Fiscal policy under the unbalanced pension system[1]

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Abstract

The ageing of the population and the imbalance of public finances force governments to carry out pension reforms in order to insure the sustainability of pension systems. The reforming of social security systems is becoming even more urgent as the government ability to cover the deficit of pension funds with transfers from federal budgets is limited. We consider an optimal combination of fiscal instruments depending on the retirement age, life expectancy and productivity of labor and compare social welfare in the case of balanced and unbalanced pension system on the basis of the overlapping generations model with endogenous interest rate. When the deficit of the pension fund is covered by the government, it is optimal to finance the deficit of pension fund by the income tax while in the case of the interior solution income tax and social contributions are perfect substitutes. In the case of balanced pension system optimal social contributions are positive and are used to finance pensions, while optimal income tax does not change with the population growth. In the case of unbalanced pension system, the maximization of welfare leads to the corner solution with zero social contributions and positive income tax, which depends on population growth, retirement age and labor productivity. An unbalanced pension system with optimal fiscal instruments allows to achieve higher social welfare due to the higher level of capital and lower equilibrium level of public debt.

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1 Introduction

Many countries are launching pension reforms in order to secure the sustainability of the social security system in the future which is at risk because of increasing expenditures on pensions. The main drivers of these dynamics are demographic changes: fertility rates below the replacement level and higher life expectancy.

Although several reforms of social security systems have been implemented, federal transfers remain one of the key sources of balancing the budget of pension funds. Moreover, they are expected to rise steadily in the future: according to the OECD estimates, this part of fiscal expenditures will increase from 9.3% of the GDP in 2010 to 11.7% of the GDP in 2050.[1] In Russia, the deficit of the pension fund is also covered by a transfer from the federal budget. This transfer was 4.3% of the GDP in 2013 and 3.4% in 2014.[2] Financing pension fund deficits out of the federal budgets has become even more complicated after the financial crisis of 2008-2009 and the European debt crisis, which started in 2010.

In the case of increasing pension expenditures pension reforms (the introduction of a higher retirement age, higher social contributions, lower pensions) can be considered as an alternative to the traditional measures of the fiscal consolidation. This research defines the optimal combination of two fiscal instruments (rate of social contributions and income tax) chosen by the social planner and compares social welfare in case of balanced and unbalanced pension system. It also specifies how this policy mix changes with the retirement age, life expectancy, labor productivity and how it depends on the type of the pension system (balanced or unbalanced).

The research of pension reforms can be classified by the type of pension system under consideration. First group consists of research of PAYG reforms (e.g. Nickel et al., 2008; Karam et al., 2010; Kilponen et al., 2006; Castro et. al., 2016; Almeida et al., 2013; Pierrard-Snessens, 2009; Marchiori-Pierrard, 2012). Others consider the switch from PAYG to fully funded pension system (Borsch-Supan et al., 2006; McGrattan-Prescott, 2015). Both types of pension systems were analyzed in Marchiori et al. (2011) and de la Croix et al. (2013). This research falls into the first category.

The analysis is based on the overlapping generations model (OLG) initially developed by Yaari (1965) and Blanchard (1985) and extended further by Buiter (1988), Giovannini (1988), Weil (1989) and Bovenberg (1993). In order to investigate the optimal policy mix we extend the model of Heijdra and Bettendorf (2006), who analyzed the economic consequences of lower pensions and a higher retirement age in an open economy with traded and non-traded sectors. They, however, consider an exogenous interest rate along with a rudimentary pension system, which allows them to analyze intergenerational

[1] OECD, Pensions at glance 2013

[2] Transfer has decreased due to the freeze of the accumulation part of the pension savings.

redistribution which is assumed to be balanced. We extend their model to investigate the unbalanced budget of a pension fund in a closed economy with an endogenous interest rate, which allows us to account for the effect of different economic characteristics (retirement age, life expectancy, labor productivity) on the capital accumulation. While Heijdra and Bettendorf (2006) consider the consequences of shocks to the welfare of each generation, we investigate the optimal subset of measures conducted by a benevolent government, which maximize the social welfare function.

Nickel et al. (2008) extend the framework of Nielsen (1994) and Heijdra and Bettendorf (2006) by considering an unbalanced pension system and assuming that firms issue equities and face adjustment costs in investment. They analyze three fiscal scenarios in an economy with decreasing population: the suspension of the public pension system and a decrease in lump-sum labor tax; the suspension of the public pension system and a decrease in distortionary corporate tax; or an increase in the retirement age. Their results suggest that the adverse consequences of pension reforms can be decreased by appropriate taxation policies. The main difference with our research is that Nickel et al. (2008) consider the government as a non maximizing entity and investigate how the predetermined changes in policy instruments would affect the transition of the main macroeconomic variables to the new equilibrium in an open economy, while we define socially optimal fiscal policy (social contributions and income tax) and compare the optimal set of policy instruments in equilibrium with both increasing and decreasing population in the closed economy.

The fact that income tax rate and social contributions are substitutes is confirmed by the comparison of optimal policy mix in the case of balanced and unbalanced pension system. In the former case social contributions are strictly positive and decreases with the population growth, while income tax rate is constant and does not depend on it. In the case of unbalanced pension system, on the contrary, the corner solution is optimal: social contributions are at zero while income tax rate decreases with the population growth. This policy mix remains optimal in the steady states with the different levels of life expectancy and labor productivity. Unbalanced pension system leads to the optimal interest rate which changes with structural characteristics of the economy while social contributions are at zero.

The paper is organized as follows. Section 2 presents an extended OLG model of Heijdra and Bettendorf (2006) with an unbalanced pension system. Section 3 covers the comparison of optimal policy mix and social welfare under balanced and unbalanced pension system. It is also shown how the optimal policy mix depends on the retirement age, life expectancy and labor productivity. Section 4 summarizes the results.

2 The model

The model of Heijdra and Bettendorf (2006) was extended by introducing an unbalanced pension system with the deficit covered by a benevolent government, which conducts fiscal policy to maximize the social welfare. A closed economy with an endogenous interest rate is considered.

The economy consists of households, firms, the government and the pension fund. Infinitely lived households maximize the present value of utility from consumption taking into account the life expectancy. They work and pay income tax throughout the life, social contributions before the retirement age and receive pensions at the retirement. Pensions are paid by the pension fund, the deficit of which is covered by the government. Public debt is financed by bonds held by the households and income tax payments.

In the model upper case variables are aggregates, lower case variables with a line ($\bar{c}$) denote individual variables, while lower case variables without any notation are aggregates per unit of efficient labor.

2.1 Households

Individual households

The representative consumer born at time υ maximizes the expected present value of instantaneous utility of consumption.

$$U(\upsilon, t) = \int_{t}^{\infty} [\ln \bar{c}(\upsilon, t)] \, \mathrm{e}^{(\rho+\beta)(t-\tau)} d\tau, \tag{1}$$

where $\bar{c}$ is personal consumption, $\rho > 0$ is the rate of time preference and $\beta \geq 0$ is the probability of death.

Following Bettendorf and Heijdra (2006) we consider PAYG pension system introduced by Nielsen (1994). The households pay income taxes during their lives, while paying social contributions t_W before the retirement age π and receiving pensions z at retirement. The threshold level π can be loosely considered as retirement age as the households continue to work after it.

Labor supply is non elastic: each household supplies one unit of labor. The households pay income tax on the labor income and receive an interest rate $r(t)$ on the financial wealth, $\bar{a}(\upsilon, t)$. The payment $\beta a(\upsilon, t)$ is the actuarially fair annuity paid by the life insurance company.[1] Interest and non-interest net labor income, $WI(\upsilon, t)$, are spent on consumption and saving. Household financial wealth consists of capital goods, $\bar{k}$, and government bonds, ($\bar{a}^G$), both denominated in terms of consumer goods.

[1] See Yaari (1965), Blanchard (1985)

The household budget constraint in terms of the consumer good is:[2]

$$\dot{\bar{a}}(v,t) = (r(\tau) + \beta)\bar{a}(v,t) + WI(v,t) - \bar{c}(v,t), \tag{2}$$

$$\bar{a}(v,t) = \bar{k}(v,t) + \bar{a}^G(v,t), \tag{3}$$

$$WI(v,t) = \begin{cases} (1-t_L)W^N(v,\tau) - t_W & \text{for } t - v \leq \pi, \\ (1-t_L)W^N(v,t) + z & \text{for } t - v > \pi. \end{cases}, \tag{4}$$

where $W^N(v,t)$ is the wage at time t of the worker born at time v.

Labor productivity decreases with the age of the worker. The worker of generation v at time t supplies $n(v,t)$ efficiency units of labor:

$$n(v,t) = E(t-v)\bar{l}(v,t), \tag{5}$$

where $\bar{l}(v,t) = 1$ is the labor hours and $E(t-v)$ is the efficiency index, which falls exponentially with the worker's age:[3]

$$E(t-v) = \omega_0 \mathrm{e}^{-\alpha(t-v)}, \tag{6}$$

where ω_0, a positive constant, equals to 1 and $\alpha > 0$ specifies the speed at which the efficiency falls with age.

At each moment t the household chooses the paths of consumption and financial assets so as to maximize the present value of lifetime utility (1) subject to budget constraint (2) and a transversality condition. The initial value of the financial assets $a(v,t)$ and the government consumption per household are taken as given.

The optimal path of household consumption is defined by the Euler equation:

$$\frac{\dot{\bar{c}}(v,t)}{\bar{c}(v,t)} = r(t) - \rho, \tag{7}$$

which specifies that in each moment consumption is proportional to the total wealth:

$$\bar{c}(v,t) = (\rho + \beta)(\bar{a}(v,t) + \bar{a}^H(v,t)), \tag{8}$$

where $\bar{a}^H$ is human wealth defined as the present value of the after-tax labor income:

$$\bar{a}^H(v,t) = \int_t^\infty WI(v,\tau)\mathrm{e}^{R(t,\tau)+\beta(t-\tau)} d\tau. \tag{9}$$

where $R(t,\tau) = \int_t^\tau r(s)ds$..

[2] A dot above the variable stands for the variable's time derivative: $\dot{\bar{a}}(v,t) = d\bar{a}(v,t)/dt$.

[3] As in Blanchard (1985).

Demography

The framework allows us to consider non-zero population growth, by distinguishing the probability of death $\beta \geq 0$, and the probability of birth, $\eta > 0$.[4] The population size $L(t)$ grows with net growth rate n_L:

$$\frac{\dot{L}(t)}{L(t)} = \eta - \beta = n_L. \tag{10}$$

Taking into account the initial condition $L(0) = 1$, the population size is:

$$L(t) = \mathrm{e}^{n_L t}. \tag{11}$$

The size of the generation born at t is proportional to the current size of the population:

$$L(\upsilon, \upsilon) = \eta L(\upsilon). \tag{12}$$

The size of each generation falls exponentially with the probability of death β:

$$L(\upsilon, t) = \mathrm{e}^{\beta(\upsilon - t)} L(\upsilon, \upsilon), t \geq \upsilon \tag{13}$$

The current size of the generation born at time υ can be obtained by substituting (11) and (12) into (13):

$$L(\upsilon, t) = \eta \mathrm{e}^{\eta \upsilon} \mathrm{e}^{-\beta t} \tag{14}$$

Aggregate household sector

The aggregate variables are defined as the integral of the variable values, specific for each living generation, weighted by the size of that generation. Aggregate consumption, for example, can be defined as follows:

$$C(t) = \int_{-\infty}^{t} L(\upsilon, t) \bar{c}(\upsilon, t) d\upsilon, \tag{15}$$

where $L(\upsilon, t)$ and $\bar{c}(\upsilon, t)$ are given by (14) and (8), respectively.

In can be shown that aggregate consumption is proportional to the household's wealth, where $A(t)$ is aggregate financial wealth and $A^H(t)$ is aggregate human wealth:

$$C(t) = (\rho + \beta) \left[A(t) + A^H(t) \right]. \tag{16}$$

The growth rate of the aggregate consumption is obtained from (15), taking into

[4]This framework was developed by Buiter (1988).

account (7) and (14):

$$\frac{\dot{C}(t)}{C(t)} = [r(t) - \rho] + \frac{\eta L(t)\bar{c}(t,t) - \beta C(t)}{C(t)}, \tag{17}$$

where $r(t) - \rho$ is the growth of individual consumption, while the second term represents the so-called generational turnover (Bettendorf and Heijdra, 2006), which depends on the demographic parameters. Aggregate consumption increases with the arrival of new agents and decreases with the death of the older generation.

The growth rate of the aggregate consumption can be simplified to:[5]

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho + \alpha + n^L - (\rho + \beta)\frac{\eta\gamma L(t) + (\alpha + \eta)A(t)}{C(t)}, \tag{18}$$

$$\gamma(t) = \frac{-d(t)}{r(t) + \beta} + (r(t) + \alpha + \beta)\left(\frac{\mathrm{e}^{-\beta\pi}}{1 - \mathrm{e}^{-\eta\pi}}\right)\left(\frac{z - d(t)}{r(t) + \beta}\right)\left(\frac{\mathrm{e}^{-r(t)\pi} - \mathrm{e}^{-n_L\pi}}{n^L - r(t)}\right). \tag{19}$$

The aggregate consumption growth, therefore, exceeds the growth of individual consumption if the net population growth is positive ($n_L > 0$) and the labor productivity decreases over time ($\alpha > 0$). It can be lower if newborns consume less or due to the redistribution from the young to the old through the pension system. In contrast to Bettendorf and Heijdra (2006) γ depends on the deficit of the pension fund, defined below.

Bettendorf and Heijdra (2006) point out that $\eta\gamma/(r + \alpha + \beta)$ can be considered as per capita deficit of the pension system. When $r > n_L$ social contributions are perceived by working households as tax on the labor income, as they are forced to save at the rate n_L which is lower then the market rate r. In case of retirees the opposite affect takes place.

Aggregate financial wealth is defined as follows:

$$A(t) = \int_{-\infty}^{t} L(v,t)\bar{a}(v,t)dv. \tag{20}$$

The definition of aggregate savings can be found by differentiating equation (20) for the aggregate financial wealth with respect to time and taking into account that the newborn generation does not have any financial wealth, $\bar{a}(t,t) = 0$:

$$\dot{A}(t) = -\beta A(t) + \int_{-\infty}^{t} L(v,t)\dot{\bar{a}}(v,t)dv \tag{21}$$

By substituting (2) in (21) we get:[6]

$$\dot{A}(t) = r(t)A(t) + WI(t) - C(t), \tag{22}$$

[5]For greater detail see Appendix 1

[6]for grater details see Appendix 2

$$WI(t) = \frac{\eta\omega_0}{\alpha+\eta}(1-t_L)F_N(k_N(t),1))L(t) + D(t), \tag{23}$$

where $F_N(k_N(t),1)$ is the marginal product of labor and $D(t)$ is the deficit of pension system.

The aggregate labor supply at time t measured in efficiency units is proportional to the population size in the corresponding period and is obtained from (5), (6), (11) and (14):

$$N(t) = \int_{-\infty}^{t} L(v,t)\bar{n}(v,t)dv = \frac{\eta\omega_0}{\alpha+\eta}L(t). \tag{24}$$

2.2 Firms

As opposed to Bettendorf and Heijdra (2006) we consider a closed economy with endogenous interest rate, important in the estimation of pensions. The output is produced according to the Cobb-Douglas technology:

$$Y = F(K,N) = K^{\varepsilon}N^{1-\varepsilon}, \tag{25}$$

where K and N represent capital and efficiency labor units. Producers maximize profit, choosing the optimal level of capital and labor:

$$\Pi(t) = Y(t) - \int_{-\infty}^{t} W^N(v,t)L(v,t)dv - W^K(t)K(t), \tag{26}$$

where $W^K(t)$ is a capital rent and $W^N(v,t)$ is the wage at time t of the worker of generation v.

The first order conditions are:

$$W^K(t) = F_K(k_N(t),1), \tag{27}$$

$$W^N(t) \equiv \frac{W^N(v,t)}{E(\tau-v)} = F_N(k_N(t),1), \tag{28}$$

where $F_K = \partial F/\partial K_N$ and $F_N = \partial F/\partial N$. $W^N(t)$ is the wage per efficiency unit of labor and $k_N(t) = K(t)/N(t)$ is the capital efficiency unit of labor.

The produced output is allocated to private consumption, investment I and government expenditures G.

$$Y(t) = C(t) + I(t) + G(t). \tag{29}$$

The optimal investment decision is based on the maximization of the net present value of cash flows from the investor's capital stock subject to the capital accumulation identity:

$$V(t) = \int_{t}^{\infty} \left[W^K(\tau)K(\tau) - I(\tau)\right] \mathrm{e}^{-R(t,\tau)} d\tau, \tag{30}$$

$$\dot{K}(t) = I(t) - \delta K(t), \tag{31}$$

where $R(t,\tau) = \int_{t}^{\tau} r(s)ds$ is a discount factor. First order condition, (32), specifies that the rental rate W^K equals the return on the capital $r(t)$ taking into account the amortization rate δ.

$$W^K(t) = r(t) + \delta \tag{32}$$

2.3 Public sector and the benevolent government

Government budget identity defines the accumulation path of public debt $A^G(t)$, which depends on the current government expenditures $G(t)$, labor tax revenues and additional expenditures coming from the deficit of the pension fund $D(t)$. It can be written as follows:

$$\dot{A^G}(t) = r(t)A^G(t) + G(t) - t_L W^N(t)N(t) + D(t). \tag{33}$$

Taking into account the transversality condition:

$$\lim_{\tau \longrightarrow \infty} A^G(\tau)\mathrm{e}^{-R(t,\tau)} = 0, \tag{34}$$

public debt is:

$$A^G(t) = \int_{t}^{\infty} \left[t_L W^N(\tau)N(\tau) - G(\tau) - D(\tau)\right] \mathrm{e}^{-R(t,\tau)} d\tau. \tag{35}$$

The key difference with the paper of Heijdra and Bettendorf (2006) is the assumption that PAYG pension system can be run on an unbalanced-budget basis, with a deficit $D(t)$.

$$t_W(1 - e^{-\eta\pi})L(t) = z\mathrm{e}^{-\eta\pi}L(t) - D(t) \tag{36}$$

The left-hand side of (36) represents the total social contributions paid by the young, while on the right-hand side are total pensions paid to the old and the surplus (or deficit) of the pension fund if the sum of social contributions and pensions do not match.

For the easier comparison we assume that the government determine the value of social contributions setting the value of ψ, which is the share of social contributions in the median wage:

$$t_W = \psi\omega_0 F_N(k(t), 1)\mathrm{e}^{-\alpha\left(\pi - \frac{1}{\alpha}\ln\left(\frac{1+\mathrm{e}^{\alpha\pi}}{2}\right)\right)}. \tag{37}$$

Social contributions, therefore, depend on the retirement age.

We define the social welfare function as the present value of the utility of all currently living and future generations weighted by their share in the population. The first term in (38) represents the welfare of young generations, while the second is the welfare of the retirees.

$$SW(t) = \int_{t}^{\infty} \int_{-\infty}^{\tau-\pi} L(\upsilon,\tau)[\ln \bar{c}(\upsilon,\tau)]e^{(\rho+\beta)(t-\tau)} d\upsilon d\tau + \tag{38}$$

$$+ \int_{t}^{\infty} \int_{\tau-\pi}^{\tau} L(\upsilon,\tau)[\ln \bar{c}(\upsilon,\tau)]e^{(\rho+\beta)(t-\tau)} d\upsilon d\tau$$

The individual consumption of the young and elder generations can be expressed as functions of their individual human wealth (a_y^H and a_o^H, respectively), which depends on the capital per efficiency units of labor.[7]

$$SW(t) = \chi \mathrm{e}^{-\eta\pi} \left[(\ln((\rho+\beta)a_o^H) + (r-\rho)\frac{\pi\eta+1}{\eta}) \right] - \tag{39}$$

$$-\chi \left[(1-\mathrm{e}^{-\eta\pi})(\ln((\rho+\beta)a_y^H)) - (r-\rho)(1-\mathrm{e}^{\eta\pi} - \eta\pi\mathrm{e}^{-\eta\pi})\eta^{-1}) \right],$$

$$\chi = \frac{\mathrm{e}^{n_L t}}{n_L - \rho - \beta}.$$

2.4 Model summary

The key equations in per capita terms are presented in Table 1 below. The endogenous variables are k, y, c, a, a^G, r, W^N,W^K, γ, n. Parameters are β, η, n_L, α and ρ. Policy instruments are π, z, t_W, t_L.

Table 1. *Summary of the Model*

Description	Analytical representation	
Dynamic equations:		
Capital	$\dot{k}(t) = ny(t) - c(t) - g(t) - (n_L+\delta)k(t)$	$(T1.1)$
Private consumption	$\dot{c}(t) = (r(t)-\rho+\alpha)c(t) - (\rho+\beta)(\eta\gamma(t) + (\alpha+\eta)a(t))$	$(T1.2)$
Public debt	$\dot{a}^G(t) = (r(t)-n_L)a^G(t) + g(t) - nt_L W^N(t) + d(t)$	$(T1.3)$
Private savings	$\dot{a}(t) = (r(t)-n_L)a(t) - c(t) + n(1-t_L)W^N(t) + d(t)$	$(T1.4)$
Static equations:		
	$\gamma(t) = \frac{-d(t)}{r(t)+\beta} + \left(\frac{\mathrm{e}^{-\beta\pi}}{1-\mathrm{e}^{-\eta\pi}}\right)\left(\frac{z-d(t)}{r(t)+\beta}\right)(r(t)+\alpha+\beta)\left(\frac{\mathrm{e}^{-r(t)\pi}-\mathrm{e}^{-n_L\pi}}{n^L-r(t)}\right),$	$(T1.5)$
Pension fund	where $t_W(1-\mathrm{e}^{-\eta\pi}) = z\mathrm{e}^{-\eta\pi} - d(t)$	
Rental rate	$W^K(t) = \varepsilon k_N(t)^{\varepsilon-1} = \varepsilon\left(\frac{k(t)}{n}\right)^{\varepsilon-1}$	$(T1.6)$
Interest rate	$r(t) = W^K(t) - \delta$	$(T1.7)$
Wage	$W^N(t) = (1-\varepsilon)y(t)$	$(T1.8)$
Output	$y(t) = k_N^{\varepsilon}(t) = \left(\frac{k(t)}{n}\right)^{\varepsilon}$	$(T1.9)$
Supplied efficiency units	$n = \frac{\eta\omega_0}{\alpha+\eta}$	$(T1.10)$
Wealth	$a(t) = k(t) + a^G(t)$	$(T1.11)$

[7] For greater detail see Appendix 3

The Eq.T1 corresponds to the accumulation of capital per capita, and it is obtained by combining (31) and (32). Eq.T2 stands for the optimal path of per capita consumption, obtained from (18) in per capita terms. Eq.T3 is the government budget constraint expressed in per capita terms, derived from the government budget constraint (33).The last dynamic equation, Eq.T4, represents the accumulation of per capita assets and is obtained from (22), taking into account (23) and (36).

Definition 1. *Given the set of policy variables* $\{g_t, t_L, t_W, z, \pi\}$ *that satisfy the government budget constraint, the set* $\left\{W_t^K, r_t, y_t, c_t, k_t, a_t, a_t^G, d_t\right\}$ *defines equilibrium, if it satisfies the optimal conditions of households and firms, (7) and (27)-(28) and equilibrium conditions for goods and capital markets.*

$$y(t) = c(t) + i(t) + g(t) \tag{40}$$

$$a(t) = k(t) + a^G(t) \tag{41}$$

As the system of dynamic equations is non-linear and cannot be solved analytically we are solving numerically the system of equations T1.1-T1.3, taking into account T1.12.[8]

To distinguish the socially optimal policy mix of measures (income tax and social contributions) we check if the resulting set of possible equilibria satisfies the stability condition of the equilibrium and the condition on the limit of public debt.[9] In the maximization problem the level of pensions and the retirement age were fixed in order to analyze optimal choice of income taxes and social contributions in the existence of pension obligations. It is known that in the OLG model with dynamically efficient equilibrium PAYG pension system worsens social welfare. If the level of pensions is chosen optimally from the maximization of social welfare, it would be optimal to set pensions and social contributions which are equal zero.

It is worth mentioning that the equilibrium with diminishing labor productivity can be dynamically inefficient if the speed of the decrease in productivity α is large enough. In this case labor income is high during the youth and falls rapidly with age, so agents save a lot during youth so the capital stock can be too large and there is overaccumulation of capital. Necessary condition for dynamic inefficiency is $\alpha < \rho$, which corresponds to the positive interest rate. The calibration used in the paper satisfies the condition of dynamic efficiency.

[8] All calculations were conducted using Matlab. Restricting $\dot{k}(t) = 0$, and $\dot{a}^G(t) = 0$ we use a root-finding method (the bisection method) to define the level of capital per capita to bring the growth of per capita consumption to zero, $\dot{c}(t) = 0$. All possible combinations of fixed and variable parameters on the initially set intervals are considered to determine the steady-state level of k^* and the corresponding combinations of parameters which bring $\dot{c}(t) = 0$.

[9] The stability condition insures that the determinant of the Jacobian matrix of the log-linearized dynamic system of equations T1.1, T1.2 and T1.4 is less than zero. In this case model is locally saddle-point stable.

3 Optimal instruments of the fiscal policy

3.1 Calibration

The optimal choice of the retirement age, social contributions and pensions points illustrates that the pension system is rudimentary. This result coincides with the results of neoclassical models with rational expectations. Households with forward looking rational expectations have an incentive to save on their own in order to smooth their consumption in retirement. Therefore we investigate optimal mix of income tax and social contributions with fixed level of pensions and the retirement age.

The parameters used for the baseline calibration are presented in Table 2. The share of pensions is fixed at 30% of the median life-time wage, while the optimal size of the mandatory social contributions, ψ, as a share of the median wage, as well as income tax are chosen optimally from the maximization of the social welfare.[1] The value of government expenditures is fixed at 25% of the GDP, a common value for the OECD countries. The productivity declines with age at the speed, α, which equals 1.25%, meaning that the worker is half as productive at the retirement age. For the results below the death rate β equals 1.25%, bringing the life expectancy to 80 years. The birth rate η varies from 1% to 3%. This range allows us to analyze both negative and positive population growth. Share of the government expenditures in the welfare function κ equals 0.5.

Table 2. *Baseline calibration*

Variable	**Symbol**	**Value**	**Source**
Rate of time preference	ρ	0.015	Heijdra, Ligthart (2006)[2]
Birth rate	η	0.0125	Heijdra, Ligthart (2006)[3]
Probability of death	β	0.0125	Heijdra, Ligthart (2006)
Output elasticity of capital	ε	0.33	Standard[4]
Speed of decline in the labor efficiency	α	0.0125	Nickel et al. (2008)[5]
Capital depreciation rate	δ	0.03	Standard[6]
Retirement age	π	60	Bettendorf, Heijdra (2006)[7]

[1] The results are robust to the change in the share of pensions.

[2] The value belongs to the interval considered by Heijdra and Ligthart (2006): from 0.01 to 0.04.

[3] In Heijdra and Ligthart (2006) birth rate varies from 0.01 to 0.04, while death rate from 0 to 0.03, in baseline calibration 0.015 and 0.01, correspondingly.

[4] In Heijdra and Ligthart (2006) elasticity with respect to capital equals to 0.35.

[5] It is consistent with the value used in Nickel et al. (2008) where this parameter equals to 0.014 with the death rate at 0.01.

[6] This value is consistent with the literature. Heijdra and Ligthart (2006) use depreciation rate equals to 0.06 with the higher rate of time preference 0.04.

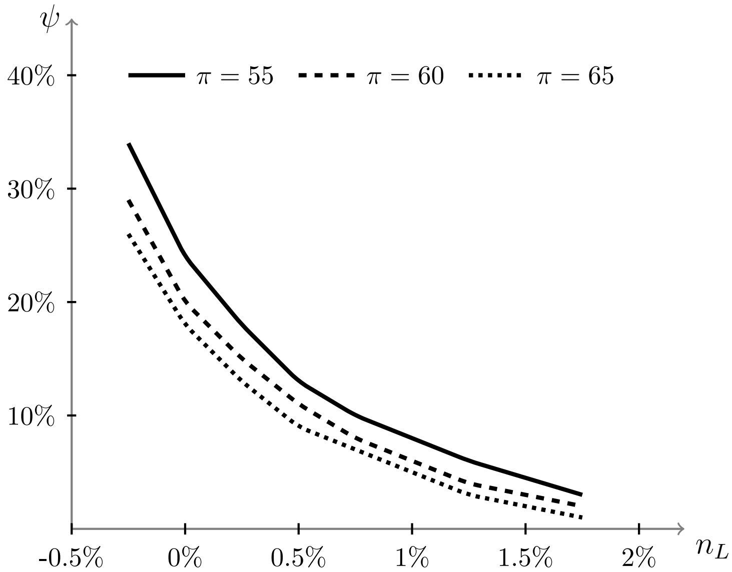
[7] Although Bettendorf and Heijdra (2006) do not use calibration in the numeric example the restriction on the retirement age postulates that it should be higher than 48.5 and 56.8 for different birth and death rates.

3.2 Choice of policy instruments depending on the retirement ages

Optimal social contributions

To focus on how the optimal social contributions depend on the retirement age we, first, consider the case with a fixed income tax ($t_L = 40\%$). Quantitative results suggest that a higher retirement age leads to lower rate of social contributions ψ. The value of social contributions t_W is affected also by the change in the median wage. Medium wage increases with the higher retirement age due to the higher capital per capita and decreases with lower working period due to the decreasing productivity of labor. The latter effect is stronger, so medium wage is decreasing with the higher retirement age. As the result of lower medium wage pensions are lower as well. Optimal level of ψ is decreasing with higher population growth as it leads to the higher share of young generations who pays social contributions.

Figure 1. *Optimal share of social contributions under different retirement age*



Optimal income tax

Next we consider the optimal choice of two instruments: income tax, t_L, and the rate of social contributions, t_W. In this case the corner solution takes place: optimal income tax rate is positive, while the rate of social contributions equals zero. The results of welfare maximization for $\pi = 60$ and $\pi = 65$ are presented in Table 3 below.[8]

Table 3. *Optimal income tax under different retirement age*

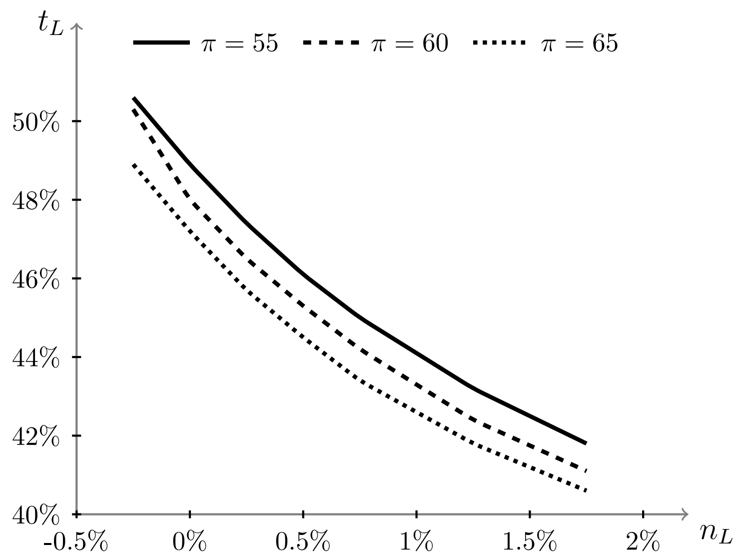
[8]The results are presented for $\beta = 1.25\%$.

n_L	-0.25%	0.00%	0.25%	0.50%	0.75%	1.25%	1.75%
$\pi = 60$							
t_L	50.3%	48%	46.5%	45.3%	44.2%	42.4%	41.1%
d_y	8.1%	7%	6%	5.2%	4.4%	3.3%	2.4%
$\pi = 65$							
t_L	48.9%	47.2%	45.7%	44.5%	43.4%	41.8%	40.6%
d_y	7.5%	6.4%	5.4%	4.6%	3.9%	2.8%	2.1%

The pension system in this case is run with a deficit as well. It decreases with population growth and retirement age. Pensions equal 30% of the median wage which changes with the longer working period, so pensions work as an automatic stabilizer.

The relationship of the optimal income tax and the birth rate is presented on Fig.2.[9] Optimal income tax exhibits the same inverse relationship as in the previous case. In this case social contributions are zero, while income tax rate is decreasing with the higher population growth: the deficit and pensions can be financed with lower income taxes when the share of young population is high.

Figure 2. *Optimal income tax under different retirement age*



Higher income tax and lower social contributions when both instruments are chosen optimally illustrates that these instruments can be considered as perfect substitutes in financing pensions and the deficit of the pension fund. Additional calculations show that an increase of ψ by 0.5% leads to the decrease in the optimal income tax rate by 0.2%. This relation can be explained by the difference in the payment period of both taxes: while income tax is paid throughout the life, social contributions are paid only up to the retirement age, and depends on the medium wage.

Income tax and social contributions are perfect substitutes only when the interior solution is considered, when both policy instruments are greater than zero. In the case when all acceptable combinations of instruments are considered the optimal mix includes strictly positive income tax rate and zero social contributions. Corner solution takes place

[9]The death rate β equals 1.25%.

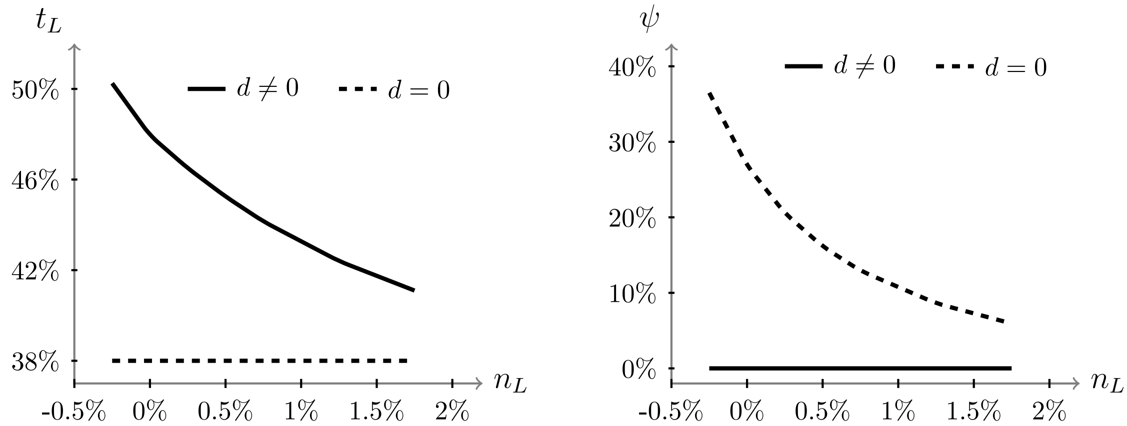
because zero social contributions leads to lower optimal income tax rate and is welfare improving. The other steady states with zero social contributions and higher income tax rate are characterized by lower social welfare.

3.3 Choice of policy instruments under balanced and unbalanced pension system

Under the balanced pension system the optimal income tax is lower than in the case of the unbalanced pension system for all growth rates, because in this case pensions are financed by social contributions and not out of the government budget. The optimal level of the income tax is constant, while the share of social contributions is decreasing with the population growth. Higher birth rate increases the value of the social contributions to the pension fund because in this case more young generations makes contributions. Therefore, the lower social contributions rate is needed to keep the pension fund balanced. Therefore, depending on the type of pension system, optimal income tax or social contributions are changing with the growth rate with the other instrument remains at its optimal level.

Figure 3 illustrates the results for the optimal level of t_L and ψ in the case of balanced and unbalanced pension system.[10]

Figure 3. *Optimal rate of the social contributions and income tax under balanced and unbalanced pension system*



In the case of balanced pension system social welfare is lower than in the case of unbalanced pension system. The former leads to the higher level of capital per capita and as the result higher social welfare. This result corresponds to the results of the OLG models with dynamically efficient steady state where PAYG pension system leads to lower social welfare.

The equilibrium with unbalanced pension system leads not only to higher social welfare but to the lower level of public debt due to the lower surpluses. Under balanced pension system tax revenues are lower: first, in this case optimal tax rate is lower as pensions are financed by social contributions; second, aggregate wage is lower due to the lower capital per capita. Public spending is also lower in the case of balanced system: due to both lower government expenditures and zero deficit of pension fund.

[10]The results for $\beta = 1.25\%$ and $\pi = 60$

3.4 Choice of policy instruments depending on the life expectancy

Next we compare steady states under the different levels of life expectancy, namely 70 and 80 years under the unbalanced pension system. Different life expectancy corresponds to β varying from $\beta = 1.43\%$ to $\beta = 1.25\%$. Income tax and social contribution rates are chosen optimally to maximize the social welfare.[11]

We have considered the fixed retirement age so that equilibria under the same birth rates were comparable. The results for the optimal income tax and share of social contributions are presented in Table 4. The optimal policy mix remains the same: positive rate of income tax and zero social contributions.

Table 4. *Optimal income tax under different life expectancy*

$\boldsymbol{\eta}$	**1%**	**1.25%**	**1.5%**	**1.75%**	**2%**	**2.5%**	**3%**
$\boldsymbol{\beta = 1.25\%}$							
n_L	-0.25%	0%	0.25%	0.5%	0.75%	1.25%	1.75%
t_L	50.3%	48%	46.5%	45.3%	44.2%	42.4%	41.1%
d_y	8.1%	7%	6%	5.2%	4.4%	3.3%	2.4%
$\boldsymbol{\beta = 1.43\%}$							
n_L	-0.43%	-0.18%	0.07%	0.32%	0.57%	1.07%	1.57%
t_L	49.7%	48%	46.5%	45.3%	44.2%	42.5%	41.1%
d_y	8.1%	7%	6%	5.2%	4.4%	3.3%	2.4%

Under the unbalanced pension system higher life expectancy leads to the lower private consumption, higher output and, as the result, higher government expenditures in the steady state. Public debt is higher under higher life expectancy due to the higher government expenditures and lower interest rate, while optimal income tax rate remains unchanged with higher β. In can be the result of the exogenous labor supply assumption.

At the same time pension system is run with deficit. Its share to the GDP is almost the same under different β because the optimal share of social contributions, ψ, while the share of pensions in the GDP is constant. Although the absolute value of pensions is increasing with the higher life expectancy due to the higher median wage, its change is proportional to the change in the output (as both output and median wage are functions of capital). I

3.5 Choice of policy instruments depending on the labor productivity

Higher productivity is modeled as the lower speed, α, with which the product of labor is falling with age. The results are presented in Table 6 for α equals 1.25%, 1.2% and $\alpha = 1.1\%$.

[11] In order to check the robustness of the results the change in β was considered for different retirement ages, for π from 55 to 70. Since the results for different retirement ages are similar, we provide here only the results for $\pi = 60$.

Table 5. *Optimal income tax under different labor productivity*[12]

n_L	-0.25%	0%	0.25%	0.5%	0.75%	1.25%	1.75%
$\alpha = 1.25\%$							
t_L	50.3%	47.95%	46.5%	45.25%	44.15%	42.4%	41.1%
d_y	8.08%	6.95%	5.99%	5.15%	4.42%	3.27%	2.39%
$\alpha = 1.1\%$							
t_L	50.4%	48.3%	46.8%	45.45%	44.35%	42.6%	41.2%
d_y	8.32%	7.17%	6.17%	5.29%	4.55%	3.36%	2.47%

Optimal income tax decreases with population growth and increases with labor productivity. Higher labor productivity results in the higher capital in the steady state and, therefore, output per capita as well as government expenditures and private consumption, although the share of private consumption remains the same. Higher productivity leads to the higher wage, increasing the level of pensions, that equal 30% of the median wage. It puts the higher pressure on the pension system and, thus, the government budget. However, despite the fact that the deficit of the pension fund is higher under higher labor productivity, public debt is lower in the equilibrium with the higher productivity due to the higher income tax payments (driven by the higher wage), which have increased more than government expenditures.

4 Conclusion

We extend the OLG model with infinitely living households developed by Bettendorf and Heijdra (2006) by introducing an unbalanced pension system, where the deficit of the pension fund is covered by the transfer from the government budget. This assumption makes income tax and social contributions interact as perfect substitutes in the financing of pensions and insuring the stability of the public debt.

It is true in the case of balanced pension system when social contributions should be strictly positive to cover the spending of pension fund. Thus, concerning the combinations of policy instruments only internal solution is considered. However, social welfare is lower under balanced pension system, while the level of public debt is higher comparing to the case of unbalanced pension system. It was also shown that under balanced pension system optimal income tax is lower than under unbalanced pension system, and it does not depend on the population growth. Social contributions on the contrary are decreasing with the population growth. When both fiscal instruments are chosen optimally, in the case of unbalanced pension system the corner solution is possible: optimal policy mix includes positive income tax rate, which decreases with the population growth, and zero social contributions.

The results also illustrate that higher retirement age leads to lower social contributions (or income tax in the case of unbalanced pension system) and can lower public debt and, thus, can be introduced as an additional measure of fiscal consolidation.

[12] The results are presented for $\pi = 60$ and $\beta = 1.25\%$.

The developed framework can be used to analyze further the optimal set of the reforms of the pension system and fiscal policy measures. The results can extend the research of the fiscal measures of consolidation and define the welfare optimal fiscal policy and pension reforms. The developed framework can be applied to the analysis of consequences of demographic changes for the public finances and in the development of the optimal consolidation measures.

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Appendix

1. Derivation of the aggregate Euler equation

The equation (18) can simplified as follows.

$$\frac{\dot{C}(t)}{C(t)} = [r(t) - \rho] + \frac{\eta L(t)\bar{c}(t,t) - \beta C(t)}{C(t)}. \tag{A1}$$

As new generations are born without any financial assets ($\bar{a}(t,t) = 0$), thus from (8) $\bar{c}(t,t) = (\rho + \beta)\bar{a}^H(t,t)$ and taking into account (16) we get:

$$\frac{\eta L(t)\bar{c}(t,t) - \beta C(t)}{C(t)} = (\rho + \beta)\frac{\eta L(t)\bar{a}^H(t,t) - \beta(A(t) + A^H(t))}{C(t)}.$$

Aggregate human wealth is defined as follows:

$$A^H(t) = \int\limits_{-\infty}^{t-\pi} \left[\bar{a}^H(\upsilon,t)\right]_{t-\upsilon>\pi} d\upsilon + \int\limits_{t-\pi}^{t} \left[\bar{a}^H(\upsilon,t)\right]_{0<t-\upsilon\leq\pi} d\upsilon,$$

where:

$$\left[\bar{a}^H(\upsilon,t)\right]_{t-\upsilon>\pi} = \int\limits_{t}^{\infty}(1-t_L)W^N(\upsilon,t)\mathrm{e}^{R(t,\tau)+\beta(t-\tau)}d\tau + \int\limits_{t}^{\infty} z\mathrm{e}^{R(t,\tau)+\beta(t-\tau)}d\tau$$

and $R(t,\tau) = \int\limits_{t}^{\tau} r(s)ds$.

$$\left[\bar{a}^H(\upsilon,t)\right]_{0<t-\upsilon\leq\pi} = \int\limits_{t}^{\infty}(1-t_L)W^N(\upsilon,t)\mathrm{e}^{R(t,\tau)+\beta(t-\tau)}d\tau - \int\limits_{t}^{\upsilon+\pi} t_W\mathrm{e}^{R(t,\tau)+\beta(t-\tau)}d\tau + \int\limits_{\upsilon+\pi}^{\infty} z\mathrm{e}^{R(t,\tau)+\beta(t-\tau)}d\tau.$$

For the simplicity of the analysis let us assume the constant interest rate r (so that equations could be applicable to the steady state).

$$\left[\bar{a}^H(\upsilon,t)\right]_{0<t-\upsilon\leq\pi} = \int\limits_{t}^{\infty}(1-t_L)W^N(\upsilon,t)\mathrm{e}^{(r+\beta)(t-\tau)}d\tau - \frac{t_W}{r+\beta}\left(1-\mathrm{e}^{-(r+\beta)(\upsilon+\pi-t)}\right) + \frac{z}{r(t)+\beta}\mathrm{e}^{-(r+\beta)(\upsilon+\pi-t)}.$$

We know that age dependent wage can be written as follows:

$$W^N(\upsilon,t)=E(\tau-\upsilon)F_N(k(t),1)=\omega_0\mathrm{e}^{-\alpha(\tau-\upsilon)}F_N(k(t),1),$$

$$\int_t^\infty(1-t_L)W^N(\upsilon,t)\mathrm{e}^{(r+\beta)(t-\tau)}d\tau=\mathrm{e}^{\alpha(\upsilon-t)}\Omega_0(t),$$

where $\Omega_0(t)$ is defined as follows:

$$\Omega_0(t)=\omega_0\int_t^\infty(1-t_L)F_N(k(t),1)\mathrm{e}^{(r+\alpha+\beta)(t-\tau)}d\tau.$$

Substituting this definition into the expressions for human wealth of workers and retirees, noted above, we get:

$$\int_{-\infty}^{t-\pi}L(\upsilon,t)\left[\bar{a}^H(\upsilon,t)\right]_{t-\upsilon>\pi}d\upsilon=$$

$$=\int_{-\infty}^{t-\pi}\eta\mathrm{e}^{\eta\upsilon}\mathrm{e}^{-\beta t}\left[\mathrm{e}^{\alpha(\upsilon-t)}\Omega_0(t)\upsilon+\frac{z}{r+\beta}\right]d\upsilon=$$

$$=L(t)\left[\frac{\eta}{\alpha+\eta}\Omega_0(t)\mathrm{e}^{-(\alpha+\eta)\pi}-\frac{z}{r+\beta}\mathrm{e}^{-\eta\pi}\right].$$

$$\int_{t-\pi}^{t}L(\upsilon,t)\left[\bar{a}^H(\upsilon,t)\right]_{0<t-\upsilon\leq\pi}d\upsilon=$$

$$=\int_{t-\pi}^{t}\eta\mathrm{e}^{\eta\upsilon}\mathrm{e}^{-\beta t}\left[\mathrm{e}^{\alpha(\upsilon-t)}\Omega_0(t)-\frac{t_W}{r+\beta}\left(1-\mathrm{e}^{-(r+\beta)(\upsilon+\pi-t)}\right)+\frac{z}{r+\beta}\mathrm{e}^{-(r+\beta)(\upsilon+\pi-t)}\right]d\upsilon=$$

$$=L(t)\left[\frac{\eta}{\alpha+\eta}\Omega_0(t)\left(1-\mathrm{e}^{-(\alpha+\eta)\pi}\right)-\frac{t_W}{r+\beta}(1-\mathrm{e}^{-\eta\beta})+\frac{\eta(t_W+z)}{r+\beta}e^{-\beta\pi}\left(\frac{\mathrm{e}^{-r\pi}-\mathrm{e}^{-n_L\pi}}{n^L-r}\right)\right].$$

Thus aggregate human wealth is:

$$A^H(t)=L(t)\left[\frac{\lambda\Omega_0(t)}{\alpha+\eta}+\eta\mathrm{e}^{-\beta\pi}\frac{t_W+z}{r+\beta}\left(\frac{\mathrm{e}^{-r\pi}-\mathrm{e}^{-n_L\pi}}{n^L-r}\right)\right],$$

where it was used that:

$$t_W(1-\mathrm{e}^{-\eta\pi})=z\mathrm{e}^{-\eta\pi}+d^{pens}(t)\Rightarrow t_W+z=\frac{z+d^{pens}(t)}{1-\mathrm{e}^{-\eta\pi}}.$$

From the expression for working-age households, taking into account that $t_W = d + (t_W + z)\mathrm{e}^{-\eta\pi}$:

$$\bar{a}^H(t,t) = \Omega_0(t) + \left(\frac{t_W + z}{r(t)+\beta}\right)\left(\mathrm{e}^{-(r(t)+\beta)\pi} - \mathrm{e}^{-\eta\pi}\right) - \frac{t_W}{r(t)+\beta} =$$
$$= \Omega_0(t) + \mathrm{e}^{-\beta\pi}\left(\frac{t_W + z}{r(t)+\beta}\right)\left(\mathrm{e}^{-r(t)\pi} - \mathrm{e}^{-n_L\pi}\right) - \frac{d(t)}{r(t)+\beta}.$$

After substituting this expression in the equation for A^H and eliminating $\Omega_0(t)$ we get:

$$\eta L(t)\bar{a}^H(t,t) = (\alpha+\eta)A^H(t) - \eta\gamma L(t),$$

where

$$\gamma = \frac{d(t)}{r+\beta} + (r+\alpha+\beta)\left(\frac{\mathrm{e}^{-\beta\pi}}{1-\mathrm{e}^{-\eta\pi}}\right)\left(\frac{z+d(t)}{r+\beta}\right)\left(\frac{\mathrm{e}^{-r\pi} - \mathrm{e}^{-n_L\pi}}{n^L - r}\right).$$

Taking into account the expression for $\bar{a}^H(t,t)$ and taking into account (16) we get:

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho + \alpha + n^L - (\rho+\beta)\frac{\eta\gamma L(t) + (\alpha+\eta)A(t)}{C(t)}.$$

2. Derivation of aggregate non-interest income

Aggregate non-interest income is defined as follows:

$$WI(t) = \int_{-\infty}^{t} L(v,t)WI(v,t)dv,$$

where $WI(v,t)$ is defined as follows:

$$WI(v,\tau) = \begin{cases} (1-t_L)W^N(v,\tau) - t_W & \text{for } \tau - v \le \pi, \\ (1-t_L)W^N(v,\tau) + z & \text{for } \tau - v > \pi. \end{cases}$$

Thus $WI(t)$ is split into parts, non-interest income of the retirees and non-interest income of the young:

$$WI(t) = \int_{-\infty}^{t-\pi} L(v,t)\left[(1-t_L)WI(v,t) + z\right]dv + \int_{t-\pi}^{t} L(v,t)\left[(1-t_L)WI(v,t) - t_W\right]dv.$$

After applying the expression for $W^N(v,t) = E(t-v)F_N(k_N(t,1)) = \omega_0\mathrm{e}^{-\alpha(t-v)}F_N(k_N(t,1))$, which comes from taking into account the definition of the efficiency index $E(\tau - v)$ and fact that the wage of particular worker born at v is equal to the marginal product of labor, adjusted for his or her productivity.

$$WI(t) = \int\limits_{-\infty}^{t-\pi} L(\upsilon,t)\left[(1-t_L)\omega_0 \mathrm{e}^{-\alpha(t-\upsilon)} F_N(k_N(t,1)) + z\right] d\upsilon +$$
$$+ \int\limits_{t-\pi}^{t} L(\upsilon,t)\left[(1-t_L)\omega_0 \mathrm{e}^{-\alpha(t-\upsilon)} F_N(k_N(t,1)) - t_W\right] d\upsilon.$$

Rearranging we get:

$$WI(t) = (1-t_L)F_N(k_N(t,1))\left[\int\limits_{-\infty}^{t-\pi} L(\upsilon,t)\omega_0 \mathrm{e}^{-\alpha(t-\upsilon)} d\upsilon + \int\limits_{t-\pi}^{t} L(\upsilon,t)\omega_0 \mathrm{e}^{-\alpha(t-\upsilon)} d\upsilon\right] +$$
$$+ z\int\limits_{-\infty}^{t-\pi} L(\upsilon,t)d\upsilon - t_W \int\limits_{t-\pi}^{t} L(\upsilon,t)d\upsilon =$$
$$= (1-t_L)F_N(k_N(t,1)) \int\limits_{-\infty}^{t} L(\upsilon,t)\omega_0 \mathrm{e}^{-\alpha(t-\upsilon)} d\upsilon +$$
$$+ z\int\limits_{-\infty}^{t-\pi} L(\upsilon,t)d\upsilon - t_W \int\limits_{t-\pi}^{t} L(\upsilon,t)d\upsilon.$$

Noting from (5) that $\bar{n}(\upsilon,t) = E(t-\upsilon) = \omega_0 \mathrm{e}^{-\alpha(t-\upsilon)}$ and applying the notion of $N(t)$ from (24) we get:

$$WI(t) = (1-t_L)F_N(k_N(t,1))\frac{\eta\omega_0}{\alpha+\eta}L(t) + z\int\limits_{-\infty}^{t-\pi} L(\upsilon,t)d\upsilon - t_W \int\limits_{t-\pi}^{t} L(\upsilon,t)d\upsilon.$$

Applying (14) we get:

$$WI(t) = (1-t_L)F_N(k_N(t,1))\frac{\eta\omega_0}{\alpha+\eta}L(t) + z\int\limits_{-\infty}^{t-\pi} \eta \mathrm{e}^{\eta\upsilon-\beta t} d\upsilon - t_W \int\limits_{t-\pi}^{t} \eta \mathrm{e}^{\eta\upsilon-\beta t} d\upsilon =$$
$$= (1-t_L)F_N(k(t,1))\frac{\eta\omega_0}{\alpha+\eta}L(t) - (1-\mathrm{e}^{-\eta\pi})t_W \mathrm{e}^{n_L t} + z\mathrm{e}^{-\eta\pi}\mathrm{e}^{n_L t}.$$

Taking into account the fact that the pension system is run on a non balanced manner, thus using (34) the equation of the aggregate income can be simplified as follows:

$$WI(t) = (1-t_L)\frac{\eta\omega_0}{\alpha+\eta}F_N(k_N(t,1))L(t) - D(t).$$

3. Social welfare

$$SW = \int_{t}^{\infty} \int_{-\infty}^{\tau-\pi} L(\upsilon,\tau)[ln\bar{c}(\upsilon,\tau)]e^{(\rho+\beta)(t-\tau)}d\upsilon d\tau +$$

$$+\int_{t}^{\infty} \int_{\tau-\pi}^{\tau} L(\upsilon,\tau)[ln\bar{c}(\upsilon,\tau)]e^{(\rho+\beta)(t-\tau)}d\upsilon d\tau.$$

Taking into account the Eurler equation:

$$\bar{c}(\upsilon,\tau) = \bar{c}(\upsilon,t)\mathrm{e}^{(r-\rho)(\tau-t)},$$

$$\bar{c}(\upsilon,\upsilon) = \bar{c}(\upsilon,t)\mathrm{e}^{(r-\rho)(\upsilon-t)},$$

$$(\rho+\beta)[\bar{a}(\upsilon,\upsilon)+\bar{a}^H(\upsilon,\upsilon)] = (\rho+\beta)[\bar{a}(\upsilon,t)+\bar{a}^H(\upsilon,t)]\mathrm{e}^{-(r(t)-\rho)(\upsilon-t)}.$$

Applying that $\bar{a}(\upsilon,\upsilon)=0$ and simplifying we get the definition for the consumption of generation υ:

$$\bar{c}(\upsilon,\tau) = (\rho+\beta)(\bar{a}(\upsilon,\tau)+\bar{a}^H(\upsilon,\tau)) = (\rho+\beta)\bar{a}^H(\upsilon,\upsilon)\mathrm{e}^{-(r-\rho)(\upsilon-t)}.$$

$$\bar{a}_o^H(\upsilon,t) = \frac{1}{r+\alpha+\beta}\left(\omega(1-\varepsilon)(1-t_L)\left(\frac{k}{n}\right)^{\varepsilon}\mathrm{e}^{\alpha(\upsilon-t)}\right)+\frac{z}{r+\beta},$$

$$\bar{a}_y^H(\upsilon,t) = \frac{1}{r+\alpha+\beta}\left(\omega(1-\varepsilon)(1-t_L)\left(\frac{k}{n}\right)^{\varepsilon}\mathrm{e}^{\alpha(\upsilon-t)}\right)-\frac{tw}{r+\beta}+\frac{tw+z}{r+\beta}\mathrm{e}^{-(r+\beta)(\upsilon+\pi-t)}.$$

Substituting the expressions of human wealth from above we get the following social welfare function as a function of the steady state level of capital per capita:

$$SW(t) = \frac{\mathrm{e}^{n_L t}}{n_L-\rho-\beta}\big[(\ln((\rho+\beta)a_o^H)+$$

$$+(r-\rho)\frac{\pi\eta+1}{\eta})\mathrm{e}^{\eta\pi}-\ln((\rho+\beta)a_y^H)(1-\mathrm{e}^{-\eta\pi})-$$

$$-(r-\rho)(1-\mathrm{e}^{\eta\pi}-\eta\pi\mathrm{e}^{-\eta\pi})\eta^{-1})\big]$$