*, i.e., no bid-ask spread and no waiting time, and the so-called *price taker* property, i.e., no impact of the investment decisions on the price process, are presumed. In a discussion of the results of this paper these assumptions can be debated. But for now, I will just briefly justify this market requirements: Short selling and perfect liquidity should not be strong assumptions for large companies' stocks under trade. Costless trading, which was in the past a strong argument of the defenders of the efficient market hypothesis to show that chartist strategies cannot work in practice (cf. Fama, 1991), might be less discussed in times of flat-rate stock trading offers. The adequate resources assumption is justified if the trader is "big enough," e.g., a mutual fund, and is not trading "too much" of the single asset under trade. The latter assumption also justifies the price taker property.

## 2.3 Literature Review on SLS Trading

The following literature review shall give an idea about why the SLS strategy is an interesting one. Barmish (2011) showed that for continuously differentiable prices $p \in \mathcal{C}^1$ it holds

$$g_{\mathcal{C}^1}^{SLS}(t) = \frac{I_0^*}{K}\left(\left(\frac{p(t)}{p(0)}\right)^K + \left(\frac{p(t)}{p(0)}\right)^{-K} - 2\right)$$

from which follows that $g(t) > 0$ for all price processes with $p(t) \in (-1, \infty) \setminus \{p(0)\}$. Note that this means, that the gain at time $t$ is independent of the process and only depends on the value of $p(t)$ at time $t$. In other words, this is an *arbitrage* strategy. Since $\mathcal{C}^1$ prices are a rather hard assumption, Barmish and Primbs (2011, 2015) showed that when the underlying price process is governed by a geometric Brownian motion (GBM)

$$p_{GBM}(t) = p_0 \cdot exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$$

(with trend $\mu > -1$, volatility $\sigma > 0$, and a Wiener process $W(t)$), the SLS strategy is not an arbitrage strategy anymore, however, for the expected gain it holds:

$$\mathbb{E}\left[g_{GBM}^{SLS}(t)\right] = \frac{I_0^*}{K}(exp(K\mu t) + exp(-K\mu t) - 2)$$

Especially it holds $\mathbb{E}[g_{GBM}^{SLS}(t)] > 0$ whenever $\mu \neq 0$ holds. Together with $I^{SLS}(0) = 0$ this is called the *robust positive expectation property.* Similar results are provided by Dokuchaev and Savkin (1998a,b, 2002, 2004); Dokuchaev (2012).

Primbs and Barmish (2013) show that the robust positive expectation property also holds when the trend $\mu(t)$ as well as the volatility $\sigma(t)$ of the GBM is time dependent. In fact, for a time-varying GBM (tvGBM) with trend $\mu(t)$ and volatility $\sigma(t)$ and the SLS trading rule it holds:

$$\mathbb{E}[g_{tvGBM}^{SLS}(t)] = \frac{I_0^*}{K}\left(exp\left(K\int_0^t \mu(s)ds\right) + exp\left(-K\int_0^t \mu(s)ds\right) - 2\right).$$

For clear, $I^{SLS}(0) = 0$ and whenever $\int_0^t \mu(s)ds \neq 0$ it holds $\mathbb{E}[g_{tvGBM}^{SLS}(t)] > 0$, too.

Iwarere and Barmish (2014) analyze the SLS strategy when prices are governed by a binomial tree (Cox-Ross-Rubinstein model) and Barmish and Primbs (2012) use a market model motivated by the CAPM. Barmish (2008) and Malekpour et al. (2013) analyze other strategies—related to the SLS rule.

Baumann (2016) generalizes the results for SLS trading to prices governed by Merton's jump diffusions model (MJDM), which is given through

$$p_{MJDM}(t) = p_0 \cdot exp\left(\left(\mu - \lambda\kappa - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)\prod_{i=1}^{N} Y_i.$$

Hereby, the GBM is extended by i.i.d. jumps $(Y_i - 1) > -1$ with jump intensity $\lambda > 0$, expected jump heigth $\mathbb{E}[Y_i - 1] = \kappa > 0$, and a number $N \sim \mathcal{P}oi(\lambda t)$ of jumps up to time $t$. Jumps are interesting in this context since they are known—in the fields of options and hedging—for making markets incomplete. However, Baumann (2016) shows that the expected gain of the SLS strategy is

$$\mathbb{E}[g_{MJDM}^{SLS}(t)] = \frac{I_0^*}{K}(exp(K\mu t) + exp(-K\mu t) - 2),$$

which is exactly the same as for the GBM. Baumann and Grüne (2016) further generalize this result to a set of price processes defined by stochastic differential equations called "essentially linearly representable prices." Barmish and Primbs (2011) give a closed formula for the variance of the SLS trading rule when prices are governed by a GBM and Baumann (2016) does this for MJDM prices as well.

Here, I want to mention again that the so-called linear long (short) trader is not necessarily long (short) when there are discontinuities, for example, when the price model allows for jumps, like MJDM, or when the model is in discrete time, as in the both papers discussed next.

Malekpour and Barmish (2016) note an interesting and especially practical problem of the SLS rule. Since the SLS strategy is calculated by use of the over-all gain, price behaviors that happened a long time ago have the same impact on the investment decision of the trader as if they happened a few days ago. Imagine a price development where in the phase after the trader entered the market the price rose a lot and then stayed nearly constant for a long time. The trader's long (short) side would have made (lost) a lot of money in the first period and then stayed approximately constant. As a consequence of the feedback loop the investment of the trader is still very high—and long—which seems to be questionable since prices stayed constant for so long. Malekpour and Barmish (2016) introduce a new strategy called Initially Long-Short (ILS) with delay as the superposition of a linear long rule with delay $I_t^{Ld} = I_0^* + K(g_t^{Ld} - g_{t-m}^{Ld})$ and a linear short rule with delay $I_t^{Sd} = -I_0^* - K(g_t^{Sd} - g_{t-m}^{Sd})$. The strategy is defined and analyzed in a discrete time setting with a time grid $\{0, 1, 2, \ldots\}$ with fixed time steps, e.g., days. The word "Initially" denotes the fact that only at the initial time one can be sure that the long (short) side is truly long (short). Among other market requirements, similar to that presumed in the work at hand, the main assumption by Malekpour and Barmish (2016) is $\mathbb{E}\left[\frac{p_t - p_{t-1}}{t_{t-1}}\right] = \mu \neq 0$, which is needed to show that the positive robust expectation property still holds. In the ILS strategy only the period gains of the last $m$ days are taken into account. While on the one hand the idea of not taking into account to old price (and so gain) developments makes the ILS rule of Malekpour and Barmish (2016) favorable to the standard SLS rule, on the other hand the "hard" delay definition seems to be a little bit problematic. Imagine a price history where $m$ days ago an important event happened at the market, for example a sudden crash, which made the short side much more important. Today, this event will be taken into account, tomorrow, this will not be the case. That means, the strategy will change substantially only because an important event happened exactly $m$ days ago, where the number $m$ is idiosyncraticly chosen by the trader. A point to think about that is not discussed in detail by Malekpour and Barmish (2016) is that the trader is assumed to be a price taker. However, the trader decides to trade, e.g., daily and the expected return on investment on a daily basis is assumed to equal $\mu$. That means, the trader indirectly influences the expected return on investment by choosing a trading frequency, which at a first glance seems to contradict the price taker property. However, this is not a problem, as shown in the next paper reviewed below.

Also Baumann and Grüne (2017) use—at first—a discrete time setting, but, with adjustable time steps $h > 0$: $\{0, h, 2h, \ldots\}$. Here it is assumed that

$$\mathbb{E}\left[\frac{p_t - p_{t-h}}{h \cdot p_{t-h}}\right] = \mu_h \neq 0$$

(which is the expected return on investment (eroi)). For the standard SLS strategy it is shown that the expected gain is

$$\mathbb{E}[g_{eroi;t}^{SLS}] = \frac{I_0^*}{K}\left[(1 + K\mu_h h)^{\frac{t}{h}} + (1 - K\mu_h h)^{\frac{t}{h}} - 2\right],$$

which is positive whenever $\mu_h \neq 0$ and $t > h$. Even in this setting the conjectural contradiction to the price taker property is given: On the one hand, the trader chooses the trading frequency $h$, on the other hand the expected return on investment $\mu_h$ has to be independent of the trader. To solve this, a (maybe more realistic) setting with an underlying continuous time price process but discrete time trades is introduced. Engineers call this a *sampled-data system.* The continuous time price process has to satisfy

$$\mathbb{E}[p(t_2)|\mathcal{F}_{t_1}] = p(t_1) \cdot exp(\mu(t_2 - t_1))$$

for all $t_2 > t_1 \geq 0$. This $\mu$, now, is independent of the trader's decision on trading time and with

$$\mu_h := \frac{exp(\mu h) - 1}{h}$$

the above theory is applicable. Finally, it is shown that when calculating the limits for $h \to 0$ the results are fully in line with the known results for the GBM and MJDM. To sum up, without assuming any fixed market model but only some core properties like the expected return on investment it is shown that the SLS rule satisfies the robust positive expectation property, i.e., a positive expected gain while zero initial investment.

# 3 Analysis of the Discounted Simultaneously Long Short Strategy

The main feature of control-based trading strategies is that though market parameters like the expected return on investment are used when analyzing the strategies, the trader does neither have to know nor to estimate them. Properties of the strategies hold for (nearly) all settings of the parameter values. The following analysis will follow Baumann and Grüne (2017) but taking into account the ideas of Malekpour and Barmish (2016) (investment decisions should not rely on market behavior long ago), Primbs and Barmish (2013) (time-varying trend and volatility), and Barmish and Primbs (2011); Baumann (2016) (expected gain and variance).

After having discussed market efficiency and control-based trading strategies, especially SLS trading, now, I construct a new, more general type of an SLS rule: the discounted SLS rule. The construction process as well as the analysis is based on refinements of the underlying time grids: Starting with discrete time price processes and thus discrete time trading, via continuous time prices but discrete trading, which is called a sampled-data system, I end with continuous prices and continuous trading. The standard SLS rule is generalized by a discounting factor $\delta$, the price process allows for time-varying parameters, and the analysis takes risk-adjusted returns into account. The mathematically proven results—either concerning all discounted SLS rules (including the standard rule) or only the standard SLS strategy—build the already mentioned puzzle to market efficiency, which is still the aim of this work.

## 3.1 The Robust Positive Expectation Property

A controller with delay has the favorable feature that too old (older then $m$ days) events do not have any influence on the strategy, but it has a questionable feature, too: An event that is $m$ days old is taken fully into account today but vanishes from the calculations after $m+1$ days. As an alternative controller type, I introduce the discounted SLS controller with discounting factor $\delta \in [0,1]$ (SLS$_\delta$). The main, and indeed the only, difference of a discounted rule to the standard rule $\ell$ is that instead of the gain $g_t^\ell$ a discounted gain

$$f_t^\ell = \sum_{i=1}^{n} I_{(i-1)h} \cdot \frac{p_{ih} - p_{(i-1)h}}{p_{(i-1)h}} \cdot \delta^{(n-i)h}$$

on a discrete time grid $\{0, h, 2h, \ldots\}$ with $h > 0$ and $t = nh$ is used. That means, the discounted SLS rule is

$$I_t^{SLS_\delta} = I_t^{L_\delta} + I_t^{S_\delta}$$

with

$$I_t^{L_\delta} = I_0^* + K f_t^{L_\delta}$$

and

$$I_t^{S_\delta} = -I_0^* - K f_t^{S_\delta}.$$

A flow diagram for the discounted SLS rule is given in Fig. 2. Note that for $\delta = 1$ this strategy is exactly the standard SLS strategy and that we fix $0^0 := 1$. The discounting factor $\delta$ specifies to which extend past information is used for calculating the current investment (cf. other economic discounting factors like, e.g., the game theoretic discounting factor in repeated games). The higher $\delta$ is, the more influence has past information; for $\delta = 1$ all available information is equally weighted, for $\delta = 0$ only the last available information is important. The discounted SLS strategy has, similar to the SLS strategy with delay, the advantage that (if $\delta < 1$) old information is not that important as new one. However, in contrast to the delay strategy the old information loses its weight gradually and not instantaneously.

The second basic novelty of this work, different to the work of Baumann and Grüne (2017), is that I allow for a time-varying trend now:

$$\mathbb{E}\left[\frac{p_t - p_{t-h}}{h \cdot p_{t-h}}\right] =: \mu_{h;t-h}$$

(For the reason of non-negative prices, $\frac{p_t - p_{t-h}}{h \cdot p_{t-h}} \geq -1$ and $\mu_{h;t-h} > -1$ has to hold.) This generalization is similar to that done by Primbs and Barmish (2013) when extending the results for standard GBMs. Additionally, we assume, analogously to Baumann and Grüne (2017), positive, stochastic prices $(p_t)_{t \in \mathcal{T}} > 0$ ($\mathcal{T} = \{0, h, 2h, \ldots, T\}$, $T = Nh$, $t = nh$), $p_0 \in \mathbb{R}^+$, and independent multiplicative growth, i.e., for all $k \in \mathbb{N}$ and all $t_0 < t_1 < \ldots < t_k \in \mathcal{T}$ it holds that

$$p_{t_0}, \frac{p_{t_1}}{p_{t_0}}, \ldots, \frac{p_{t_k}}{p_{t_{k-1}}}$$

are stochastically independent. This is—depending on the definition used—the *weak form of the market efficiency hypothesis.* Note that this stochastic independence holds when applying any measurable function on the growth rates, too. Again, there seems to be a contradiction to the price taker property: While on the one side $h$ is chosen by the trader, on the other side the trend $\mu_{h;t}$ depends on $h$. But, as shown by Baumann and Grüne (2017) this problem can easily be solved either by use of so-called sampled-data systems or by calculating the limits for $h \to 0$.

In the following, I will show that the positive robust expectation property does not hold in general anymore (an example is given later in this section), but at least in two special cases. First, I note that for the expected price it holds

$$\mathbb{E}[p_t] = p_0 \cdot \prod_{i=1}^{n} (\mu_{h;ih} h + 1)$$

and

$$\mathbb{E}[p_{t_2} | \mathcal{F}_{t_1}] = p_{t_1} \cdot \prod_{i=n_1+1}^{n_2} (\mu_{h;ih} h + 1).$$

I start the analysis of the discounted SLS strategy with its long side. By the definition of $I_t^{L_\delta}$ and $f_t^{L_\delta}$ it follows

$$\frac{I_t^{L_\delta} - I_{t-h}^{L_\delta}}{h \cdot I_{t-h}^{L_\delta}} = K \cdot \frac{p_t - p_{t-h}}{h \cdot p_{t-h}}$$

and so

$$\mathbb{E}\left[\frac{I_t^{L_\delta} - I_{t-h}^{L_\delta}}{h \cdot I_{t-h}^{L_\delta}}\right] = K \mu_{h;t-h}.$$

Note that these formulae are independent of $\delta$. It follows

$$\mathbb{E}[I_t^{L_\delta}] = I_0^* \cdot \prod_{i=1}^{n} (K \mu_{h;ih} h + 1).$$

Again by the definition of $I_t^{L_\delta}$ it follows:

$$\mathbb{E}[f_t^{L_\delta}] = \frac{I_0^*}{K} \left( \prod_{i=1}^{n} (K \mu_{h;ih} h + 1) - 1 \right)$$

By substituting $I_0^* \to -I_0^*$ and $K \to -K$ the formula for $\mathbb{E}[f_t^{S_\delta}]$ follows.

Next, I investigate whether $\mathbb{E}[f_t^{L_\delta} + f_t^{S_\delta}]$ is positive or not. The reader my ask why am I interested in the expected sum of the discounted gain of the short and the long side of the discounted SLS strategy? The answer is because it holds

$$g_t^\ell = f_t^{\delta;\ell} + (1-\delta)\sum_{i=1}^{n-1} f_{ih}^{\delta;\ell}$$

and so

$$\mathbb{E}[g_t^{SLS_\delta}] = \mathbb{E}[g_t^{L_\delta} + g_t^{S_\delta}] = \mathbb{E}\left[(f_t^{L_\delta} + f_t^{S_\delta}) + \sum_{i=1}^{n-1}(1-\delta)(f_{ih}^{L_\delta} + f_{ih}^{S_\delta})\right].$$

That means, when $\mathbb{E}[f_t^{L_\delta} + f_t^{S_\delta}] > 0$ for all $t$, then $\mathbb{E}[g_t^{SLS_\delta}] > 0$, too. And this is what is really of interest, it is the positive robust expectation property.

Unfortunately, $\mathbb{E}[f_t^{L_\delta} + f_t^{S_\delta}] > 0$ is not true for all $t$ and all $(\mu_{h;t})_t$. This can be seen by rewriting

$$\begin{aligned}\mathbb{E}[f_t^{L_\delta} + f_t^{S_\delta}] &= \frac{I_0^*}{K}\left(\prod_{i=1}^{n}(K\mu_{h;ih}h+1) + \prod_{i=1}^{n}(-K\mu_{h;ih}h+1) - 2\right) \\ &= \frac{2I_0^*}{K}\sum_{\substack{\alpha\subset\{1,\dots,n\}\\ |\alpha|\ even\\ |\alpha|\neq 0}}\prod_{j\in\alpha} K\mu_{h;jh}h\end{aligned}$$

An example where this sum is negative is easy to find when setting $n=2$ with $\mu_{h;h} > 0$ and $\mu_{h;2h} < 0$. In this example,it holds $\mathbb{E}[f_{2h}^{L_\delta} + f_{2h}^{S_\delta}] = 2KI_0^*h^2\mu_{h;h}\mu_{h;2h} < 0$ and $\mathbb{E}[f_h^{L_\delta} + f_h^{S_\delta}] = 0$. It follows that $\mathbb{E}[g_{2h}^{SLS_\delta}] = \mathbb{E}[f_{2h}^{L_\delta} + f_{2h}^{S_\delta}] + (1-\delta)\mathbb{E}[f_h^{L_\delta} + f_h^{S_\delta}] < 0\ \forall\delta\in [0,1]$.

However, there are (at least) two special cases where $\mathbb{E}[f_t^{L_\delta} + f_t^{S_\delta}] > 0$ holds. (i) First, when $n > 1$ and $\mu_{h;t} \geq 0$ for all $t$ and $\mu_{h;t} > 0$ for at least two points of time $t$ or when $\mu_{h;t} \leq 0$ for all $t$ and $\mu_{h;t} < 0$ for at least two points of time $t$ (since $|\alpha|$ is *even*). That means, whenever $(\mu_{h;nh})_{n\in\{1,\dots,N\}}$ is non-negative (non-positive), $\mathbb{E}[f_t^{L_\delta} + f_t^{S_\delta}]$ is non-negative. When additionally there exists $\nu\subset\{1,\dots,N\}$ with $|\nu|\geq 2$ so that $(\mu_{h;jh})_{j\in\nu}$

is positive (negative), it holds that $\mathbb{E}[f_t^{L_\delta} + f_t^{S_\delta}]$ is positive. The settings of Baumann and Grüne (2017) and Malekpour and Barmish (2016), i.e., $\mu$ or $\mu_h$ *const.* and non-zero, are a special case of case (i).

(ii) Second, when letting $h \to 0$ (i.e., $n \to \infty$) one can use the continuously compounded interest rate formula, which is a Vito Volterra style product integral, to see

$$\mathbb{E}[f_t^{L_\delta} + f_t^{S_\delta}] = \frac{I_0^*}{K}\left(exp\left(\int_0^t K\mu(s)ds\right) + exp\left(\int_0^t -K\mu(s)ds\right) - 2\right),$$

which is positive whenever $\int_0^t \mu(s)ds \neq 0$. Here, it has to be ensured that all stochastic integrals are well defined, e.g., by assuming the price $p(t)$ to be a càdlàg semi-martingale, which—by construction—makes $\frac{I^\ell(t)}{p(t)}$ predictable ($\ell \in \{L_\delta, S_\delta\}, \delta \in [0, 1]$).

The setting of Primbs and Barmish (2013) is a special case of case (ii) and all the results using GBMs or MJDM are special cases of the cases (i) and (ii). In this case (ii) $\mu(t)$ has to be a Riemann integrable function.

That means during every time interval with positive expected returns or negative expected returns a trader using the discounted SLS rule can expect positive gains. Only when the expected return $\mu$ switches from rising to falling or vice versa the trader has to expect a loss. When increasing the trading frequency to continuous trading—which is in times of high frequency trading nearly a realistic assumption—and $\mu(t)$ is Riemann integrable, the measure of time points when $\mu$ is switching its direction goes to zero.

Mostly, in market efficiency literature, it is assumed that the price process is a random walk around its fundamental value. When allowing the fundamental value to be non-constant and assuming it to be not "too wild" (i.e., $\mu(t)$ has to be Riemann integrable and $\int_0^t \mu(s)ds \neq 0$) the SLS$_\delta$ trader can—when trading fast enough—expect a positive gain for all $t$ and all discounting factors $\delta \in [0, 1]$. And this should not be true in an efficient market.

Even when comparing the expected gain of the SLS rule with that one of a buy-and-hold strategy, it turns out, that for all $t$ with $\bar{\mu}(t) \in (-1, 0) \cup (B_{eg}(K), \infty)$ the SLS rule is the dominant one (see Fig. 3). The value $B_{eg}(K)$ is a depending on $K$ and it holds: $B_{eg}(K) \to 0$ for $K \to \infty$. Note that $\bar{\mu}(t) \notin [0, B_{eg}(K)]$ does not mean that the SLS is only dominant for special price paths, which would not be a result deserving attention. Since $\bar{\mu}(t) = \int_0^t \mu(s)ds$ with $\mu(t) = \mathbb{E}\left[\frac{dp(t)}{p(t)}\right]$ is the expected gain of the price path that depends on changes in the fundamentals and all results so far concern expectations, the price paths are allowed to be random walks around the fundamental value when $\bar{\mu}(t)$ satisfies the condition.

### 3.2 Risk-Adjusted Expected Return

For sure, there are some possible critics to this result. The assumption that there are short time trends in expected returns (that can be caused by changes in fundamentals) should not be a point of critics. The argument that the trader in practice has to achieve a positive gain on average when there are trading costs, in times of flat-rate trading offers is not really a solution of the puzzle. The same is true for the continuous trading assumption when considering high frequency trading. However, there is one argument against the discounted SLS rule that puzzles me: the risk-adjustment.

Classically, the risk argument is given by the defenders of the market efficiency hypothesis when someone finds an external variable that allows for estimating higher expected returns of an asset. Then it is said that this external variable is just a better proxy for measuring risk and so it is concluded that the asset under investigation is more risky, which allows the asset to be more profitable (on average) without being a counter example to market efficiency. In the setting of this paper, this is not applicable since there is only one asset under analysis. Here, only different trading strategies are considered. The only way to apply the risk adjustment argument to the $\text{SLS}_\delta$ rule is to use volatility (standard deviation; which is not a risk measure in the sense of mathematical finance), which we will do next. At the end of the paper, the risk of the SLS rule and other definitions of it (cf. skewness) are discussed again. But for now, I rely on the most common way.

For the remainder of this section, the analysis will be restricted to the standard SLS rule, i.e., we set $\delta = 1$. For calculating the standard deviation of the SLS strategy, an assumption on the volatility of the underlying price process is needed. Analogous to the definition of the trend, it is set

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2\right] =: \sigma^2_{h;t-h} > 0.$$

Note that also here a market parameter, namely $\sigma^2_{h;t}$, depends on $h$, which is chosen by the trader. However, the same argument as for $\mu_{h;t}$ holds (cf. Baumann and Grüne, 2017).

With this assumption it follows

$$\mathbb{E}[p_t^2] = p_0^2 \cdot \prod_{i=1}^{n}((\sigma^2_{h;ih} + 2\mu_{h;ih})h + 1)$$

and

$$\mathbb{E}[p_{t_2}^2|\mathcal{F}_{t_1}] = p_{t_1}^2 \cdot \prod_{i=n_1+1}^{n_2}((\sigma^2_{h;ih} + 2\mu_{h;ih})h + 1).$$

Again, I start the analysis of the SLS strategy with its long side. Using the definition of $I_t^L$ and $g_t^L$ leads to

$$\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)^2 = \frac{K^2}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2$$

and so

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)^2\right] = K^2\sigma^2_{h;t-h}.$$

It holds

$$\mathbb{E}[(I_t^L)^2] = I_0^{*2} \cdot \prod_{i=1}^{n}((K^2\sigma^2_{h;ih} + 2K\mu_{h;ih})h + 1).$$

Again by the definition of $I_t^L$ it follows:

$$\mathbb{E}[(g_t^L)^2] = \frac{I_0^{*2}}{K^2}\left(\prod_{i=1}^{n}((K^2\sigma_{h;ih}^2 + 2K\mu_{h;ih})h + 1) - 2\prod_{i=1}^{n}(K\mu_{h;ih}h + 1) + 1\right)$$

Once more, by substituting $I_0^* \to -I_0^*$ and $K \to -K$ the formula for $\mathbb{E}[g_t^S]$ follows. For calculating the standard deviation of the SLS strategy's gain, the "mixed" expectation of the long and the short side $\mathbb{E}[g_t^L g_t^S]$ are needed, too.

It holds:

$$\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)\left(\frac{I_t^S - I_{t-h}^S}{I_{t-h}^S}\right) = -\frac{K^2}{h}\left(\frac{p_t - p_{t-h}}{p_{t-h}}\right)^2$$

and

$$\mathbb{E}\left[\frac{1}{h}\left(\frac{I_t^L - I_{t-h}^L}{I_{t-h}^L}\right)\left(\frac{I_t^S - I_{t-h}^S}{I_{t-h}^S}\right)\right] = -K^2\sigma_{h;t-h}^2$$

With that it follows

$$\mathbb{E}[I_t^L I_t^S] = -I_0^{*2}\cdot\prod_{i=1}^{n}(-K^2\sigma_{h;ih}^2 h + 1).$$

Now, by the definitions of $I_t^L$ and $I_t^S$ it follows:

$$\mathbb{E}[g_t^L g_t^S] = \frac{I_0^{*2}}{K^2}\left(\prod_{i=1}^{n}(-K^2\sigma_{h;ih}^2 h + 1) - \prod_{i=1}^{n}(K\mu_{h;ih}h + 1) - \prod_{i=1}^{n}(K\mu_{h;ih}h + 1) + 1\right)$$

Now, all components needed for the calculation of $\mathbb{E}[(g^{SLS}(t))^2] = \mathbb{E}[(g^L(t))^2] + 2\mathbb{E}[g^L(t)g^S(t)] + \mathbb{E}[(g^S(t))^2]$ and $Var(g^{SLS}(t)) = \mathbb{E}[(g^{SLS}(t))^2] - (\mathbb{E}[g^{SLS}(t))])^2$ are known. To keep the computation simple, I calculate the limit for continuous time trading $h \to 0$ and define $\bar{\mu}(t) := \int_0^t \mu(s)ds$ and $\overline{\sigma^2}(t) := \int_0^t \sigma^2(s)ds$ (of course, $\sigma^2(t)$ has to be Riemann integrable as well). By use of the Vito Volterra style product integral, it follows:

$$\begin{aligned}
\mathbb{E}[(g^{SLS}(t))^2] &= \mathbb{E}[(g^L(t))^2 + (g^S(t))^2 + 2g^L(t)g^S(t)] \\
&= \frac{I_0^{*2}}{K^2}\Big(exp(K^2\overline{\sigma^2}(t) + 2K\bar{\mu}(t)) - 2exp(K\bar{\mu}(t)) + 1 \\
&\quad + exp(K^2\overline{\sigma^2}(t) - 2K\bar{\mu}(t)) - 2exp(-K\bar{\mu}(t)) + 1
\end{aligned}$$

$$+ 2(exp(-K^2\overline{\sigma^2}(t)) - exp(K\bar{\mu}(t)) - exp(-K\bar{\mu}(t)) + 1)\Big)$$

Combining the results for $\mathbb{E}[g^{SLS}(t)] = \frac{I_0^*}{K}(exp(K\bar{\mu}(t))+exp(-K\bar{\mu}(t))-2)$ and $\mathbb{E}[(g^{SLS}(t))^2]$ leads to the formula for the SLS rule's variance:

$$\begin{aligned} Var(g^{SLS}(t)) = \frac{I_0^{*2}}{K^2}\Big(&(exp(K^2\overline{\sigma^2}(t)) - 1)(exp(2K\bar{\mu}(t)) + exp(-2K\bar{\mu}(t))) \\ &+ 2(exp(-K^2\overline{\sigma^2}(t)) - 1)\Big) \end{aligned}$$

This expression fits exactly the results obtained by Baumann (2016) for MJDM (and the GBM).

It is easy to see that for the expected gain of a simple buy-and-hold (bnh) strategy with initial investment $I_0^*$ it holds

$$\mathbb{E}[g^{bnh}(t)] = I_0^*(exp(\bar{\mu}(t)) - 1)$$

and for the respective variance

$$Var(g^{bnh}(t) = I_0^{*2}exp(2\bar{\mu}(t))(exp(\overline{\sigma}(t)) - 1),$$

for example by using the results for $g^L(t)$ and setting $K = 1$. As suggested by Malkiel (1973), I am going to compare the risk-adjusted returns of SLS rule with that one of a buy-and-hold strategy. For any strategy $\ell$ let

$$rar(\ell;t) := \frac{\mathbb{E}[g^\ell(t)]}{\sqrt{Var(g^\ell(t))}}$$

be the risk-adjusted return of this strategy at time $t$.

Even when comparing the risk-adjusted return of the SLS rule with that one of a buy-and-hold strategy, similar to the results for the expected gain, it turns out that for all $t$ with $\bar{\mu}(t) \in (-1, 0) \cup (B_{rar}(K), \infty)$ the SLS rule is the dominant one, too, see Figs. 4, 5, and 6. The value $B_{rar}(K)$ is again depending on $K$ but now it holds: $B_{eg}(K) \rightarrow 0$ for $K \rightarrow 0$. Also in this case, note that $\bar{\mu}(t) \notin [0, B_{rar}(K)]$ means that the SLS is dominant the result is true for all price paths (i.e., random walks) around the fundamental value when $\bar{\mu}(t)$, which depends on the changes of the fundamentals, satisfies the condition.

## 4 Discussion & Conclusion

In the past, most puzzles for market efficiency came from empirical data and statistical methods. The puzzle presented in the work at hand is a purely theoretical, mathematical one. I proved that for all price processes, which are random walks, it holds:

- The expected gain of the discounted $\text{SLS}_\delta$ strategy for all discounting factors $\delta \in [0, 1]$, which includes the standard SLS rule ($\delta = 1$), the expected feedback trading gain is positive when $(\bar{\mu}_{h;t})_t > 0$ or when $(\bar{\mu}_{h;t})_t < 0$ in discrete time.

- The expected gain of the discounted $\text{SLS}_\delta$ strategy for all discounting factors $\delta \in [0, 1]$, which includes the standard SLS rule ($\delta = 1$), the expected feedback trading gain is positive when $\bar{\mu}(t) \neq 0$ in continuous time.

- The expected gain of the standard SLS rule surpasses the expected gain of a simple buy-and-hold strategy for all $t > 0$ with $\bar{\mu}(t) \notin [0, B_{eg}(K)]$, with $B_{eg}(K) \rightarrow 0$ for $K \rightarrow \infty$ in continuous time.

- The risk-adjusted return of the standard SLS rule exceeds the risk-adjusted return of a simple buy-and-hold strategy for all $t > 0$ with $\bar{\mu}(t) \notin [0, B_{rar}(K)]$, with $B_{rar}(K) \rightarrow 0$ for $K \rightarrow 0$ in continuous time.

- Both, the expected gain and the risk-adjusted return of the standard SLS rule exceed the expected gain and the risk-adjusted return of a simple buy-and-hold strategy, respectively, for all $m\bar{u}(t) \in (-1, 0) \cup (max\{B_{eg}(K), B_{rar}(K)\}, \infty)$.

That means, an SLS trader can expect positive gain (even in discrete time) on all arbitrary small intervals where the trend is not changing its sign. Only for that points of time where the trend changes its sign, the SLS trader is facing negative expected gain. Note that the price path itself can change its slope arbitrarily often. When the trend path is to some extent smooth and trading frequency is increased, the points of time where the trend changes its sign do carry less (or, when going to continuous time, even no) weight. Also when the SLS rule is compared to a buy-and-hold rule the expected gain as well as the risk-adjusted return are higher for the SLS rule except for finite intervals of $\bar{\mu}(t)$. Especially this interval for the risk-adjusted return can easily be chosen arbitrary small.

Clearly, there are some assumptions to discuss. For example trading costs would decrease the expected gain of the SLS rule. However, as can be seen in Figs. 4, 5, and 6, the gap between the risk-adjusted returns of the SLS and the buy-and-hold rule is widening heavily when decreasing $K$ or increasing the absolute value of $\bar{\mu}$. Thus, in times of flat-rate trading offers, trading costs are not that important anymore. Continuous time trading is a hard assumption. But since the results of this work do not rely on any price path but only on the trend process and there are high frequency trading possibilities, only a very hard no-trending assumption could invalidate these results. For example, one had to assume that for every point of time with an positive (negative) price trend, for ever arbitrary small interval after that point of time, the price trend has to be non-positive (non-negative). However, this would also imply that there are absolutely no identifiable trends in fundamental values. Adequate resources, perfect liquidity, short selling, and the price taker property are on modern stock exchanges, when both the trader and the traded asset are big enough and $I_0^*$ and $K$ are chosen small enough, can be seen as justified.

If one asked me to give an answer to that puzzle, the only—more or less—satisfying answer I could give is that the risk measure is inappropriate (maybe *skewness* would be better). But there are two problems: First, this only works, when market efficiency is defined via risk-adjusted returns only (and not when it is defined via expected gain). And second, I would run in a problem very similar to the joint hypothesis problem: I suggest for nearly every trading strategy one could find a risk measure so that the risk-adjusted return is higher for the buy-and-hold rule—and one so that is lower. And the other way around, I suggest also that for nearly all risk measures one can find a trading strategy that beats the buy-and-hold strategy and one that is beaten by it. No one could say *whether the risk measure or the market efficiency hypothesis is wrong.* Thus, I rely on a standard definition of risk-adjustment.

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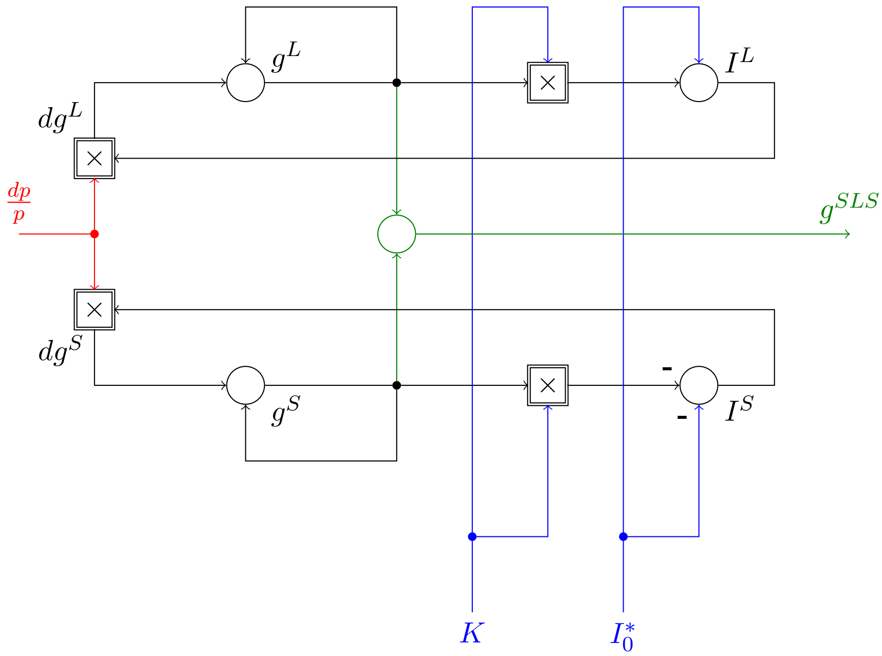


Figure 1: Flow diagram for the standard SLS controller with input (or disturbance) variable return on investment $\frac{dp}{p}$, i.e., price, and output variable gain $g^{SLS}$. The SLS trader's parameters are $K > 0$ and $I_0^* > 0$.

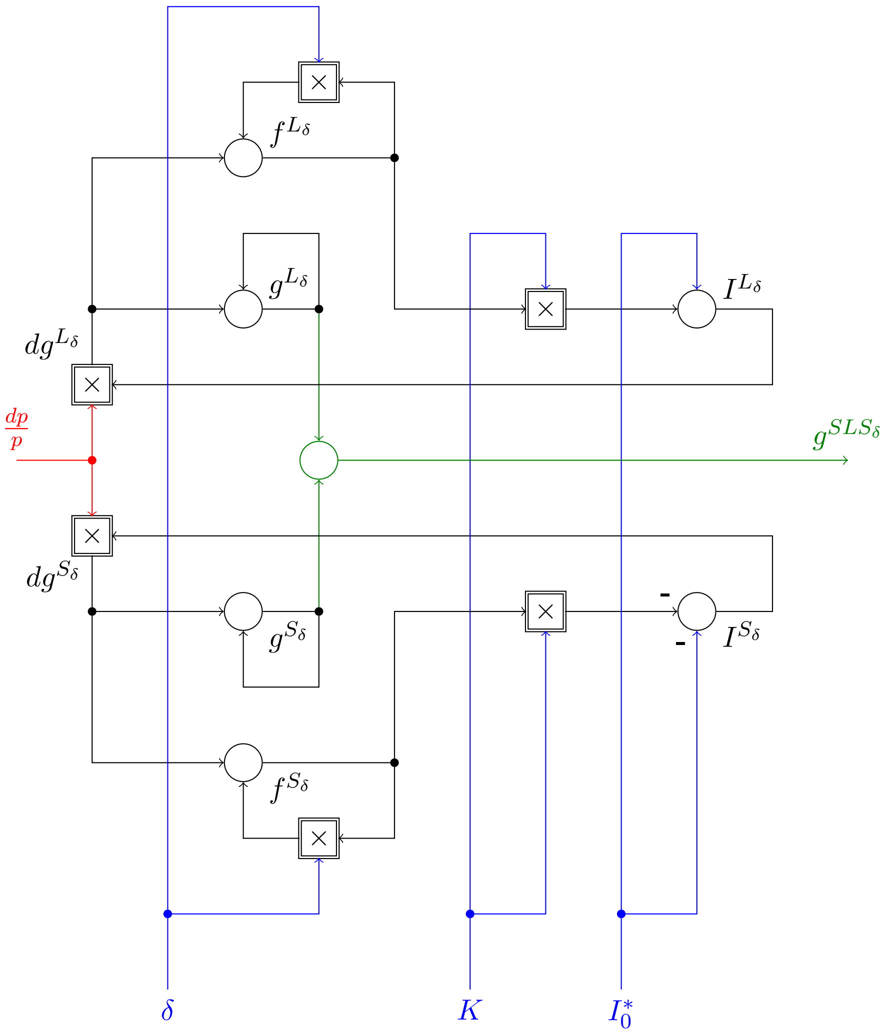


Figure 2: Flow diagram for the discounted SLS controller with input (or disturbance) variable return on investment $\frac{dp}{p}$, i.e., price, and output variable gain $g^{SLS_\delta}$. The SLS$_\delta$ trader's parameters are $K > 0$, $I_0^* > 0$, and $\delta \in [0, 1]$.

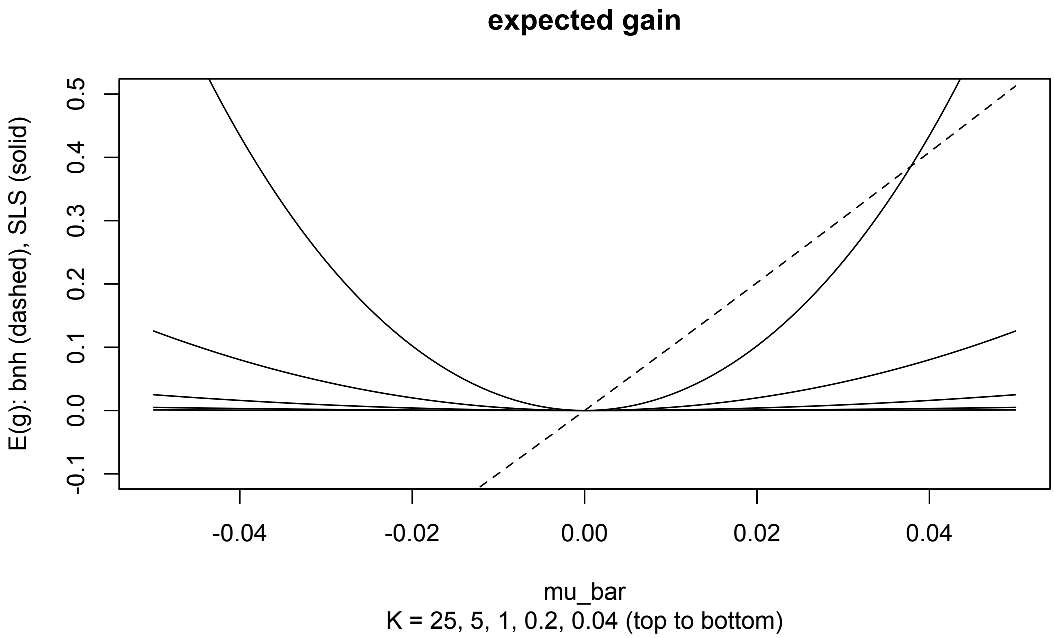


Figure 3: Return on investment of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = 25, 5, 1, \frac{1}{5}, \frac{1}{25}$ (from the top to the bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is in $\bar{\mu} \in [-5\%, 5\%]$.

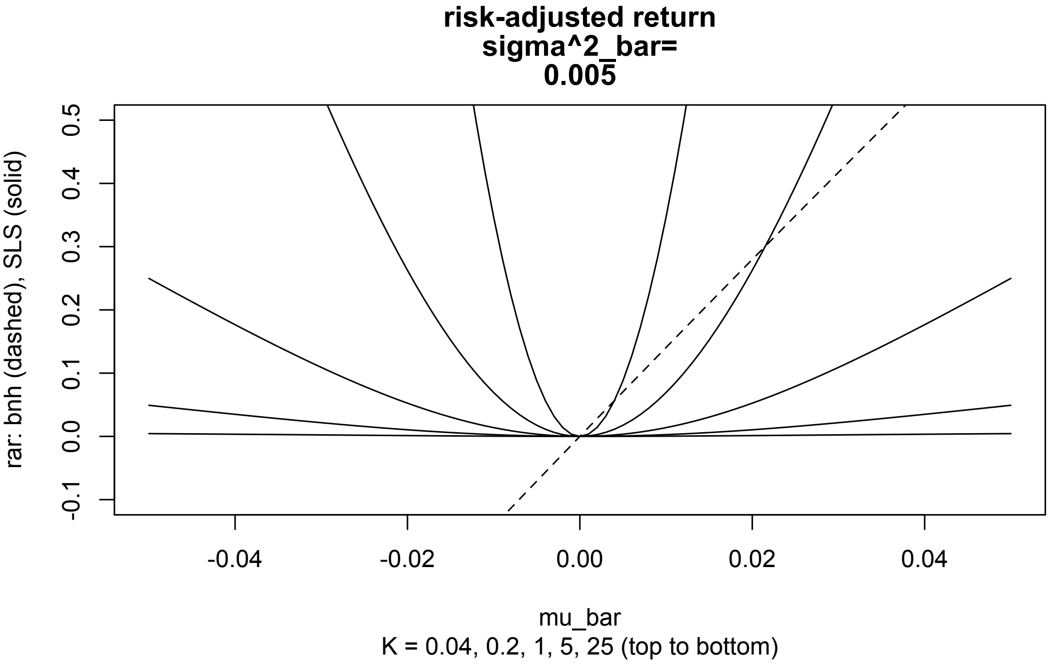


Figure 4: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{25}, \frac{1}{5}, 1, 5, 25$ (from the top to the bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is in $\bar{\mu} \in [-5\%, 5\%]$ and the average volatility is $\overline{\sigma^2} = 0.5\%$.

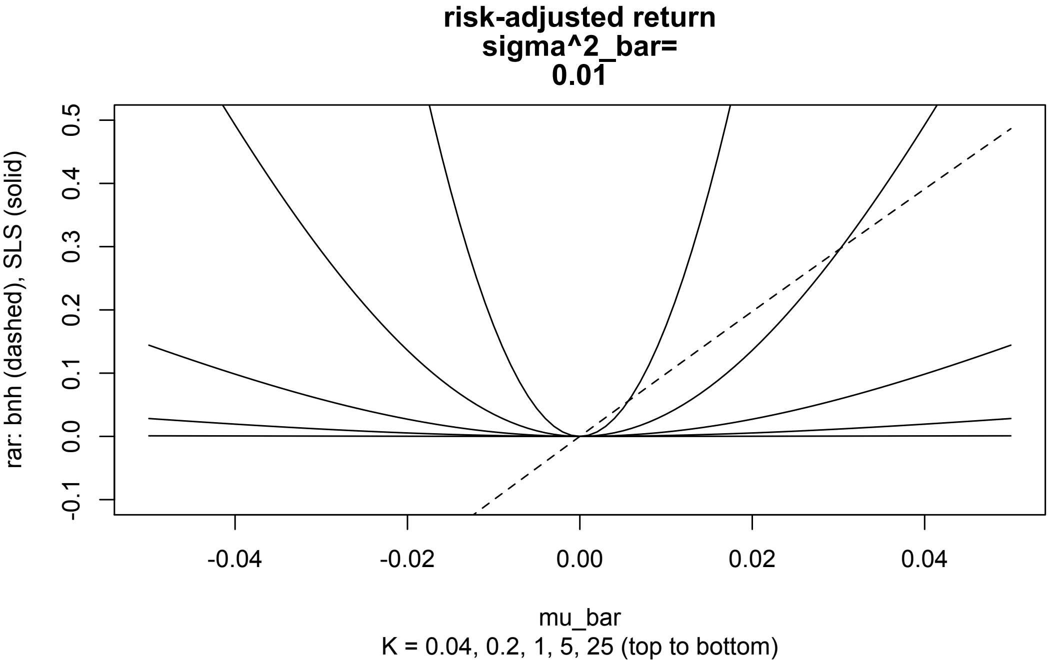


Figure 5: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{25}, \frac{1}{5}, 1, 5, 25$ (from the top to the bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is in $\bar{\mu} \in [-5\%, 5\%]$ and the average volatility is $\overline{\sigma^2} = 1\%$.

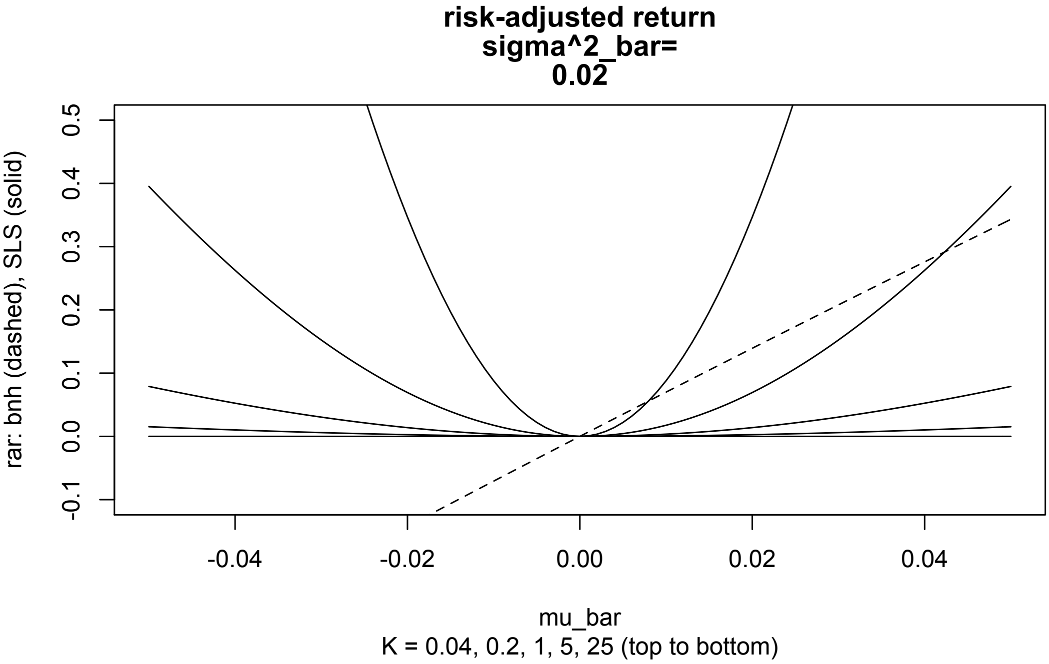


Figure 6: Risk-adjusted return of different SLS strategies (solid lines) with $I_0^* = 10$ and $K = \frac{1}{25}, \frac{1}{5}, 1, 5, 25$ (from the top to the bottom) compared to the risk-adjusted return of a simple buy-and-hold strategy (dashed line) with initial investment 10. All returns are adjusted with the respective standard deviation. The average trend is in $\bar{\mu} \in [-5\%, 5\%]$ and the average volatility is $\overline{\sigma^2} = 2\%$.