Perils of unconventional monetary policy[1]

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Abstract

Unconventional monetary policy, by relaxing restrictions on the composition of the balance sheet of the central bank, compromises control over the stochastic path of inflation. If the composition of the portfolio is unrestricted (either left to market demands or governed by a rule that depends on expected inflation rates) then a unique path of inflation cannot be implemented. This is the case under pure quantitative easing where the target is the size of real money balances. In contrast, credit easing policies restricts the composition of the portfolio by targeting a specific expansion in the maturity profile of bonds bought, and so can implement a unique path of inflation.

Keywords: unconventional monetary policy; path of inflation.

JEL Codes: D50; E31; E52

Conventional monetary policy restricts assets on the balance sheet of the central bank to short-term Treasury Bills. Much analysis takes this as given and, as a result, the importance of restrictions on the central bank asset portfolio has typically been overlooked. Unconventional monetary policy relaxes the restrictions on the central bank asset portfolio and allows for assets of varying maturity and risk profiles. In this paper we explore what this (potentially) less-restricted portfolio means for the ability of the central bank to control the stochastic path of inflation.

We consider a stochastic cash-in-advance economy with flexible prices and a perfect, in particular complete, asset market, and we restrict attention to trades in securities of one-period maturity; trades in long-lived assets can duplicate such trades, as in Kreps (1982), and allow for a role for the maturity structure of debt, as in Cochrane (2001) or Angeletos (2002). Well-founded criticisms of the fiscal theory of the price level in Buiter (2002) and Drèze and Polemarchakis (2000) notwithstanding, we make the assumption of non-Ricardian seigniorage policy for the central bank; this, to remain in an environment that, under conventional monetary policy, yields a determinate price level. Our conclusion, that, surprisingly, has gone unnoticed, is that monetary policy, that sets a path of short-term, nominal interest rates, determines the path of expected or average inflation, but not the distribution of possible paths of inflation. The key result is that the stochastic path of inflation is determined by the adjustment of the portfolio of the monetary authority over time in response to market forces and expectations. Without adequate restrictions on the asset portfolio, indeterminacy is pervasive.

In other words, future inflation outcomes depend on the portfolio choices today. This is because, as uncertainty unfolds, market forces and expectations determine the value of the assets on the balance sheet. Variations in the value of the balance sheet drive the stochastic path in which money is injected or withdrawn which determines the path of inflation. Once portfolio allocations are selected, the distribution of inflation outcomes is unique. In fact, with full knowledge of the structure of the economic environment, the central bank could choose the portfolio weights to target a specific distribution of inflation. However, if the portfolio allocations are left unrestricted, the distribution of inflation is not pinned down.

Under a conventional (restricted) central bank asset portfolio, there is a unique stochastic distribution of inflation outcomes even though central banks typically target expected inflation. That is, even though the restrictions on the portfolio of assets was likely driven more by risk aversion of the central bank, it actually meant that the central bank could target expected inflation and keep the distribution of inflation anchored. However, when a central bank shifts to unconventional policies, as was the case in response to hitting the zero lower bound (ZLB), there will be change in the distribution of inflation outcomes driven by the change in the central bank portfolio. The nature of the change in the stochastic inflation distribution depends on the nature of the change in portfolio policy.

While a common feature of unconventional policies is that these policies have expanded the central bank balance sheet, central banks have employed two distinct approaches to these policies which we discuss in detail below; namely, quantitative easing (QE) and credit easing (CE). QE focuses only on the expansion of the central bank liabilities and does not restrict the asset composition of the balance sheet. By contrast, CE targets a specific allocation of assets, much like conventional monetary policy that restricts open market operations to Treasury bills. Under CE, it is the explicit target for the composition of the balance sheet that allows the monetary authority to target the stochastic path of inflation: the target for the composition of the portfolio guarantees the necessary restrictions to obtain determinacy of the inflation distribution and limit the de-anchoring of the inflation distribution.

We also show that even policy rules which select the portfolio weights as a function of forecast inflation (or any other future nominal variable), do not overcome the indeterminacy problem. This may be surprising as the policy rule might seem like a restriction, but we show it effectively leaves portfolio weights unrestricted.

The indeterminacy of QE in our benchmark model is nominal; while the central bank loses the control of inflation, the indeterminacy does not affect the attainable equilibrium allocations. If the central bank were to switch to a money supply, rather than interest rate, policy, the indeterminacy would be real: it would affect real allocations. More importantly, the indeterminacy would be real if prices were sticky or the asset market is incomplete, as in Bai and Schwarz (2006).

There is a vast and important literature on indeterminacy of monetary equilibria: Sargent and Wallace (1975) pointed out the indeterminacy of the initial price level under interest rate policy; Lucas and Stokey (1987) derived the condition for the uniqueness of a recursive equilibrium with money supply policy; Woodford (1994) analysed the dynamic paths of equilibria associated with the indeterminacy of the initial price level under money supply policy. In this paper, we give the exact characterisation of recursive equilibria under quantitative easing with interest rate policy.

The fiscal theory of the price level in Woodford (1994) takes it for granted that a monetary authority trades exclusively in short-term, nominally risk-free bonds. This is also the case in Dubey and Geanakoplos (2003), who argue that "outside money" suffices to eliminate the indeterminacy that prevails in economies with nominally denominated assets; an important claim, because, as noted by Cass (1984, 1985) and analysed in depth in Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989), when the asset market is incomplete, nominal indeterminacy has real effects. Here, we highlight the importance of the composition of the portfolio of the monetary authority for the determinacy of the path of prices, even, the determinacy of the price level.

The possible multiplicity of stochastic inflation paths at equilibrium was clear in Bloise, Drèze, and Polemarchakis (2005) and Nakajima and Polemarchakis (2005b); but there, the specification was Ricardian, equilibria were indeterminate, and the point was to demonstrate that the indeterminacy can be parametrised by the price level and a nominal martingale measure. Magill and Quinzii (2014b) developed the argument that inflationary expectations can serve as an alternative parametrisation, which is more interesting. Drèze and Polemarchakis (2000) pointed out the need for "comprehensive monetary policy" that sets the stochastic path of the term structure of interest rates (or, equivalently, all state-contingent short-term rates) in order to determine the path of inflation. This theme was later developed in Adao, Correia, and Teles (2014), and Magill and Quinzii (2014a). Importantly, in this argument, the way out of indeterminacy involved targets or restrictions on the returns of assets. Our point here is that the the composition of the balance sheet of the monetary authority matters as an instrument of immediate policy relevance.

Our argument does not derive from the infinity of the horizon or the stability of a steady state; Benhabib and Farmer (1999) is a useful survey of this literature. In particular, it applies to a finite horizon, which explains that it applies to recursive equilibria. Although, as long as fiscal policy is Ricardian, the coefficient in the Taylor rule does not change the degree of indeterminacy, it affects the number of locally bounded equilibria as in Woodford (1999) and Benhabib, Schmitt-Grohe, and Uribe (2001); Benhabib, Schmitt-Grohe, and Uribe (2002) examined the interaction of non-Ricardian fiscal policy with the Taylor rule that yields a unique equilibrium. Carlstrom and Fuerst (1998) discussed the indeterminacy of sticky-price equilibria when the nominal interest rate is zero. here, as we show, feedback rules, that set interest rates or the composition of the balance sheet as a function of future variables, are not sufficient to obtain a determinate inflation path. "Simple" inflation processes may only be compatible with conventional monetary policy.

In Curdia and Woodford (2011), if the portfolio under unconventional policies has the same risk-profile (or is collinear) as the portfolio under conventional policy, the unconventional policies may have real effects in the presence of segmented markets. In practice central banks have accommodated trade to include non-collinear assets (bonds of longer maturities or private sector liabilities). It is the trade in these assets that we focus on and our formulation is in the spirit of Eggertsson and Woodford (2003) where the central bank chooses a portfolio among a set of state-contingent assets.

In practice, unconventional policies are employed during times of crisis, when interest rates may be constrained at the zero lower bound, and they target an increase of the size of the balance sheet. We do not consider quantitative aspects of unconventional monetary policy or reasons that motivate changes in the size of the balance sheet. We focus on the implications of changes in the composition of risky assets in the portfolio of the central bank for the stochastic path of inflation. However, we show that interest rates hitting the effective zero lower bound does not alter our main finding.

The portfolio balance channel operates when bonds of different maturities are not perfect substitutes and traders have have maturity-specific bond demands. In this setting, the maturity structure of outstanding debt can affect term premia. Theoretical models describing the portfolio balance channel such as Vayanos and Vila (2009) and Hamilton and Wu (2012) neglect the consequences of variations in the composition of the monetary authority portfolio on the stochastic path of inflation. We show that as the composition of the portfolios of monetary-fiscal authorities determine the stochastic path of prices, they also determine the nominal stochastic discount factor. Independent of changes in expectations about the path of short-term interest rates, the correlation between the discount factor and asset prices, and nominal exchange rates, then generates risk premia and biases whose size and direction corresponds to the chosen portfolio composition.

Since the global financial crisis of 2007, there is an emerging view that variations in the capital account should be examined if not managed, and that these variations may stem from the monetary policy of trading partners (see, for example, Rey (2013)). Our results contributes to this view by highlighting that QE proliferates indeterminacy in central bank portfolios and consequently the path of exchange rates, and, if markets were incomplete within each country, then fluctuations in central bank portfolios would resonate abroad not only by affecting the nominal exchange rate, but also directly to asset prices and premia globally. Furthermore, if central banks set interest rates according to a Taylor-type rule that accounts for changes in the nominal exchange rate[1], and trading partners conducted QE, then they would not be able to guarantee the desired outcomes can be implemented. In other words, QE by trading partners would manifest itself as indeterminacy of both nominal and real risk-premia, globally, and more importantly, even in countries that conducted traditional monetary policy or CE.

[1] For a discussion on such rules see Taylor (2001).

1 Unconventional monetary policy in practice

Before turning to the model, we first explore the practical adoption of unconventional monetary policy in major economies. There is general agreement on the objective of unconventional policies as a mechanism to support credit and liquidity. However the implementation of policies vary as Bernanke (2009) explains:

> "*The Federal Reserve's approach to supporting credit markets is conceptually distinct from quantitative easing (QE), the policy approach used by the Bank of Japan from 2001 to 2006. Our approach–which could be described as 'credit easing' (CE)–resembles quantitative easing in one respect: It involves an expansion of the central bank's balance sheet. However, in a pure QE regime, the focus of policy is the quantity of bank reserves, which are liabilities of the central bank; the composition of loans and securities on the asset side of the central bank's balance sheet is incidental.*"

Heterogeneity of central bank implementation of unconventional policies which makes our theoretical distinction between the extreme cases of pure QE and pure CE particularly relevance. While no central bank has pursued pure QE or pure CE, some central banks have pursued policies much closer to CE and others policies much closer to QE. The key characteristic that makes the Fed's policies closer to CE is that it sought to expand it's balance sheet while committing to a specific asset composition in doing so.[2] As part of this, the Federal Reserve Bank of New York published how (in terms of portfolio weights) the total Large Scale Asset Purchases (LSAP) program purchases would be distributed across maturity sectors. Moreover, each two-week-long round of asset purchases would begin every-other Wednesday when the SOMA Desk would announce, along with the specific days on which it would be conducting the auctions, the maturity sectors in which it would be buying over the subsequent two weeks. In our model the indeterminacy is resolved by ex-ante restrictions on the composition of assets held on the central bank balance sheet.

[2] The Federal Reserve reduced the target federal funds rate to effectively zero and also implemented a number of other programs and policies which led to significant changes to the Federal Reserve's balance sheet. For a discussion of Fed policies see, for example, Bernanke (2009), Goodfriend (2011), Reis (2009) and Fawley and Christopher (2013).

The Fed's approach contrasts somewhat with some other major central banks easing policies. During the first round of QE undertaken between 2001 and 2006, the Bank of Japan (BOJ) set new operational targets for monetary policy in terms of the central bank reserves held by financial intermediaries (called Current Account Balances). To achieve these targets, it made outright purchases of a long-term Japanese government bonds, stocks held by commercial banks (from October 2002 to September 2003) and ABS (July 2003 to March 2006), but the specific portfolio of these assets was not the target of the central bank. Recent unconventional policies by the BOJ have targeted lending to banks rather than the outright purchase of assets from secondary markets though Fawley and Christopher (2013) argue that the BOJ was mainly concerned with generating reserves and provided limited restrictions on the range of assets. One may argue that the assets purchased by the BOJ were motivated by expectations of prices and premia. We show that even such a policy is insufficient to rule out indeterminacy. Ugai (2007) and Maeda et al. (2005) discuss the details of the BOJ experience with QE in more detail.

Recent unconventional policies by the ECB have also targeted lending to banks rather than the outright purchase of assets from secondary markets. Nonetheless, Fawley and Christopher (2013) argues that these programs may also be "considered pure QE in the sense that they targeted reserves and typically accepted a wide range of assets as collateral". The Bank of England's QE scheme was similar to the early BOJ scheme. The Monetary Policy Committee set an overall target for the amount of assets purchased through the Asset Purchase Facility (APF), the composition of assets was not the target. However, the Bank of England set some restrictions on the asset portfolio (medium- and long-term gilts) making the APF somewhat closer to the Fed, and CE, than the BOJ.[3]

2 The analytical argument

Monetary policy involves *quantitative easing* if open market operations extend to unrestricted portfolios of government bonds of different maturities or

[3]Established in January 2009, the APF was first used as a tool of monetary policy in March 2009. Initially the APF would buy high-grade corporate bonds and government gilts with maturity 5-25 years, but more recently the APF only bought conventional gilts, though over a slightly extended maturity range starting at three years. In any given purchase operation, a broad range of assets, such as gilts with maturities 10-25 years, were up for purchase; a reverse auction determined the allocation and prices, remained determined by market conditions.

bonds issued by the private sector. It involves *credit easing* if open market operations extend beyond treasuries, but still target a specific composition for the balance sheet of the monetary-fiscal authority; as a limit case, monetary policy is *conventional* when open market operations are restricted to short term, nominally risk-free assets (Treasury bills).

Fiscal policy is *Ricardian* if it is restricted to satisfy an intertemporal budget constraint or transversality condition; equivalently, if public debt vanishes for all possible, equilibrium or non-equilibrium, values of prices and interest rates. It is *non-Ricardian*, if it is not restricted to satisfy an intertemporal budget constraint; in particular, outside money or initial liabilities of the public towards the private sector are not taxed back.

Quantitative easing generates indeterminacy indexed by a *nominal pricing measure* over states of the world. This measure determines the distribution of rates of inflation, up to a moment that is determined by the risk-free rate and non-arbitrage. Ricardian policy leaves the initial price level indeterminate as well. Determinacy and, by extension, monetary and financial stability, obtain under credit easing or monetary policy that is conventional. The indeterminacy is nominal only as long as prices are flexible, monetary policy sets nominal rates of interest, and the asset market is (effectively) complete; otherwise, there are, generically, real effects. It is worth pointing as our analysis considers a process of continual re-balancing of the monetary-fiscal authority balance sheet, our argument applies equally to the unwinding of quantitative easing as well as to the initiation of it. What is essential is the type of policy that determines the stochastic evolution of the balance sheet.

In Section 2.1 and 2.2 we characterize unconventional monetary policy under pure quantitative easing where the composition of the assets traded by the central bank is unrestricted. We show the indeterminacy inherent in a stochastic economy and link it to the mix of interest and non-interest bearing assets traded by the monetary-fiscal authority. In Section 2.3 we show that this is not a consequence of non-stationary equilibria or of exogenous interest rate paths, and we make explicit the role of the composition of the portfolio of the monetary-fiscal authority portfolio in the determination of stochastic inflation rates. In the presence of pure quantitative easing, interest-rate feedback rules are insufficient to obtain determinacy. We then show that pure credit easing policies (which set portfolio weights exogenously) obtains determinacy while policies that allow for feedback rules determining the composition of assets is insufficient to rule out indeterminacy. Finally, restricting attention to "simple" inflation processes may only be compatible with conventional monetary policy.

2.1 Stochastic Monetary Model

Consider a 3-period monetary model in which activity extends over dates $t = 0, 1$, while a final third date, $t = 2$, serves for accounting purposes.[4] Uncertainty over states of the world, $s \in \{1 \ldots, s, \ldots S\}$, is realised at $t = 1$; each state occurs with probability $f(\cdot)$. These states could be purely extrinsic or they could refer to some other "fundamentals" such as monetary policy shocks across states). To maintain notational consistency with later sections of this paper, date events at date t is denoted s^t, with s^0 being one element, $s^1|s^0$ being one of S elements and $s^2|s^1$ being one element.

There is a continuum of utility-maximising households, distributed uniformly over [0, 1]; at dates 0 and 1, the household supplies $l(s^t)$ units of labour to produce perishable output $y(s^t) = l(s^t)$ in exchange for competitive nominal wages $w(s^t)$, while consumption is $c(s^t)$ and the price level is $p(s^t)$. As real wages are 1, equilibrium nominal wages equal the price level at each-date event.

Utility is derived from consumption of goods and leisure and the intertemporal utility is separable and that flow utility function is continuously differentiable, strictly increasing, strictly concave and satisfies boundary conditions. The lifetime utility of the household is

$$u(c(s^0), 1 - l(s^0)) + \beta \sum_{s^1} u(c(s^1), 1 - y(s^1)) f(s^1). \tag{1}$$

Households cannot use their labour income to purchase goods but must instead use cash obtained from the asset market. The asset market is assumed to open (and close) before the goods market, and so any cash proceeds from the sale of output must be carried over to the next period.[5]

A household enters date 0 with nominal wealth $\tau(s^0)$; the asset market is then open and cash $\hat{m}(s^0)$ and a complete set of contingent claims are traded. The household purchases an amount $\theta(s^1|s^0)$ of the elementary security that pays one unit of currency in state s^1 and zero otherwise; the price of this contingent claim is $q(s^1|s^0)$. While the asset market is open, the household faces the budget constraint

$$\hat{m}(s^0) + \sum_{s^1|s^0} q(s^1|s^0)\theta(s^1|s^0) \leq \tau(s^0). \tag{2}$$

[4] In particular, debts are settled in this final period and there is no uncertainty after period $t = 1$.

[5] Our timing and monetary structure closely follows the cash-in-advance models in Lucas and Stokey (1987) and Nakajima and Polemarchakis (2005b).

$r(s^0)$ is the nominally risk-free one-period interest rate ate date 0; the no arbitrage condition yields

$$\frac{1}{1+r(s^0)} = \sum_{s^1} q(s^1|s^0). \tag{3}$$

The cash-in-advance constraint (CIA) implies that the household must have sufficient cash $(\hat{m}(s^0))$ to cover its purchases:

$$p(s^0)c(s^0) \leq \hat{m}(s^0). \tag{4}$$

The cash brought by the household into the next period $(m(s^0))$ is

$$m(s^0) = \hat{m}(s^0) - p(s^0)z(s^0) \tag{5}$$

where we $z(s^0) = c(s^0) - l(s^0)$ is the net demand of the household at date 0 (with a similar definition for $z(s^1|s^0)$).

Equation (5) and (4) can be combined to write the cash-in-advance constraint as[6]

$$m(s^0) \geq p(s^0)y(s^0). \tag{6}$$

Multiplying both sides of (6) by $\frac{r(s^0)}{1+r(s^0)}$, together with the requirement that $r(s^0) \geq 0$, we can express our CIA constraint in the more useful form which holds with equality:[7]

$$\frac{r(s^0)}{1+r(s^0)}m(s^0) = \frac{r(s^0)}{1+r(s^0)}p(s^0)y(s^0) \tag{7}$$

At the start of the second period, nature determines the state and household wealth is:

$$\tau(s^1|s^0) = m(s^0) + \theta(s^1|s^0) \tag{8}$$

Equations (8) and (5) can be used to substitute $\hat{m}(s^0)$ and $\theta(s^1|s^0)$ out of (2). Using (7) to substitute out $m(s^0)$ and applying the definition of net demand $(z(s^0) = c(s^0) - y(s^0))$, we derive a single equation ((9) below) which represents the constraints (budget and CIA) facing the household in period 0:

$$p(s^0)z(s^0) + m(s^0) + \sum_{s} q(s^1|s^0)\theta(s^1|s^0) \leq \tau(s^0) \tag{9}$$

[6]Rewriting (5) as $\hat{m}(s^0) = m(s^0) + p(s^0)c(s^0) - p(s^0)y(s^0)$, and using (4), (6) follows.

[7]If $r(s^0) = 0$ then both sides are zero while if $r(s^0) > 0$, then utility-maximising households will not wish to hold excess cash between periods as doing so would forego a positive return.

In period 2, the household behaviour is similar except that uncertainty has been fully resolved. Similar reasoning yields a single constraint facing the household:

$$p(s^1|s^0)z(s^1|s^0) + m(s^1|s^0) + \frac{1}{1+r(s^1|s^0)}\theta(s^2|s^1) \leq \tau(s^1|s^0) \tag{10}$$

subject to the cash-in-advance constraint

$$m(s^1|s^0) \geq p(s^1|s^0)y(s^1|s^0). \tag{11}$$

When $t = 2$, the final period, all debts are repaid such that nominal wealth at the end of the period cannot be negative:

$$\tau(s^2|s^1) = m(s^1|s^0) + \theta(s^2|s^1) \geq 0 \tag{12}$$

Given that preferences are such that households strictly prefer more to less, there will be no slack in (12).

Using (10), together with (12) holding with equality, we substitute $\tau(s^1|s^0)$ out of (9) to yield the lifetime budget constraint facing household i:

$$p(s^0)z(s^0) + m(s^0)\frac{r(s^0)}{1+r(s^0)} + \sum_s q(s^1|s^0)z(s^1|s^0) + \sum_s q(s^1|s^0)m(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)} \leq \tau(s^0) \tag{13}$$

The optimization problem of household i is to choose $c(s^0)$, $c(s^1|s^0)$, $y(s^0)$ and $y(s^1|s^0)$ so as to maximise (1) subject to (13), (6) and (11).

Optimisation yields the following three household first order conditions for an optimum:

$$\frac{\frac{\partial u[c(s^0),1-l(s^0)]}{\partial c(s^0)}}{\frac{\partial u[c(s^0),1-l(s^0)]}{\partial l(s^0)}} = 1 + r(s^0) \tag{14}$$

$$\frac{\frac{\partial u[c(s^1|s^0),1-l(s^1|s^0)]}{\partial c(s^1|s^0)}}{\frac{\partial u[c(s^1|s^0),1-l(s^1|s^0)]}{\partial l(s^1|s^0)}} = 1 + r(s^1|s^0) \tag{15}$$

$$\frac{\frac{f(s^1|s^0)\partial u[c(s^1|s^0),1-l(s^1|s^0)]}{\partial c(s^1|s^0)}}{\frac{\partial u[c(s^0),1-l(s^0)]}{\partial c(s^0)}} = \tilde{q}(s^1|s^0) = \frac{q(s^1|s^0)p(s^1|s^0)}{p(s^0)} \tag{16}$$

Equations (14) and (15) are standard intratemporal conditions which equate the ratio of marginal utilities from leisure and consumption with the gross interest rate. Equation (16) is the intertemporal Euler equation which equates the ratio of the expected marginal utilities of consumption across periods with the appropriately defined relative price ratio.

The consolidated monetary-fiscal authority sets the nominal interest rate $r(s^0)$ and $r(s^1|s^0)$, manages its asset portfolio by choosing the securities which it trades $(\delta(s^1|s^0))$. The monetary fiscal authority enters with liabilities $T(s^0)$. The monetary fiscal authority prints money $M(s^0)$ which it introduces to the economy either via open market operations in the asset market. This yields the following conditions for period 0 and period 1:

$$M(s^0) + \sum_s q(s^1|s^0)B(s^1|s^0) = T(s^0) \tag{17}$$

$$T(s^1|s^0) = M(s^0) + B(s^1|s^0) \tag{18}$$

$$M(s^1|s^0) + \frac{1}{1 + r(s^1|s^0)}B(s^2|s^1) = T(s^1|s^0) \tag{19}$$

$$T(s^2|s^1) = M(s^1|s^0) + B(s^2|s^1) \tag{20}$$

We write the composition of the monetary fiscal authority portfolio in each security $(B(s^1|s^0))$ as a security specific share $(\delta(s^1|s^0))$ times a measure of the total size of the holdings (D_1):

$$B(s^1|s^0) = \delta(s^1|s^0)D_1 \tag{21}$$

The present-value budget constraint is

$$M(s^0)\frac{r(s^0)}{1 + r(s^0)} + \sum_{s^1|s^0} q(s^1|s^0)M(s^1|s^0)\frac{r(s^1|s^0)}{1 + r(s^1|s^0)} = T(s^0). \tag{22}$$

If initial liabilities, $T(s^0)$, vary with prices to satisfy (22) then the fiscal-policy regime is Ricardian, otherwise it is non-Ricardian. The distinction between the two regimes results in either the initial price level being indeterminate or not. As our focus is on the distribution of second period prices, the fiscal-policy regime is not important for our results.

Conventional and unconventional measures are defined as:

Definition 1. Conventional monetary policy trades only of risk-free securities $(\delta(s^1|s^0) = \frac{1}{s},\ \forall s)$.

Definition 2. QE policies leave the portfolio shares unrestricted, or, equivalently, depend on expected inflation rates.

Definition 3. CE policies restrict the portfolio shares, or, equivalently, do not depend on expected inflation rates. CE does not trade only risk-free securities.

In addition to agents optimizing, the following conditions must hold across households in equilibrium:

$$c(s^0) = y(s^0) \qquad c(s^1|s^0) = y(s^1|s^0) \tag{23}$$

$$m(s^0) = M(s^0) \qquad m(s^1|s^0) = M(s^1|s^0) \tag{24}$$

$$\tau(s^0) = T(s^0) \qquad \tau(s^1|s^0) = T(s^1|s^0) \tag{25}$$

$$\tau(s^2|s^1) = T(s^2|s^1) \tag{26}$$

Equilibrium requires that excess demand vanishes $z(s^0) = c(s^0) - y(s^0) = 0$ and $z(s^1|s^0) = c(s^1|s^0) - y(s^1|s^0) = 0$. Using this and the first order conditions (14) and (15) we obtain the allocation. Given the allocation and (16) we obtain the real price of the state-contingent bond, $\tilde{q}(s^1|s^0)$. From the present value budget constraint and cash-in-advance constraints we obtain an expression for the real value of initial nominal wealth. Under a non-Ricardian fiscal policy, this then determines the initial price level. Without further restrictions, the indexed bond price cannot be uniquely decomposed and the distribution of inflation rates is left undetermined.

Let a tilde over a variable denote the real value of the variable. The second period household budget constraint and market clearing gives us in equilibrium

$$p(s^1|s^0)z(s^1|s^0) + m(s^1|s^0)\frac{r(s^1|s^0)}{1 + r(s^1|s^0)} = \theta(s^1|s^0) + m(s^0) = \delta(s^1|s^0)D_1 + m(s^0) \tag{27}$$

$$1 + \pi(s^1|s^0) = \frac{\delta(s^1|s^0)\tilde{D}_1 + \tilde{m}(s^0)}{\tilde{m}(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)}}. \tag{28}$$

If $\delta(s^1|s^0)$ does not depend on the expected rate of inflation, we have S relationships between the expected inflation rates and the real scale of debt and the real value of money balances.[8]

The no-arbitrage condition (3) is used to express security prices $q(s^1|s^0)$ as depending on the nominal equivalent Martingale measure $\nu(s^1|s^0)$ in the

[8] On the other hand, suppose that the portfolio weights depend on the expected inflation rate and are of the form $\delta(s^1|s^0) = \frac{1+\pi(s^1|s^0)}{\sum_{s^1|s^0} 1+\pi(s^1|s^0)} \times constant_s$ such that $\sum_{s^1|s^0} \delta(s^1|s^0) = 1$. In this case, the state-contingent inflation rates cancel out and what remains is the sum. It will be clear in what follows that this precludes uniquely determining the state-contingent inflation rates.

form:

$$q(s^1|s^0) = \frac{\nu(s^1|s^0)}{1+r(s^0)} \tag{29}$$

$$\nu(s^1|s^0) = \tilde{q}(s^1|s^0)\frac{1+r(s^0)}{1+\pi(s^1|s^0)}. \tag{30}$$

The sum of the martingale probabilities being 1. If interest rates are positive real money balances are $\tilde{m}(s^0) = y(s^0)$ and $\tilde{m}(s^1|s^0) = y(s^1|s^0)$ and we obtain one equation to determine the scale of debt.

With the real scale of debt in hand, and (28), we obtain the distribution of inflation rates as a function of the portfolio weights. If policy is conventional or one of credit easing, then the portfolio weights are *restricted* and policy uniquely determines the path of inflation. On the other hand, under quantitative easing, the portfolio weights are *unrestricted* (either free or depends on expected inflation), and a unique path of inflation cannot be implemented by policy.

In Eggertsson and Woodford (2003) when real money balances are not saturated, the central bank portfolio uniquely determines the path of prices. This is because the composition of the portfolio depends on current state variables and is *restricted*. If QE focuses exclusively on expansion of real money balances, then the assumption made in their paper is a strong one. A relaxation of this assumption leaves the portfolio *unrestricted* even when the zero lower bound is not binding, the central bank loses control of the path of inflation.

Although our derivations above used positive interest rates, this was only for analytical convenience and our results remain robust to scenarios where the economy is either temporarily or permanently at the zero lower bound. One may also be concerned that our analysis precludes the expansion of real money balances through unconventional policy. When interest rates are positive, the path of real money balances are determined solely from the path of the real interest rate. In contrast, when interest rates are zero, the composition of assets held by the monetary-fiscal authority determines the path of real money balances.

Temporary Zero Lower Bound

Suppose that the economy is temporarily at the zero lower bound; at date 0 interest rates are zero but in the second period the economy exits and interest rates are positive. If date 0 interest rates were 0, then real money balances may be greater than real income at date 0; we lose one equation.

The real monetary fiscal-authority present-value budget constraint when date 0 interest rates are 0 is

$$\sum_s \tilde{q}(s^1|s^0)\tilde{M}(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)} = \frac{W(s^0)}{p(s^0)}. \tag{31}$$

and under a non-Ricardian policy gives us the initial price level.

The real date-zero budget constraint in equilibrium gives

$$\tilde{M}(s^0) + \tilde{D}_1 \sum_s \frac{\tilde{q}(s^1|s^0)}{1+\pi(s^1|s^0)}\delta(s^1|s^0) = \tilde{w}(s^0) \tag{32}$$

Note that $\frac{\tilde{q}(s^1|s^0)\delta(s^1|s^0)}{1+\pi(s^1|s^0)} = \tilde{q}(s^1|s^0)\delta(s^1|s^0)\frac{z(s^1|s^0)+\tilde{M}(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)}}{\delta(s^1|s^0)\tilde{D}_1+\tilde{M}(s^0)}$ and if interest rates are positive real money balances are $\tilde{m}(s^0) = y(s^0)$ and $\tilde{m}(s^1|s^0) = y(s^1|s^0)$. Furthermore $\frac{W(s^0)}{p(s^0)} = \tilde{w}(s^0)$ and $\tilde{M}(s^1|s^0) = \tilde{m}(s^1|s^0)$. In equilibrium, (32) becomes

$$\tilde{m}(s^0) + \tilde{D}_1 \sum_s \tilde{q}(s^1|s^0)\delta(s^1|s^0)\frac{\tilde{m}(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)}}{\delta(s^1|s^0)\tilde{D}_1+\tilde{m}(s^0)} = \sum_s \tilde{q}(s^1|s^0)\tilde{m}(s^1|s^0)\frac{r(s^1|s^0)}{1+r(s^1|s^0)} \tag{33}$$

and, together with the no-arbitrage condition, the real scale of debt can be solved uniquely if monetary policy is restricted. Importantly, the real value of date 0 money balances depends on the portfolio shares; balance policy determines both the size of real money balances and the path of inflation. Note that the Ricardian/non-Ricardian fiscal policy distinction is not driving the results, and only requires that fiscal policy is chosen to be compatible with equilibrium.

Permanent Zero Lower Bound

If interest rates are zero in both periods, we require a more fully articulated fiscal policy though again, the Ricardian/non-Ricardian policy distinction is not important. Let the monetary-fiscal authority levy indexed transfers at all date-events, in addition to the initial nominal liabilities.

The equilibrium monetary-fiscal authority real budget constraints now become, with all interest rates zero,

$$\tilde{m}(s^0) + \tilde{D}_1 \sum_{s^1|s^0} \tilde{q}(s^1|s^0)\delta(s^1|s^0) + g(s^0) = \frac{T(s^0)}{p(s^0)} \tag{34}$$

$$(1+\pi(s^1|s^0))g(s^1|s^0) = \tilde{D}_1\delta(s^1|s^0) + \tilde{m}(s^0) \tag{35}$$

and the real present-value budget constraint is

$$g(s^0) + \sum_{s^1|s^0} \tilde{q}(s^1|s^0) g(s^1|s^0) = \frac{T(s^0)}{p(s^0)}, \tag{36}$$

where g is the real value of the indexed nominal transfer. The household optimality conditions remain the same. The real scale of debt and the real value of date 0 money balances is solved from:

$$\tilde{m}(s^0) + \tilde{D}_1 \sum_{s} \tilde{q}(s^1|s^0)\delta(s^1|s^0) = \sum_{s} \tilde{q}(s^1|s^0) g(s^1|s^0) \tag{37}$$

$$\sum_{s^1|s^0} \tilde{q}(s^1|s^0) g(s^1|s^0) \frac{1 + r(s^0)}{\tilde{D}_1 \delta(s^1|s^0) + \tilde{m}(s^0)} = 1 \tag{38}$$

where the second equation is the no-arbitrage condition. Finally the inflation rate is given by (35). Hence, in contrast to conventional policy or credit easing, if the portfolio weights are unrestricted under QE (either free or depends on expected inflation rates) then the path of inflation cannot be uniquely determined from policy.

Date 0 real money balances is given by $\sum_{s^1|s^0} \frac{\tilde{q}(s^1|s^0)g(s^1|s^0)}{\frac{\sum_s \tilde{q}(s^1|s^0)g(s^1|s^0) - \tilde{m}(s^0)}{\sum_{s^1|s^0} \tilde{q}(s^1|s^0)\delta(s^1|s^0)}\delta(s^1|s^0) + \tilde{m}(s^0)} = \frac{1}{1+r(s^0)}$. Loosely speaking, date 0 real money balances are increasing on the correlation between the portfolio shares and the present value of state-contingent real indexed transfers. To be explicit, our results remain robust to a world where unconventional policies simultaneously expand the size of the balance sheet and alter the composition of assets held on it.

Our results on the uniqueness of prices at the zero lower bound contrasts with Eggertsson and Woodford (2003) as fiscal transfers in their model are purely nominal and the presence of indexed transfers there would result in a link between the composition of the portfolio and inflation in their setting.

2.2 A stochastic dynamic economy

For ease of exposition, we now examine infinite horizon equilibria to show that our results do not depend on a finite horizon or on not restricting attention to stationary equilibria. As our results do not depend on interest rates, for analytical convenience we assume interest rates are positive throughout this section.

Time, t, is discrete, and it extends into the infinite future: t $= 1, \ldots$. Events, s^t, at each date are finitely many. An immediate successor of a date-event is $s^{t+1}|s^t$, and, inductively, a successor is $s^{t+k}|s^t$. Conditional on

s^t, probabilities of successors are $f(s^{t+1}|s^t)$ and, inductively, $f(s^{t+k}|s^t) = f(s^{t+k}|s^{t+k-1})f(s^{t+k-1}|s^t)$.

At a date-event, a perishable input, labor, $l(s^t)$, is employed to produce a perishable output, consumption, $y(s^t)$, according to a linear technology:

$$y(s^t) = a(s^t)l(s^t), \quad a(s^t) > 0.$$

A representative individual is endowed with 1 unit of leisure at every date-event. He supplies labor and demands the consumption good, and he derives utility according to the cardinal utility index $u(c(s^t), 1 - l(s^t))$ that satisfies standard monotonicity, curvature and boundary conditions. The preferences of the individual over consumption-employment paths commencing at s^t are described by the separable, von Neumann-Morgenstern intertemporal utility function

$$u(c(s^t), 1 - l(s^t)) + \mathrm{E}_{s^t} \sum_{k>0} \beta^k u(c(s^{t+k}|s^t), 1 - l(s^{t+k}|s^t)),$$

where $0 < \beta < 1$. Balances, $m(s^t)$, provide liquidity services. Elementary securities, $\theta(s^{t+1}|s^t)$, serve to transfer wealth to and from immediate successor date-events. The price level is $p(s^t)$, and the wage rate is $w(s^t) = a(s_t)p(s^t)$, as profit maximization requires. The nominal, risk-free interest rate is $r(s^t)$.

At each date-event, the asset market opens after the uncertainty, s^t, has realized, and, as a consequence, purchases and sales in the markets for labor and the consumption good are subject to standard cash-in-advance constraints; the effective cash-in-advance constraint is[9]

$$a(s^t)p(s^t)l(s^t) \leq m(s^t).$$

Prices of elementary securities are

$$q(s^{t+1}|s^t) = \frac{\nu(s^{t+1}|s^t)}{1 + r(s^t)},$$

with $\nu(\cdot|s^t)$ a “nominal pricing measure” or transition probabilities, which guarantees the non-arbitrage relation

$$\sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) = \frac{1}{1 + r(s^t)}.$$

Inductively,

$$\nu(s^{t+k}|s^t) = \nu(s^{t+k}|s^{t+k-1})\nu(s^{t+k-1}|s^t), \quad k > 1,$$

[9] Nakajima and Polemarchakis (2005b) provide an explicit derivation.

and the implicit price of revenue at successor date-events is

$$q(s^{t+k}|s^t) = \frac{\nu(s^{t+k}|s^{t+k-1})}{1 + r(s^{t+k-1}|s^t)} q(s^{t+k-1}|s^t), \quad k > 1.$$

The individual has initial wealth $\tau(s^1) = \omega$. Initial wealth constitutes a claim against the monetary-fiscal authority; alternatively, it can be interpreted as outside money. It is exogenous in a non-Ricardian specification. In a Ricardian specification, it is set endogenously so as to satisfy the transversality condition imposed on monetary-fiscal policy.

The flow budget constraint is

$$p(s^t)z(s^t) + m(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)\theta(s^{t+1}|s^t) \leq \tau(s^t),$$

where $z(s^t) = c(s^t) - a(s^t)l(s^t)$ is the effective excess demand for consumption.

Wealth at successor date-events is

$$\tau(s^{t+1}|s^t) = \theta(s^{t+1}|s^t) + m(s^t),$$

and, after elimination of the trade in assets, the flow budget constraint reduces to

$$p(s^t)z(s^t) + \frac{r(s^t)}{1 + r(s^t)} a(s^t)p(s^t)l(s^t) + \sum_{s_{t+1}} q(s_{t+1}|s^t)\tau(s_{t+1}|s^t) \leq \tau(s^t).$$

Debt limit constraints are

$$-\tau(s^t) \leq \sum_{k>0} \sum_{s^{t+k}|s^t} q(s^{t+k}|s^t) \frac{1}{1 + r(s^t)} a(s^{t+k})p(s^{t+k}).$$

Alternatively, $\tilde{m}(s^t) = (1/p(s^t))m(s^t)$ are real balances, $\tilde{\tau}(s^t) = (1/p(s^t))$ $\tau(s^t)$ is real wealth, $\pi(s^{t+1}|s^t) = (p(s^{t+1})/(p(s^t)) - 1$ is the rate of inflation, and

$$\tilde{q}(s^{t+1}|s^t) = q(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t)) = \frac{\nu(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t))}{1 + r(s^t)}$$

are prices of indexed elementary securities.

Real wealth at successor date-events is

$$\tilde{\tau}(s^{t+1}|s^t) = \left(\frac{\theta(s^{t+1}|s^t) + m(s^t)}{p(s^t)} \right) \frac{1}{1 + \pi(s^{t+1}|s^t)},$$

and the flow budget constraint reduces to

$$z(s^t) + \frac{r(s^t)}{1+r(s^t)}a(s^t)l(s^t) + \sum_{s_{t+1}} \tilde{q}(s_{t+1}|s^t)\tilde{\tau}(s_{t+1}|s^t) \leq \tilde{\tau}(s^t).$$

First order conditions for an optimum are

$$\tfrac{\partial u(c(s^t),1-l(s^t))}{\partial c(s^t)} = \tfrac{\partial u(c(s^t),1-l(s^t))}{\partial l(s^t)} \left(\tfrac{a(s^t)}{1+r(s^t)}\right)^{-1},$$

$$\beta f(s^{t+1}|s^t)\tfrac{\partial u(c(s^{t+1}),1-l(s^{t+1}))}{\partial c(s^{t+1})}\tilde{q}(s^{t+1}|s^t)^{-1} = \tfrac{\partial u(c(s^t),1-l(s^t))}{\partial c(s^t)},$$

and the transversality condition is

$$\lim_{k\to\infty} \sum_{s^{t+k}|s^t} \tilde{q}(s^{t+k}|s^t)\tilde{\tau}(s^{t+k}|s^t) = 0.$$

The monetary-fiscal authority sets rates of interest and accommodates the demand for balances. It supplies balances, $M(s^t)$, and trades in elementary securities subject to a flow budget constraint that, after elimination of the trade in assets, reduces to

$$T(s^t) \leq \frac{r(s^t)}{1+r(s^t)}M(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t)T(s^{t+1}|s^t),$$

where $T(s^t)$ and, similarly, $T(s^{t+1}|s^t)$ are obligations towards the private sector; initial obligations are $\Omega = T(s^1)$. Ricardian policy imposes on the monetary-fiscal authority the transversality condition

$$\lim_{k\to\infty} \sum_{s^{t+k}|s^t} q(s^{t+k}|s^t)T(s^{t+k}|s^t) = 0$$

or, equivalently, as prices vary, it sets the initial claims of the private sector as

$$\Omega = \frac{r(s^1)}{1+r(s^1)}M(s^1) + \sum_{t>0}\sum_{s^t|s^1} \frac{r(s^t|s^1)}{1+r(s^t|s^1)}q(s^t|s^1)M(s^t|s^1).$$

For equilibrium, it is necessary and sufficient that the excess demand for output vanishes:

$$z(s^t) = c(s^t) - a(s^t)l(s^t) = 0.$$

From the first order conditions for an optimum, this determines the path of employment and consumption:

$$\frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial l(s^t)} \left(\frac{a(s^t)}{1+r(s^t)}\right)^{-1},$$

and, in turn, the prices of indexed elementary securities:

$$\beta f(s^{t+1}|s^t)\frac{\partial u(c(s^{t+1}), 1-l(s^{t+1}))}{\partial c(s^{t+1})}\tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)}.$$

The initial price level serves to guarantee that, at equilibrium, the transversality condition of the monetary-fiscal authority holds. If monetary-fiscal policy is Ricardian, the price level remains indeterminate. If it is non-Ricardian, in that initial claims are given, then the equilibrium path of nominal asset prices determines the present-discounted value of unindexed transfers and so the initial price level.

More importantly, without further restrictions, as is the case under QE, the decomposition of equilibrium asset prices into an inflation process, $\pi(\cdot|s^t)$, and a nominal pricing measure, $\nu(\cdot|s^t)$, remains indeterminate: if the nominal pricing measure, $\nu(\cdot|s^t)$, is specified arbitrarily, the inflation process, $\pi(\cdot|s^t)$, adjusts to implement the equilibrium; that is, to satisfy

$$\tilde{q}(s^{t+1}|s^t) = \frac{\nu(s^{t+1}|s^t)(1+\pi(s^{t+1}|s^t))}{1+r(s^t))}.$$

The determinacy in Woodford (1994) highlights the importance of the present value of the monetary-fiscal authority budget constraint in the determination of the price level. We examine here, and what is often overlooked, is that the stochastic evolution of government wealth is essential for the determination of the stochastic path of prices. That Woodford (1994) restricts attention to conventional policy, in which case the portfolio of the monetary-fiscal authority is composed solely of Treasury bills, obscures this second point. Our results are not dependent on the infinite horizon of the economy, or the open-endedness of the policies that we describe. In previous versions of this paper we showed that our results remain valid in a finite horizon economy.

It may be confusing that we abstract from fiscal transfers after the initial period; we only do so because their implications are straightforward and do not affect the argument. It is worth pointing out, however, that the dichotomy between the nominal pricing measure and the initial price level that obtains when transfers are indexed, no longer holds when transfers are not indexed: the nominal pricing measure, indeterminate under quantitative easing, affects the aggregate volume of claims against the monetary-fiscal authority and, as a consequence, the initial price level as well.

Under QE, the nature of the interest-elasticity of money demand does not determine the stationary equilibrium path, though it determines the stability of the path. Following Sims (1994), the introduction of a portfolio of securities (rather than the single risk-free bond that was considered) under a policy of QE leaves the difference equations, that otherwise determine the unique path of money, to depend on the state-contingent return on the portfolio of the monetary-fiscal authority and the portfolio-to-money supply ratio. A given (stationary) distribution of portfolio returns, that satisfies the no-arbitrage condition given by the fixed short-term nominal interest rate, then corresponds to a stationary distribution of portfolios, even for the fiscal policy rules that were considered there. Put simply, adequate consideration of the interest-elasticity of money demand guarantees a stationary distribution, but not a unique one.

2.3 A stationary economy

In this section we examine the nature of stationary equilibria, abstracting from consideration of the stability of stationary equilibria and, therefore, also from the interest-elasticity of money demand. We show that the argument extends to stationary economies and stationary equilibria or steady states.

The resolution of uncertainty follows a stationary stochastic process. Elementary states of the world are s, finitely many, and transition probabilities are $f(s'|s)$.

Rates of interest, $r(s^1|s^0)$, determine the path of consumption, $c(s^1|s^0)$, and employment, $l(s^1|s^0)$, at equilibrium, which, in turn, determine the prices of indexed elementary securities:

$$\beta f(s'|s)\frac{\partial u(c(s'),1-l(s'))}{\partial c(s')}\tilde{q}(s'|s)^{-1} = \frac{\partial u(c(s^1|s^0),1-l(s^1|s^0))}{\partial c(s^1|s^0)}$$

or

$$\tilde{Q} = \beta Du(s^1|s^0)^{-1}FDu(s').$$

Here,

$$Du(s^1|s^0) = diag\left(\dots,\frac{\partial u(c(s^1|s^0),1-l(s^1|s^0))}{\partial c(s^1|s^0)},\dots\right)$$

is the diagonal matrix of marginal utilities of consumption, and

$$F = (f(s'|s)) \quad \text{and} \quad \tilde{Q} = (\tilde{q}(s'|s))$$

are, respectively, the matrices of transition probabilities and of prices of indexed elementary securities.

With

$$\tilde{y} = (\dots\frac{r(s^1|s^0)}{1+r(s^1|s^0)}a(s^1|s^0)l(s^1|s^0)\dots)'$$

the vector of net, real expenditures on balances at equilibrium, real claims against the fiscal-monetary authority at the steady state,

$$\tilde{\tau} = (\ldots \tilde{\tau}(s^1|s^0), \ldots)',$$

are determined by the equation

$$\tilde{y} + \tilde{Q}\tilde{\tau} = \tilde{\tau} \quad \text{or} \quad \tilde{\tau} = (I - \tilde{Q})^{-1}\tilde{y};$$

since $0 < \beta < 1$,

$$(I - \tilde{Q})^{-1} = \sum_{k=0}^{\infty} \tilde{Q}^k = \sum_{k=0}^{\infty} \beta^k Du(s^1|s^0)^{-1} F^k Du(s'),$$

and, since F is a Markov transition matrix, while $\tilde{y} \gg 0$, the real claims against the monetary-fiscal authority at the steady state are strictly positive:

$$\tilde{\tau} \gg 0.$$

Non-Ricardian monetary-fiscal policy determines the initial price level by setting exogenously the level of initial nominal claims; otherwise, the price level remains indeterminate.

More importantly, the decomposition of equilibrium asset prices into an inflation process, $\pi(\cdot|s)$, and a nominal pricing measure, $\nu(\cdot|s)$, remain indeterminate:

$$\tilde{Q} = R^{-1} N \otimes \Pi,$$

where $\otimes$ denotes the Hadamard product. Here,

$$R = diag(\ldots, (1 + r(s^1|s^0)), \ldots)$$

is the diagonal matrix of interest factors, and

$$N = (\nu(s'|s)) \quad \text{and} \quad \Pi = ((1 + \pi(s'|s)))$$

are, respectively, the Markov transition matrix of "nominal pricing transition probabilities" and the matrix of inflation factors.

Alternative specifications of the stochastic process of inflation serve to characterize the set of equilibria and to highlight the role of the balance sheet policy of the monetary-fiscal authority.

The role of the balance sheet policy is the focus of the analysis here; it was not dealt with in Drèze and Polemarchakis (2000) or Bloise, Drèze, and Polemarchakis (2005).

QE: In the absence of restrictions on the balance sheet of the monetary fiscal authority, which is the case under QE, the set of steady state equilibria is indexed by the nominal pricing transition probabilities, $\nu(\cdot|s)$, that can be set arbitrarily, while the inflation factors, $\pi(\cdot|s)$, adjust to implement the equilibrium; alternatively, the inflation factors are set arbitrarily, up to a scale effect, and the nominal pricing transition probabilities adjust to implement the equilibrium.

The argument is as follows: with

$$(1+\pi(s'|s)) = h(s^1|s^0)\gamma(s'|s),$$

an arbitrary (for the moment) decomposition of the inflation process into a term (the scale effect) that depends only on the current state and a term of (relative) inflation factors and is Markovian, the equilibrium condition takes the form

$$\tilde{Q} = R^{-1}N \otimes H\Gamma;$$

here, H be the diagonal matrix of the $h(s^1|s^0)$ and Γ the matrix of the $\gamma(s'|\, s)$.

Given Γ, there are H, N that guarantee equilibrium; the argument is straightforward:

$$\tilde{Q} = R^{-1}N \otimes \Pi \Rightarrow \tilde{Q} = R^{-1}N \otimes H\Gamma \Rightarrow \tilde{Q} \oslash \Gamma = (R^{-1}H)N,$$

the last step, since H is a diagonal matrix.[10]

Since N is Markovian if and only if it is non-negative and $N\mathbf{1}_S = \mathbf{1}_S$,

$$(\tilde{Q} \oslash \Gamma)\mathbf{1}_S = (R^{-1}H)\mathbf{1}_S,$$

which allows us to solve for $h(s^1|s^0)$.

With H, Γ in hand, we can solve for N, that shall indeed, be Markovian.

If N is given, there are H, Γ (or, equivalently, Π) that guarantee equilibrium.

Taylor rules: We now show that the indeterminacy obtained is not ruled out by interest-feedback rules. Any process can be written uniquely as

$$(1+\pi(s'|s)) = h(s^1|s^0)\gamma(s'|s), \quad \gamma(s'|s) = \frac{\delta(s'|s)}{f(s'|s)}, \quad \sum_{s'} \delta(s'|s) = 1,$$

in which case,

$$h(s^1|s^0) = \mathrm{E}_{s'}(1+\pi(s'|s));$$

[10] $\oslash$ denotes Hadamard division.

With $r(s^1|s^0)$ not set exogenously, but as a function of $h(s^1|s^0)$, this is a Taylor (1993) rule, and indeterminacy persists. In other words, policy that specifies the path of nominal interest rates as a function of expected inflation, does not pin down the stochastic path of inflation.[11]

Evidently, with $r(s^1|s^0)$ not set exogenously, but as a function of $h(s^1|s^0)$, equilibrium requires solution of the equation

$$\frac{h(s^1|s^0)}{1+r((h(s^1|s^0))} = \sum_{s'\in S} \frac{f(s'|s)}{\gamma(s'|s)} \beta \frac{\frac{\partial u(c(h(s')),1-l(h(s')))}{\partial c(h(s'))}}{\frac{\partial u(c(h(s^1|s^0)),1-l(h(s^1|s^0)))}{\partial c(s^1|s^0)}},$$

where the allocation, as a function of $h(s^1|s^0)$, is solved from the individual optimality conditions. If a solution to this system of equations exists and is unique, for example if the function/rule is linear, then the solution still depends on the (arbitrarily chosen) Γ.

CE: Alternatively,

$$(1+\pi(s'|s)) = h(s^1|s^0)\gamma(s'|s), \quad \gamma(s'|s) = \frac{\delta(s'|s)}{\tilde{\tau}(s')}, \quad \sum_{s'} \delta(s'|s) = 1,$$

in which case, $\delta(s'|s)$ are portfolio weights that determine the composition of assets in the balance sheet of the monetary-fiscal authority.

Monetary-fiscal policy conducted as CE sets the composition of the balance sheet; that is, it sets explicit positive portfolio weights, $\delta(s'|s) > 0$; claims against the monetary-fiscal authority in real terms, $\tilde{\tau}(s^1|s^0)$, are determined, at the steady-state, by fundamentals, and, as a consequence, under CE, the matrix Γ is determined.

Since

$$N\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad H\mathbf{1}_S = (R\tilde{Q} \oslash \Gamma)\mathbf{1}_S,$$

the Markov transition matrix, N, is well defined ($h \gg 0$) and determinate; it follows that the equilibrium is determinate as well.

Under conventional monetary-fiscal policy, the portfolio of the monetary-fiscal authority consists of Treasury bills, nominally risk-free bonds of short maturity. Here, this corresponds to one-period nominally risk-free bonds: $\delta(s'|s) = 1/S$.

Determinacy obtains for arbitrary, but, importantly, portfolio weights in the balance sheet of the monetary-fiscal authority that only depend on fundamentals and/or realized variables at s. If the portfolio weights are

[11] That the Taylor rule does not depend on realized rates of inflation is appropriate for (stochastic) steady-state equilibria.

chosen by policy to depend on endogenous nominal variables at s', such as the expected stochastic rate of inflation, then indeterminacy obtains. To be explicit, consider, for example, that the portfolio weights depended on expectations of the future nominal value of wealth: $\delta(s'|s) = [\tilde{\tau}(s')(1 + \pi(s'|s))]/[\sum_{s'} \tilde{\tau}(s')(1 + \pi(s'|s))] \Rightarrow h(s^1|s^0) = \sum_{s'} \tilde{\tau}(s')(1 + \pi(s'|s))$. In a model where there are long-dated securities, setting portfolio weights as a function of expected nominal asset prices would have the same outcome. This contrasts with Magill and Quinzii (2014b) and Adao, Correia, and Teles (2014), where explicit targets for asset prices, independent of equilibrium, pin down portfolio weights.

Simple inflation processes: It is instructive to consider whether restricting the inflation process to depend endogenously only on either the current or future state is compatible with an equilibrium policy choice. Suppose that the inflation process, which is endogenous, is restricted to take the form

$$(1 + \pi(s'|s)) = h(s^1|s^0)b(s'),$$

where $b(s^1|s^0) > 0$ is positive function of the fundamentals of the economy determined at the steady state and, as a consequence,

$$N \otimes \Pi = HNB.$$

Here,

$$b = (\ldots, b(s^1|s^0), \ldots), \quad \text{and} \quad h = (\ldots, h(s^1|s^0), \ldots),$$

and B and H are the associated diagonal matrices.

Then,

$$\tilde{Q} = R^{-1}N \otimes \Pi \quad \Leftrightarrow \quad R\tilde{Q}B^{-1} = HN,$$

which determines the inflation process as well as nominal pricing probabilities, since

$$N\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad N = \left(diag(R\tilde{Q}B^{-1}\mathbf{1}_S)\right)^{-1} R\tilde{Q}B^{-1},$$

a Markov transition matrix, as required.

This is indeed the case under conventional monetary policy.

Real wealth at successor date-events is

$$\tilde{\tau}(s') = \left(\frac{\theta(s'|s) + m(s^1|s^0)}{p(s)}\right) \frac{1}{1 + \pi(s'|s)},$$

and conventional monetary policy requires that

$$\theta(s'|s) = \theta(s^1|s^0)$$

or

$$(1+\pi(s'|s)) = \underbrace{\left(\frac{\theta(s^1|s^0)+m(s^1|s^0)}{p(s)}\right)}_{h(s^1|s^0)} \underbrace{\frac{1}{\tilde{\tau}(s')}}_{b(s')}.$$

Suppose, instead, that inflation is restricted to depend endogenously only on the future state,

$$(1+\pi(s'|s)) = h(s')b(s^1|s^0),$$

and, as a consequence,

$$N \otimes \Pi = BNH.$$

In this case,

$$\tilde{Q} = R^{-1}N \otimes \Pi \quad \Leftrightarrow \quad B^{-1}R\tilde{Q} = NH,$$

and

$$N\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad N = B^{-1}R\tilde{Q}\left(diag((B^{-1}R\tilde{Q})^{-1}\mathbf{1}_S)\right)$$

that need not be positive. In other words equilibrium inflation may be restricted to depend endogenously only on the current state but not only on the future one. However such a restriction precludes analysis of the effects of unconventional monetary policy on changes in the composition of the balance sheet of the monetary-fiscal authority and their subsequent determination of the stochastic path of inflation.

A large open economy

There are two countries in the world, home and foreign, each inhabited by a representative agent. Foreign variables, both macro and those relating to foreign agents, will be denoted with an asterisk (*). The transactions of agents in the home and foreign country will be denoted with a subscript "h" and "f," respectively. It suffices to specify explicitly mostly only the constraints and variables relevant for the home agent and country.

At a date-event, a perishable non-tradable input, labor, $l(s^t)$, is employed to produce a perishable domestic tradable output, consumption, $y(s^t)$, according to a linear technology. The representative home individual is endowed with 1 unit of leisure at every date-event. He supplies non-tradable labor and demands the tradeable consumption good, and he derives utility according to the cardinal utility index $u(c(s^t), 1-l(s^t))$ that satisfies standard monotonicity, curvature and boundary conditions. The preferences of the individual over consumption-employment paths commencing at s^t are described by the separable, von Neumann-Morgenstern, intertemporal utility function. Balances, $m_h(s^t)$ and $m_f(s^t)$ provide liquidity services in the home and foreign country respectively. Elementary securities, $\theta(s^{t+1}|s^t)$, serve to transfer wealth to and from immediate successor date-events. The price level is $p(s^t)$, and the wage rate is $w(s^t) = a(s_t)p(s^t)$, as profit maximisation requires. The nominal, risk-free interest rate is $r(s^t)$. As the goods produced in each country are perfect substitutes, the law-of-one-price holds and determines the exchange rate $e^*(s^t) = p(s^t)/p^*(s^t)$.

At each date-event, the asset (and currency) market opens after the uncertainty, s^t, has realized, and, as a consequence, purchases and sales in the markets for labor and the consumption good are subject to standard cash-in-advance constraints; the effective cash-in-advance constraint is[12]

$$a(s^t)p(s^t)l(s^t) \leq m_h(s^t), \quad 0 \leq m_f^*(s^t).$$

Prices of elementary securities in the domestic country are

$$q(s^{t+1}|s^t) = \frac{\nu(s^{t+1}|s^t)}{1 + r(s^t)},$$

with $\nu(\cdot|s^t)$ the domestic "nominal pricing measure,". Note that the nominal prices of elementary securities and the "nominal pricing measure" are unique to the currency in which they are denominated. As there are a complete set of state-contingent bonds in each currency, the prices of securities which deliver currency in the same state are related by the following no-arbitrage condition

$$\frac{\nu^*(s^{t+1}|s^t)}{\nu(s^{t+1}|s^t)} \left\{ \frac{1 + r(s^t)}{1 + r^*(s^t)} \right\} = \frac{e^*(s^{t+1}|s^t)}{e^*(s^t)}.$$

In other words, the path of nominal exchange rates depends on the ratio of the "nominal pricing measure" across countries and implies the uncovered interest parity condition

$$\frac{1 + r(s^t)}{1 + r^*(s^t)} e^*(s^t) = \sum_{s^{t+1}|s^t} \nu(s^{t+1}|s^t) e^*(s^{t+1}|s^t).$$

This gives the risk-neutral expected exchange rate. As markets are complete, variations in the nominal equivalent martingale measure in each country only have nominal effects on the implicit premium in the exchange rate[13].

[12] Nakajima and Polemarchakis (2005a) provide an explicit derivation.

[13] The difference between the risk neutral and objective expected exchange rate.

If there were nominal rigidities or frictions which prevented the law of one price from holding, then the covariance between the nominal equivalent martingale measure and nominal exchange rate, and hence the premium in expected exchange rates, would imply different allocations of (real) resources.

The individual has initial nominal wealth $\tau_h(s^t)$ and $\tau_f(s^t)$ in each country. Initial wealth constitutes a claim against the respective monetary-fiscal authority; alternatively, it can be interpreted as outside money. It is exogenous in a non-Ricardian specification. In a Ricardian specification, it is set endogenously so as to satisfy the transversality condition imposed on monetary-fiscal policy.

The flow budget constraint is[14]

$$\begin{aligned} p(s^t)c_h(s^t) + e^*(s^t)p^*(s^t)c_f(s^t) + m(s^t) \\ + \sum_{s^{t+1}|s^t} \left\{ q(s^{t+1}|s^t)\theta_h(s^{t+1}|s^t) + e^*(s^t)q^*(s^{t+1}|s^t)\theta_f(s^{t+1}|s^t) \right\} \\ \leq p(s^t)a(s^t)l(s^t) + \tau_h(s^t) + e^*(s^t)\tau_f(s^t). \end{aligned}$$

Debt limit constraints are

$$\tau_h(s^t) + e^*(s^t)\tau_f(s^t) \geq -\sum_{k>0} \sum_{s^{t+k}|s^t} q(s^{t+k}|s^t) \frac{1}{1+r(s^t)} a(s^{t+k})$$

or, equivalently,

$$\lim_{k\to\infty} \sum_{s^{t+k}|s^t} q(s^{t+k}|s^t) \left\{ \tau_h(s^{t+k}|s^t) + e^*(s^{t+k}|s^t)\tau_f(s^{t+k}|s^t) \right\} \geq 0.$$

Wealth at successor date-events is

$$\tau_h(s^{t+1}|s^t) = \theta(s^{t+1}|s^t) + m(s^t) \quad \text{and} \quad \tau_f(s^{t+1}|s^t) = \theta_f(s^{t+1}|s^t),$$

and, after elimination of the trade in assets and using the law-of-one-price, the flow budget constraint reduces to

$$p(s^t)z(s^t) + \frac{r(s^t)}{1+r(s^t)} a(s^t)p(s^t)l(s^t) + \sum_{s_{t+1}} q(s_{t+1}|s^t)\tau(s_{t+1}|s^t) \leq \tau(s^t),$$

where $z(s^t) = c(s^t) - a(s^t)l(s^t)$ is the effective excess demand for consumption, $c(s^t)$ is the sum of consumption at home and abroad and $\tau(s^t) = \tau_h(s^t) + e^*(s^t)\tau_f(s^t)$.

[14] Foreign money balances are dominated by foreign bonds and are zero in equilibrium, while the effective cash-in-advance constraint guarantees that domestic money balances are positive.

Alternatively, $\tilde{m}(s^t) = (1/p(s^t))m(s^t)$ are real balances, $\tilde{\tau}(s^t) = (1/p(s^t))$ $\tau(s^t)$ is real wealth, $\pi(s^{t+1}|s^t) = (p(s^{t+1})/(p(s^t)) - 1$ is the rate of inflation, and

$$\tilde{q}(s^{t+1}|s^t) = q(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t)) = \frac{\nu(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t))}{1 + r(s^t)}$$

are prices of indexed elementary securities[15].

Real wealth at successor date-events is

$$\tilde{\tau}(s^{t+1}|s^t) = \left(\frac{\theta(s^{t+1}|s^t) + m(s^t) + e^*(s^{t+1}|s^t)\theta(s^{t+1}|s^t)}{p(s^t)}\right)\frac{1}{1 + \pi(s^{t+1}|s^t)},$$

and the flow budget constraint reduces to

$$z(s^t) + \frac{r(s^t)}{1 + r(s^t)}a(s^t)l(s^t) + \sum_{s_{t+1}} \tilde{q}(s_{t+1}|s^t)\tilde{\tau}(s_{t+1}|s^t) \leq \tilde{\tau}(s^t).$$

First order conditions for an optimum are

$$\frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial l(s^t)}\left(\frac{a(s^t)}{1 + r(s^t)}\right)^{-1},$$

$$\beta f(s^{t+1}|s^t)\frac{\partial u(c(s^{t+1}), 1 - l(s^{t+1}))}{\partial c(s^{t+1})}\tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)},$$

and the transversality condition is

$$\lim_{k \to \infty} \sum_{s^{t+k}|s^t} \tilde{q}(s^{t+k}|s^t)\tilde{\tau}(s^{t+k}|s^t) = 0.$$

The monetary-fiscal authority in each country sets domestic one period rates of interest and accommodates the demand for domestic balances. It supplies domestic balances, $M(s^t)$, and trades in elementary securities subject to a flow budget constraint as in the closed economy case. For equilibrium, it is necessary and sufficient that the excess demand for output vanishes:

$$z(s^t) + z^*(s^t) = c(s^t) + c^*(s^t) - a(s^t)l(s^t) - a^*(s^t)l^*(s^t) = 0,$$

which determines the path of employment and consumption for each household:

$$\frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial l(s^t)}\left(\frac{a(s^t)}{1 + r(s^t)}\right)^{-1};$$

[15]From the no-arbitrage condition for assets, this is also the same in the foreign country.

in turn, this determines the prices of indexed elementary securities:

$$\beta f(s^{t+1}|s^t)\frac{\partial u(c(s^{t+1}), 1 - l(s^{t+1}))}{\partial c(s^{t+1})}\tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)}.$$

The initial price level serves to guarantee that, at equilibrium, the transversality condition of the monetary-fiscal authority holds. If monetary-fiscal policy is Ricardian, the price level remains indeterminate.

More importantly, without further restrictions, as is the case under QE, the decomposition of equilibrium asset prices into an inflation process, $\pi(\cdot|s^t)$, and a nominal pricing measure, $\nu(\cdot|s^t)$, remains indeterminate: if the nominal pricing measure, $\nu(\cdot|s^t)$, is specified arbitrarily, the inflation process, $\pi(\cdot|s^t)$, adjusts to implement the equilibrium; that is, to satisfy

$$\tilde{q}(s^{t+1}|s^t) = \frac{\nu(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t))}{1 + r(s^t)}.$$

Furthermore, the path of the nominal exchange rate remains indeterminate. Arbitrary nominal pricing measures in each country determine the stochastic future exchange rate to satisfy

$$\frac{\nu^*(s^{t+1}|s^t)}{\nu(s^{t+1}|s^t)}\left\{\frac{1 + r(s^t)}{1 + r^*(s^t)}\right\}e^*(s^t) = e^*(s^{t+1}|s^t).$$

A stationary economy

The argument extends to stationary economies and stationary equilibria or steady states.

The resolution of uncertainty follows a stationary stochastic process. Elementary states of the world are s, finitely many, and transition probabilities are $f(s'|s)$.

Rates of interest, $(r(s^1|s^0), r^*(s^1|s^0))$ determine the path of consumption, $(c(s^1|s^0), c^*(s^1|s^0))$ and employment, $(l(s^1|s^0), l^*(s^1|s^0))$ at equilibrium, which, in turn, determine the prices of indexed elementary securities:

$$\beta f(s'|s)\frac{\partial u(c(s'), 1 - l(s'))}{\partial c(s')}\tilde{q}(s'|s)^{-1} = \frac{\partial u(c(s^1|s^0), 1 - l(s^1|s^0))}{\partial c(s^1|s^0)}$$

or

$$\tilde{Q} = \beta Du(s^1|s^0)^{-1}FDu(s').$$

Note that the prices of indexed elementary securities is independent of the country. The nominal elementary securities, and hence martingale measures, across countries differ in their stochastic rates of inflation (and consequently the no-arbitrage condition).

Here,

$$Du(s^1|s^0) = diag(\ldots, \frac{\partial u(c(s^1|s^0), 1 - l(s^1|s^0))}{\partial c(s^1|s^0)}, \ldots)$$

is the diagonal matrix of marginal utilities of consumption, and

$$F = (f(s'|s)) \quad \text{and} \quad \tilde{Q} = (\tilde{q}(s'|s))$$

are, respectively, the matrices of transition probabilities and of prices of indexed elementary securities.

For the home household,

$$\tilde{m} = (\ldots \frac{r(s^1|s^0)}{1 + r(s^1|s^0)} a(s^1|s^0) l(s^1|s^0) \ldots)$$

is the vector of net, real balances at equilibrium,

$$\tilde{z} = (\ldots z(s^1|s^0) \ldots)$$

is the vector of excess demands and the real wealth at the steady state is given by

$$\tilde{\tau} = (\ldots \tau(s^1|s^0), \ldots).$$

$\tilde{\tau}$ is determined by by the equations

$$\tilde{z} + \tilde{m} + \tilde{Q}\tilde{\tau} = \tilde{\tau} \quad \text{or} \quad \tilde{\tau} = (I - \tilde{Q})^{-1} [\tilde{z} + \tilde{m}] \,.$$

$$\tilde{z}^* + \tilde{m}^* + \tilde{Q}\tilde{\tau}^* = \tilde{\tau}^* \quad \text{or} \quad \tilde{\tau}^* = (I - \tilde{Q})^{-1} [\tilde{z}^* + \tilde{m}^*] \,.$$

The real wealth of the monetary-fiscal authorities in the home country, $\tilde{T}$, is determined by

$$\tilde{M} + \tilde{Q}\tilde{T} = \tilde{T} \quad \text{or} \quad \tilde{T} = (I - \tilde{Q})^{-1} \tilde{M},$$

where $\tilde{M} = (\ldots \frac{r(s^1|s^0)}{1+r(s^1|s^0)} a(s^1|s^0) l(s^1|s^0) \ldots)$ and, since F is a Markov transition matrix, while $\tilde{M} \gg 0$, the real claims against the monetary-fiscal authority at the steady state are strictly positive:

$$\tilde{T} \gg 0.$$

Note that the real claims against the monetary-fiscal authorities can only be jointly determined, $\tilde{T} + \tilde{T}^* = \tilde{\tau} + \tilde{\tau}^*$.

As we have solved the entire real economy without nominal variables, the initial price level in each country remains indeterminate. More importantly, the decomposition of equilibrium asset prices into an inflation process, $\pi(\cdot|s)$, and a nominal pricing measure, $\nu(\cdot|s)$, remain indeterminate in each country. For the home country:

$$\tilde{Q} = R^{-1} N \otimes \Pi.$$

Here,

$$R = diag(\ldots, (1 + r(s^1|s^0)), \ldots)$$

is the diagonal matrix of interest factors, and

$$N = (\nu(s'|s)), \qquad \Pi = ((1 + \pi(s'|s))), \quad \text{and} \quad E = \big(e^*(s'|s)/e^*(s^1|s^0)\big)$$

are, respectively, the matrices of "nominal pricing transition probabilities", inflation factors and exchange rate factors. The stochastic growth rates of nominal exchange rates are given by[16]

$$E = \Pi \oslash \Pi^*$$

In the absence of restrictions on the balance sheet of the monetary fiscal authority, which is the case under QE, the set of steady state equilibria is indexed by the nominal pricing transition probabilities, $\nu(\cdot|s)$, that can be set arbitrarily; the inflation factors, $\pi(\cdot|s)$, and exchange rate factors, $e(s'|s)/e(s^1|s^0)$, then adjust to implement the equilibrium.

In each country, the composition of the balance sheet of the monetary-fiscal authority can be described by portfolio weights, $\delta(s'|s)$ (that is, $0 < \delta(s'|s) \leq 1$, and $\sum_{s'} \delta(s'|s) = 1$), and scale factors $h(s^1|s^0)$, such that

$$h(s^1|s^0)\delta(s'|s) = \tilde{T}(s')(1 + \pi(s'|s));$$

this is the case since the inflation factor is the rate of exchange of output between a date-event and an immediate successor.

The equilibrium condition, then reduces to

$$\tilde{Q} = R^{-1} N \otimes H\Gamma = R^{-1} H N \otimes \Gamma.$$

Here,

$$H = diag(\ldots, h(s^1|s^0), \ldots)$$

is the diagonal matrix of scale factors, and

$$\Gamma = \left(\frac{\delta(s'|s)}{\tilde{T}(s')} \right)$$

[16] Entry-by-entry multiplication is $\otimes$, while $\oslash$ is entry-by-entry division.

is the matrix of portfolio weights relative to the payoff of the balance sheet.

Monetary-fiscal policy conducted as CE sets the composition of the balance sheet; that is, it sets positive portfolio weights, $\delta(s'|s) > 0$; claims against the monetary-fiscal authority in real terms, $\tilde{T}(s^1|s^0)$, are determined, at the steady-state, by fundamentals, and, as a consequence, under CE, the matrix Γ is determined.

Since

$$N\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad H = (R\tilde{Q} \oslash \Gamma)\mathbf{1}_S,$$

the Markov tradition matrix, N, is well defined ($h \gg 0$) and determinate; it follows that the equilibrium is determinate as well.

Under conventional monetary-fiscal policy, the portfolio of the monetary-fiscal authority consists of treasury bills, nominally risk-free bonds of short maturity. Here, this corresponds to one-period nominally risk-free bonds: $\delta(s'|s) = 1/S$. □

Concerning the indeterminacy that obtains, further remarks are in order:

1. Our results under unconventional quantitative policies remain valid if the law of one price failed to hold, as in Corsetti and Pesenti (2005), or if there were pricing rigidities. However, in these cases the indeterminacy may have real effects.

2. The indeterminacy under QE obtained is not a consequence of deviations from steady-state equilibria and will not be eliminated by an interest rate feed-back rule, such as a "Taylor rule". This will be discussed at the end of the following section to avoid repetition. The non-stationary equilibria results presented above allow for extreme paths inflation and exchange rates. As the Fisher equation only guarantees an expected rate of inflation, it is entirely possible that there are paths of ever increasing inflation and a path of ever decreasing inflation (deflation), and consequently large stochastic changes in nominal exchange rates, and is reminiscent of the literature on speculative hyperinflation such as Obstfeld and Rogoff (1983).

3. Our requirement that the present-value budget constraints of the monetary-fiscal authority in each country be satisfied individually is not innocuous. Equilibrium only requires that the individual household budget constraints are satisfied, and as a consequence, only the joint budget constraint of the two government budget constraints will be satisfied. In that case the non-Ricardian assumption only guarantees that the present value of the monetary liabilities of both central banks, weighted by the exchange rate, equals the initial nominal wealth, also weighted

by the exchange rate. As a consequence, neither the price levels in each country nor the exchange rate is determinate. This is the point of Dupor (2000). Here, the non-Ricardian assumption in each country results in the price-level in each country to be uniquely determined. The subsequent indeterminacy is then restricted to the indeterminacy of the stochastic path of inflation, and is convenient to identify the role that QE plays in generating this indeterminacy.

4. A managed exchange rate, satisfying uncovered interest parity, will either transmit or eliminate the indeterminacy. If, for arguments sake, the home country conducts traditional monetary policy (and has a determinate path of inflation), then the foreign country may partake in quantitative easing and provided that they also target a path of the exchange rate, then the law-of-one-price guarantees that foreign prices are also determinate. If, however, the home country also conducts quantitative easing, then management of the path of the exchange rate leaves the rates of inflation in each country indeterminate. This is because the law-of-one-price only determines the ratio of prices across countries to equal the nominal exchange rate, but the (stochastic) levels are left free.

5. Our argument allows monetary-fiscal authorities to arbitrarily select the composition of initial assets, and independently of the initial quantity of money and price level, which are determined by the initial fiscal liabilities. Our argument is valid when the monetary-fiscal authority attempts to affect the initial quantity of money by purchasing assets with newly printed money: this would be analagus to increasing the outstanding liabilities that need to be returned through seignorage profits

6. The argument holds for the policies of unwinding of quantitive easing that are dependent on realized rates of inflation. This will be made more explicit in the following section, but intuitively, the monetary-fiscal authority here are faced with a new portfolio every period, due to the one-period contracts we focus on. This implies that the degrees of indeterminacy are $S - 1$ in each country and state. Hence, even if the initial portfolio composition is fixed, the consequent evolution of the portfolio (ie the unwinding phase) is not and indeterminacy will result.

It is worth pointing out that the indeterminacy we obtain is not a consequence of the stochastic nature of our economy per se, but rather that, given the uncertainty, the non-colliniarity of assets traded by the monetary-fiscal authority. In a related note, McMahon et al. (2012), we examine the consequences of the recent European Central Bank (ECB) policy on purchasing the debt of member countries (Outright Monetary Transactions, or OMT). If the bonds purchased by the ECB are not expected to default, which such a policy is in fact designed to support, then the bonds of the member countries are collinear and there is no requirement to provide ex-ante restrictions on the composition of assets held by the ECB. If however such a policy cannot prevent default, then the bonds are no longer collinear and short-term interest rates may no longer be sufficient to determine the path of Eurozone inflation.

Unconvential Monetary Policy and Premia

In cash-in-advance specifications, liquidity costs generate a wedge between cash and credit goods, and consequently affect marginal utilities and equilibrium prices. This generates a positive correlation between the (real) stochastic discount factor and expected nominal interest rates, and, as a consequence, a real risk premium that causes the term structure of interest rates to lie above levels predicted by the pure expectation hypothesis. In a closed economy, Espinoza et al. (2009) show that the risk-premia generated by the non-neutrality of monetary policy exist in addition to the ones derived from the stochastic distribution of endowments as presented in Lucas (1978) and Breeden (1979). They provide a potential explanation for the Term Premium Puzzle[17]. In an open economy, the argument extends, whereby the path of nominal interest rates in each country can affect real risk-premia on the path of nominal exchange rates as in Peiris and Tsomocos (2015)[18]. This is in

[17] There is a large literature on the difficulties of the uncovered interest parity holding empirically. The forward premium anomaly, as documented by Fama (1984), Hodrick (1987), and Backus et al. (1995) among others, states that when a currency's interest rate is high, that currency is expected to appreciate. Roughly speaking, the expected change in the exchange rate is constant and interest differentials move approximately one-for-one with risk premia.

[18] In that paper, there are two countries each inhabited by a representative agent and who must use domestic money for domestic trades, such as in the present paper. A cash-in-advance structure means that nominal interest rates affect the wedge between the marginal utilities of income and expenditure. Furthermore, markets are incomplete and agents may default upon their nominal obligtations. Monetary policy, by altering the wedge, affects the volume of real trade, and hence marginal utilities, default probabilities and implied risk neutral probabilities. Consequently, there is a covariance between nominal exchange rates, and real and nominal premia which affects the difference between the risk-neutral and objective expectation of future exchange rates. In the present paper, this difference is generated purely by altering the composition of assets traded by the monetary-fiscal authorities in each country.

contrast to equilibrium models where monetary policy is neutral, as in Lucas (1982), where, as risk premia are constant, interest rate differentials move one-for-one with the expected change in the exchange rate. We extend this literature by showing how the composition of the monetary-fiscal authority balance sheet, in addition to policy setting the path of interest rates or money supplies, affects premia in the bond and currency markets. The premia that we obtain is purely nominal though our results extend to economies with incomplete markets and price rigidities, in which case the premia would also be real.

Term Premia:

Our analysis utilises the stationary equilibrium results obtained in the previous section.

Consider the price, in the home country, of a two-period nominally riskless bond, at state s:

$$q_2(s^1|s^0) = \sum_{s'} q(s'|s) \sum_{s''|s'} q(s''|s') = \frac{1}{1 + r(s^1|s^0)} \sum_{s'|s} \frac{\nu(s'|s)}{1 + r(s'|s)}$$

In other words, the forward rate gives the risk-neutral expectation of the future one-period interest rates:

$$q_2(s^1|s^0)(1 + r(s^1|s^0)) = \sum_{s'|s} \frac{\nu(s'|s)}{1 + r(s'|s)}.$$

The term premia are then described by

$$\sum_{s'|s} \frac{\nu(s'|s) - f(s'|s)}{1 + r(s'|s)}.$$

The stationary distribution of the term premia is

$$N \otimes R - F \otimes R = [((R\tilde{Q} \oslash \Gamma)\mathbf{1}_S)^{-1} R\tilde{Q} \oslash \Gamma - F] \otimes R.$$

Recall that Γ is the matrix of portfolio weights relative to the payoff of the balance sheet of the monetary-fiscal authority. Hence, given the fundamentals of the economy, and a given path of one-period interest rates, the term premia depends on the composition of the monetary-fiscal authority balance sheet. More precisely, a correlation is generated between the nominal martingale measure and nominal interest rates which results in risk-neutral pricing being systematically biased (from subjective pricing alone).

Currency Premia:

Recall the Uncovered Interest Parity equation in state s

$$e^*(s^1|s^0)\frac{1+r(s^1|s^0)}{1+r^*(s^1|s^0)} = \sum_{s'|s} \nu(s'|s)e^*(s'|s),$$

where $\sum_{s'|s} \nu(s'|s)e^*(s'|s)$ is the risk neutral expectation of exchange rates. The realised distribution of exchange rates implies an (objective) expectation of $\sum_{s'|s} f(s'|s)e^*(s'|s)$. The difference between these two will be the currency premium. The stationary distribution of the premium is:

$$\begin{gathered} N \otimes E - F \otimes E = N \otimes \Pi \oslash \Pi^* - F \otimes \Pi \oslash \Pi^* = \\ R\tilde{Q} \oslash \Pi^* - F \otimes \Pi \oslash \Pi^* = (R\tilde{Q} - F \otimes \Pi) \oslash \Pi^* = \\ (R\tilde{Q} - F \otimes H\Gamma) \oslash (H^*\Gamma^*) = \\ (R\tilde{Q} - F \otimes (R\tilde{Q} \oslash \Gamma)\mathbf{1}_S\Gamma) \oslash ((R^*\tilde{Q}^* \oslash \Gamma^*)\mathbf{1}_S\Gamma^*). \end{gathered}$$

This is entirely in terms of real variables and nominal interest rates and portfolio weights set by policy. Furthermore there is a clear separation between home and foreign variables and policy parameters. It follows then that stationary portfolio weights chosen in each country correspond to varying premia in the currency markets. The sign and magnitude of the premium can be chosen arbitrarily by appropriate choices of nominal interest rates and portfolio weights. Note that varying the nominal interest rates results in the premium having a real (risk) component while varying the portfolio weights affects the stationary distribution of inflation and exchange rates which is purely nominal. From the equation, it is clear that the joint distribution of interest rates and inflation across countries matters in addition to the mean and variance of the inflation process in each country.

We have considered only interest rate targeting; the results do extend to policies that target the paths of money supplies. In that case, although the path of money is given by policy, fluctuations in demands for assets affect the path of interest rates and changes in the composition of monetary-fiscal authority portfolio has real effects. That is, in a money growth targeting regime, the path of real risk-premia depends on the composition of assets held by the monetary-fiscal authority.[19]

[19] Nakajima and Polemarchakis (2005a) show this in a closed economy. Alvarez et al. (2009) consider an open economy similar to ours, but with segmented participation in the asset market; in the the absence of a credit good, monetary policy is otherwise neutral. If the monetary-fiscal authority portfolio is left unrestricted, introducing a credit good may mean that the correlation between interest rates and risk premia depend both on the path of money and the evolution of the composition of the monetary-fiscal authority portfolio.

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