DYNAMIC MODELS WITH HETEROGENEOUS AGENTS: The Case of Contests

By Andreas Hefti and Martin Grossmann*

December 31, 2015

Abstract

Analyzing economic models with possibly many arbitrarily heterogeneous agents is an elusive task already in the static case, and even more so in dynamic settings. We propose a novel, systematic approach to analyze such models building on the notion of aggregate-taking behavior. The usefulness of this approach is demonstrated by analyzing a two-stage contest. Our new set of tools allows us to study how changes in the contest design, particularly variations the prize structure or the intensity of the contest, affect the distribution of the equilibrium success chances and profit inequality, without the need to resort to Computer simulations.

JEL Classification: D72; D82; C73; C62; C65

Keywords: Heterogeneity; Inequality; Contest Theory; Political Economy; Aggregate-Taking Behavior

Preliminary Version – PLEASE DO NOT CIRCULATE

*Corresponding author: Andreas Hefti, Bluemlisalpstrasse 10, CH-8006 Zurich, andreas.hefti@econ.uzh.ch. Author affiliations. Hefti: Department of Economics, University of Zurich and School of Management and Law, Zurich University of Applied Sciences. Grossmann: Department of Business Administration, University of Zurich. We wish to thank Armin Schmutzler, David Wettstein, Shuo Liu, Georg Noeldeke and Nick Netzer as well as the participants of the Swiss-Basel Theory Meeting 2014 and EEA 2014 for many valuable comments.

1 Introduction

Economic activity frequently is of a dynamic nature and most of the time involves heterogeneous agents. On a daily basis, individuals and firms compete for scarce goods, opportunities, positions, and status. Yet, most theoretical analysis need to impose symmetry conditions in order to describe the respective economic behavior and the resulting equilibrium prediction. The main problem, if one does not wish to rely on Computer simulations, is that many models quickly become highly intractable with heterogeneous agents already in a static setting. Things usually do not get simpler in dynamic versions.

In this article we propose a systematic approach to analyzing models with heterogeneous agents. The model is flexible both with respect to the number of heterogeneous agents and the degree of asymmetry among the agents. We evaluate the model in the special but important case of a dynamic contest, while the general setting is almost immediately applicable to any model where some form of "market share" (be it probabilistic or effective) is part of the participant's choice variables.

We choose the contest setting as our primary application because many economic interactions have been related to contests, where contestants expend scarce resources and are rewarded based on their relative efforts. Often contests exhibit a multi-stage structure where the winner advances to subsequent stages and the losers are eliminated. Such multi-stage elimination tournaments that offer prizes on each stage are ubiquitous. For example, politicians compete in multi-stage election processes (Glazer and Gradstein (2005), Klumpp and Polborn (2006)), R&D laboratories compete in patent race contests (Loury (1979), Taylor (1995)), employees compete in promotion contests (Rosen (1986), Bognanno (2001)), and athletes in multi-stage championships (Szymanski (2003)).

While it seems intuitive that stage-wise prizes in multi-stage contests should have similar incentive effects, we challenge this intuition in a two-stage contest setting, and show that they, in fact, have exactly *opposing* effects on payoffs and success distributions. Higher first-stage prices tend to equate success chances, efforts and payoffs, while higher second-stage prizes increases any pre-existing inequality among participants. We illustrate that this result is true independent of the inter-temporal connection between the two stages. Specifically, we investigate contests in which first-stage efforts do not carry over to the second stage (the standard case) and contests with carryovers.

In many contests, the extent of how carryovers exist is not obvious. For example, consider a firm that has a vacant CEO position on the international level. To fill this position the firm will choose among senior managers that have been successful in prior promotional contests on the national levels. Part of the human capital of these managers (such as the talent, academic qualification, experience in leadership and general professional experience) has often been built up on an earlier stage of their career and may affect the chance to advance in his/her job in the future. But one can also argue that the manager's effort today mainly impacts his/her chance to be promoted today. The degree to which the previously accumulated human capital carries over from the national stage to the international stage determines the extent of carryovers. Similarly, in architectural competitions, architecture offices compete with a certain design proposal on preliminary stages, and in case of success, on subsequent stages. The architecture firm Herzog & de Meuron, for instance, started their business in 1978 with smaller projects. After building up their human capital they advanced to the "Champions League" of global architecture competition in 1995 converting the Bankside Power Station into the Tate Gallery of Modern Art in London. The previously acquired reputation and experience have had an impact that Herzog & de Meuron got the opportunity in later years to construct many meaningful buildings worldwide. However, earlier efforts are not sufficient for today's success. Convincing projects submissions and high efforts today are necessary to win new project orders. But not only in the world of business do we find contests in which contestants past accumulated capital influences the outcome of competitions on different stages. Also in sports, clubs invest in its roster with long-term contracts and they compete in multiple subsequent contests. For example, a soccer club that wins the domestic league competition qualifies to the UEFA Champions League in the subsequent season. The team quality in the Champions League is similar to the previous season as part of the team's characteristics (e.g., the core team members, the game play, the team manager, etc) usually does not significantly change. But, of course, clubs are able to re-invest in new players and selectively improve the team strength. In all of these examples, today's (human) capital stock has an impact on today's performance but probably also on tomorrow's success.

If carryovers exist and contestants are rational, contestants will anticipate the importance of today's contest and the potential future opportunities as a winner of the first-stage contest. In this paper, we apply a new set of tools that allow us to systematically study this and related questions in case of an imperfectly discriminatory, two-stage elimination contest with asymmetric contestants. Particularly, we analyze how changes in the first-stage and second-stage contest prize affect effort levels, the distribution of equilibrium success chances and payoffs. Noting that contests are special types of aggregative games (Corchon (1994)), we facilitate the analysis by resorting to aggregate-taking behavior (ATB) of the contestants. ATB means that the contestants choose their effort level by taking the aggregate effort level in the contest as given, while this aggregate is endogenously determined in equilibrium. We use ATB because (i) it is plausible that the contestants have an idea of some aggregate (or average) effort level, rather than the effort level of individual contestants if many contestants are involved, (ii) ATB-equilibria approximate Nash equilibria if the number of contestants grows large and, most importantly, (iii) this approach leads to high analytical tractability.

Related literature Our article contributes to the growing literature on dynamic contests, in particular, on multi-stage elimination contests. Starting with Rosen (1986) seminal article about elimination tournaments, the literature on multi-stage has mainly focus on contests with homogeneous contests (e.g., Gradstein and Konrad (1999), Stein and Rapoport (2004), Fu and Lu (2012)) - heterogeneities among contestants have largely been neglected. Exceptions that examine heterogeneity in multi-stage, imperfectly discriminating contests are Stein and Rapoport (2004), Harbaugh and Klumpp (2005), Klumpp and Polborn (2006) and Stracke (2013). Stein and Rapoport (2004) develop a two-stage contest model of group rent seeking. However, they consider only heterogeneity between groups and assume homogeneity within each group, while we allow for heterogeneity between and within groups (contests). Harbaugh and Klumpp (2005) allow only for two types of contestants and in Klumpp and Polborn (2006) the same contestants repeatedly compete in the first stage. Stracke (2013) considers a sequential elimination contest without carryovers. As there are only two types of contestants in the model an extensive analysis of the impact of heterogeneity is partly limited. Another branch of the contest literature, namely the literature on perfectly discriminating contests (so-called all-pay auctions) has made further progress in examining heterogeneity. For example, Moldovanu and Sela (2006) analyze the optimal design of contests with heterogeneous contestants when the contest designer wants to maximize either total effort or the highest effort. Groh et al. (2012) focus on how the allocation of player types in the first stage (so-called seeding) affects the contest properties. Finally, the existing literature on multi-stage contests has mostly focused on contests without carryovers. Only few articles examine contests with carryovers (e.g. Schmitt et al. (2004) or Grossmann et al. (2011)). However, as the previous examples show the existence of carryovers depends on the specific context. Therefore, we propose a framework that allows analyzing contests with and without carryovers.

The remainder of the article is structured as follows. In section 2 we introduce the general form of the contest we want to study in this article as well as the ATB equilibrium concept. Section 3 presents our distribution tools, which we then apply various versions of a general two-stage contest in section 4. Section 7 concludes. The appendix (section A) contains further theoretical results and the longer proofs omitted in the main text.

2 A baseline contest model

Continuum agents We develop our model assuming a unit mass of continuum agents, where the agent population corresponds to $[0,1]$. We work with continuum agents because of two reasons.[1] First, continuum agents simplify the formal analysis because they give probability densities rather than atomistic distributions. Second, assuming continuum agents in our setting is, in fact, without loss of generality, which we show in appendix A.1. We capture heterogeneity among agents by an increasing (hence integrable) function $c : [0,1] \to \mathbb{R}_+$, where we will interpret $c(i)$ as agent i's cost (efficiency) parameter. If c is a step function, this means that all agents sitting on the same step are homogeneous to each other.[2]

[1] Continuum agents have been widely used. For example, the famous Dixit-Stiglitz Love of variety model uses a continuum of firms representing the different product varieties, which further plays a major role in the Melitz-model of international trade.

[2] That is, the steps partition the population into equivalent cost types, and all members of an equivalence class will choose the same behavior.

Example: Fixed-prize contests Suppose that a unit mass of agents compete to obtain a single prize worth of $V > 0$. The prize could be, e.g., obtaining a research grant, winning a political election, or winning a sport championship. Each agent i has means to influence his chance of seizing the prize, e.g. by allocating more time to writing the grant proposal, by expending more resources for lobbying activities, or by hiring new players for a team. Agent i's endeavor to win the contest is quantified by his **effort level** $e(i) \geq 0$. Efforts are costly as measured by the cost function $c(i)C(e(i))$, with $C' > 0$. How likely an agent is to seize the prize is random, but higher effort $e(i)$ increases i's success chances, and higher efforts of the other contestants decreases i's chances. Let $e(-i)$ denote the vector of efforts of all agents other than i. The expected payoff of agent i is

$$\Pi(i) = P\left(e(i), e(-i)\right) V - c(i)C(e(i)) \tag{1}$$

The essential primitive in (1) is the **Contest Success Function** (CSF) $P\left(e(i), e(-i)\right) \in [0, 1]$, sometimes also called the contest technology, which captures how individual efforts map into success probabilities. Each agent non-cooperatively and simultaneously chooses his effort $e(i)$ to maximize $\Pi(i)$. A flexible and important specification is provided by the class of Tullock CSF's:[3]

$$P\left(e(i), e(-i)\right) = \frac{e(i)^{1/\eta}}{\int e(s)^{1/\eta} ds}, \qquad \eta > 0 \tag{2}$$

The Tullock CSF has three intuitive and useful features: i) Only relative efforts matter (zero-homogeneity), ii) individual efforts matter relatively to a sum-aggregative measure $\int e(s)^{1/\eta}$ of total efforts and iii) the "noise" parameter η flexibly controls how discriminatory the contest is. A low value of η makes the CSF very sensitive to possibly small effort differences.[4] That is, the contest becomes more discriminatory (less noisy) for lower values of η. Most of our results do not depend on the precise value of η, while the assumption of a fixed noise parameter greatly simplifies the analysis.

Finally, it turns out to be more convenient to analyze an equivalent model, in which η enters the cost function rather than the CSF. To illustrate, let $C(e(i)) = e(i)$ and set $t(i) \equiv e(i)^{1/\eta}$, where we interpret $t(i)$ as i's effective (technology-adjusted) efforts. Using (2), payoff (1) can be restated as

$$\Pi(i) = \frac{t(i)}{T} V - c(i)t(i)^{\eta}, \qquad T \equiv \int t(s) \tag{3}$$

[3]In our setting, integrability of $e(i)$ will follow from the fact that $e(\cdot)$ is monotone.

[4]To see this think of $e(s) > 0$ as a finite step function and rewrite (2) as $\frac{1}{\int \left(\frac{e(s)}{e(i)}\right)^{1/\eta} ds}$. For $\eta \to 0$ the contest approaches a perfectly discriminatory contest (essentially an all-pay auction). At the other extreme, for $\eta \to \infty$, success becomes effort-independent in thus that if $e(s) > 0$ almost everywhere the CSF approaches the uniform distribution for any positive effort level $e(i) > 0$.

2.1 Aggregate-taking behavior (ATB)

Analyzing a contest with heterogeneous agents and a possibly endogenous prize structure $V = \hat{V}(t(i), \int t(s)ds, i)$ is a major formal challenge. However, such a prize structure is implied by the dynamic nature of two-stage contests. With endogenous prizes the efforts not only determine expected revenue via individual success chances, but also via directly affecting the value of the prize to the winner. To study the effects of certain exogenous variations of the prize function $V(\cdot)$ on the resulting distribution of success chances, payoffs and efforts in presence of heterogeneous agents, we solve the model by adopting the concept of **aggregate-taking behavior**. ATB means that an optimizing agent takes as given the aggregate effort level T in judging how effective an additional unit of effort is at increasing his expected benefit. One crucial upshot of ATB is tractability: While T is a parameter to the individual agent, it is an endogenous equilibrium variable to the model. This turns out to be of great convenience when analyzing the equilibrium system.[5] Besides technical justifications, ATB may be a realistic behavioral assumption in many situations. For example, the agents involved in a contest may not know what individual efforts the other agents choose nor what their cost types are, but they sill may have a good estimate of the aggregate (or average) effort level T in the contest. Such reasons explain why ATB has already been popular in economics (consider e.g. the Global-Games Literature, Dixit-Stiglitz monopolistic competition or traditional Walrasian price-taking equilibrium). Finally, there is a growing literature studying ATB more theoretically.[6]

XXX bis hier

2.2 ATB equilibria

Let $p(i) \equiv t(i)/T$ denote agent i's chance of seizing a prize worth of $V(\cdot)$. Note that by choosing individual efforts $t(i)$, each agent actually determines her conjectured success chance $p(i)$. To analyze the model, we can take a convenient shortcut, and assume that agents directly choose $p(i)$, given the equilibrium constraint that $\sum p(i) = 1$. Specifically, consider (3) with a generalized prize worth of $V = \hat{V}(t(i), T, i)$. The problem of agent i then is to solve

$$\max_{p(i) \geq 0} \Pi(i) = p(i)V(p(i), T, i) - \frac{1}{\eta} c(i)p(i)^{\eta} T^{\eta} \qquad \eta > 1 \tag{4}$$

where $V(p(i), T, i) \equiv \hat{V}(p(i)T, T, i)$, and $1/\eta$ is for normalization purpose. In an ATB equilibrium, each agent chooses his success probability $p(i)$ to maximize (4), and success probabilities integrate up to one.[7]

[5] If the number of agents n grows large, ATB equilibria approximate conventional Nash equilibria. While conventional Nash-behavior makes it virtually impossible to analyze the two-stage model with general heterogeneity, such a model possibly allows for numerical evaluations. We used such simulations to check if our main results could be reversed by Nash behavior, for which we found no evidence in the numerical data.

[6] A couple of recent papers have addressed ATB from a theoretical perspective. For example, Jensen (2010) considers aggregative games and best-reply potentials. Alos-Ferrer and Ania (2005) study the evolutionary stability of aggregative games, or Hefti (2014b) examines the connections between stability and uniqueness in sum-aggregative games.

[7] This implies that $\sum t(s) = T$.

Heterogeneity In the following we either let the cost coefficient function c be a finite step function or a strictly increasing C^2-function:[8]

- **Class I** consists of all increasing, right-continuous step functions for which $\exists i_0 \in (0,1)$: $i < i_0 \leq j \Rightarrow c(i) < c(j)$
- **Class II** consists of all strictly increasing functions $c \in C^2([0,1],[1,\bar{c}])$.

We refer to the final qualification for class I functions as the **somewhere strictly increasing (SI)** property, which means that heterogeneity matters in a non-zero-measure way, and is equivalent to the requirement that $c(i)$ is not constant on $(0,1)$.[9] Class I functions capture the relevant case of finitely many different cost types (e.g. finitely many agents). Intuitively, one could think of every cost type k being represented by an agent i_k, who solves problem (4) for his entire group.[10] An important motivation for studying class II type function is the possibility to use calculus methods to analyze the equilibrium distribution of success chances. Our main results hold for both classes.

We are now ready to formally state the definition of an ATB equilibrium.

Definition 1 (ATB-equilibrium) *An ATB equilibrium is a bounded function $p : [0,1] \to \mathbb{R}_+$ and a number $T \in (0,\infty)$ such that*

i) $p(i)$ *solves problem* (4) *for all* $i \in [0,1]$

ii) $\int_0^1 p(i)di = 1$

Henceforth, we refer to an ATB equilibrium just as an equilibrium. It should be intuitively clear that if $c(i)$ is a step function capturing agent groups $k = 1,...,K$ with measures $\gamma_1,...,\gamma_K$, $\sum_k \gamma_k = 1$, then so is $p(i)$, meaning that $p([0,1]) = \{p(i_1),...,p(i_K)\}$ and $\int_0^1 p(i)di = \sum_{k=1}^K \gamma_k p(i_k)$. Hence in case of class I finding the equilibrium requires solving a $(K+1)$-system of equations in the unknowns $p(i_1),...,p(i_K)$ and T.

Existence and uniqueness In this section we prove existence and uniqueness of ATB equilibria in an abstract contest with an endogenous prize. The way how we prove this result is representative for all later proofs in the two-stage contest model developed in section 4. Moreover, we obtain some preliminary insights on how the nature of spillovers in the prize function V might affect the equilibrium success distribution and related variables in presence of heterogeneous agents. For now we concentrate on the

[8] The functions in these classes are integrable.
[9] Class II functions also satisfy SI.
[10] It is straightforward to verify that identical agents also behave identically in our setting.

case of identity-independent prize values, i.e. we consider[11]

$$\Pi(i) = p(i)V(p(i),T) - \frac{1}{\eta}c(i)p(i)^{\eta}T^{\eta} \qquad \eta > 1 \tag{5}$$

Suppose that $\Pi(i)$ is continuous in $(p(i),T)$ on $[0,\infty)\times[0,\infty)$, and a C^2-function of $(p(i),T)$ on $(0,\infty)\times(0,\infty)$. Let $g(p(i),T) \equiv V(p(i),T) + p(i)V_1(p(i),T)$, $\mu_T(p(i),T) \equiv \frac{g_2(p(i),T)T}{g(p(i),T)}$, and denote a solution to $g(p(i),T) = c(i)p(i)^{\eta-1}T^{\eta}$ (the FOC pertaining to (5)) by $p(i;T)$. We interpret $g(p(i),T)$ as the marginal revenue of the contest to agent i with success goal $p(i)$ and aggregate effort T.

Assumption 1 *For $i \in [0,1]$ the following is satisfied:*

(A1) For $T > 0$: $g(0,T) > 0$, and $g(\cdot,T)$ bounded from above and strongly quasiconcave in $p(i) > 0$

(A2) $g(1,0) > 0$, and $g(1,\cdot)$ bounded from above and $\mu_T((p(i);T),T) < \eta$.

We have not imposed any assumption on the direction of how $p(i)$ and T affect marginal revenues. Also note that a ceteris paribus increase in T always raises marginal costs. This follows because maintaining a certain scoring target then requires more effort.
These two assumptions assert the existence of a unique and interior equilibrium.[12]

Proposition 1 (Existence of ATB equilibria) *Consider a contest with payoff function* (5) *satisfying assumption 1. Then there exists a unique ATB equilibrium $(p(i),T)$. All equilibrium payoffs are positive, and the success distribution $p(\cdot)$ is a bounded, decreasing and strictly positive density.*

The proof evolves in two steps, corresponding to the two requirements of an ATB equilibrium, and we explain them on intuitive grounds here. First, assumption (A1) means that investing at least a bit in the contest is always attractive ($g(0,T) > 0$), but the gains from investing more are limited ($g(\cdot,T)$ bounded). Therefore an optimizer $p(i;T) > 0$ exists for any given $T > 0$ and any $i \in [0,1]$. By strong quasiconcavity, $p(i;T)$ must be unique. However, T could be such that $\int p(i;T) = 1$ is violated. (A2) assures that a unique $\hat{T} > 0$ exists such that $\int_0^1 p(i;\hat{T})di = 1$ is satisfied. To understand the meaning of (A2) more intuitively, note that (A2) is satisfied if marginal revenue satisfies $g(p,T) > 0$ and is bounded from above $\forall T$. In such a case even the best agent seizes to invest as T (and therefore costs) grow arbitrarily large ($\lim_{T\to\infty} p(i;T) = 0$), and even the worst agent urges to invest as costs fall to zero ($\lim_{T\to\infty} p(i;T) = \infty$). Together with continuity these two properties then assure existence of $\hat{T}$ as desired. Uniqueness follows from $\mu_T((p(i);T),T) < \eta$, which says that the direct (positive) effect of $dT > 0$ on marginal cost must exceed the direct effect on marginal revenues, and therefore $p(i;T)$ (and thus $\int_0^1 p(i;T)di$) decreases in T.[13]

[11] We will see that a two-stage contest with perfect carryovers exactly implies such a structure for the stage I contest.
[12] We concentrate here on interior equilibria, as our later two-stage contest model usually features only such.
[13] This conditions also rules out the possibility of "perverse" comparative statics.

It is a trivial consequence of the proof of the above proposition that if all agents are homogeneous, i.e. $c(i) = c > 0$, so that the contest is a symmetric game, a unique symmetric equilibrium exists, and $p(i) = 1$ $\forall i \in [0, 1]$.[14]

Equilibrium properties To gain some insight on how the direction of direct or aggregate spillovers matters for the equilibrium distribution of $p(i)$ and $\Pi(i)$, we now compare relative success chances $\frac{p(i)}{p(j)}$ of any two different cost types to the corresponding technology adjusted cost coefficient $\left(\frac{c(j)}{c(i)}\right)^{\frac{1}{\eta-1}}$. The ratio $\frac{p(i)}{p(j)}$ has been coined **competitive balance** (CB) in sports economics featuring two-player contest (e.g. Szymanski (2003)). The CB is a measure of how unpredictable (or how fair) a two-player contest is. For now we adopt this terminology[15], and denote with $\text{CB}_{ij} \equiv \frac{p(i)}{p(j)}$ the CB between i and j.
Because we also consider finitely many different cost types (class I heterogeneity), we need to formally distinguish between different agents and different cost types. $\forall i$ define the sets $G(i_0) \equiv \{s \in [0, 1] : c(s) = c(i_0)\}$. Hence $G(i)$ identifies the cost type of agent i $(= c(i))$ or, put differently, i's equivalence class, i.e. $G(i)$ consists of all agents that sit on the same step of $c(\cdot)$, and any agent $j > i$: $j \notin G(i)$ is of a higher cost type.

Proposition 2 (Equilibrium properties of $(p(i), \Pi(i))$**)** *Given assumption 1 and $j > i$ with $j \notin G(i)$, the equilibrium satisfies*

a) No leap-frogging: $c(i) < c(j) \Leftrightarrow t(i) > t(j) \Leftrightarrow p(i) > p(j) \Leftrightarrow \Pi(i) > \Pi(j)$

b) $g_1(p, T) \gtreqless 0 \; \forall p > 0 \Longrightarrow CB_{ij} \gtreqless \left(\frac{c(j)}{c(i)}\right)^{\frac{1}{\eta-1}}$

c) $V_1(p, T)(\eta - 2) - pV_{11}(p, T) \gtreqless 0 \; \forall p > 0 \Longrightarrow \frac{\Pi(i)}{\Pi(j)} \gtreqless CB_{ij}$

The no leap-frogging property says that the contest preserves the order induced by the cost types or, equivalently, that $\text{CB}_{ij} > 1$ iff i and j represent different cost types and $\text{CB}_{ij} = 1$ iff they represent the same cost type. Intuitively, this holds because a better type can always match a worse type's effort and thereby earns more revenue.[16] The contest-technology parameter η influences the magnitude of the CB for given cost types. A more discriminatory contest (lower η) means that small differences in efforts imply large differences in success chances. Therefore small initial cost advantages translate into unboundedly large equilibrium differences in winning probabilities as $\eta \downarrow 1$.
Concerning the role of spillovers, b) shows that the CB corresponds to the technology-adjusted inverse cost ratio for any two different cost types if spillovers are purely aggregative (g depends only on T,

[14]This logically corresponds to the case of $n > 1$ identical atomistic agents, each with equilibrium success chances of $p^d(i) = 1/n$.
[15]In section 3 we generalize the notion of competitive balance as a ratio condition to a property of the outcome distribution $p(i)$, called functional competitive balance (FCB), and later show that two-stage contests with the constant elasticity technology frequently generate outcome distributions with the FCB property.
[16]Hefti (2014a) finds a tight connection between no leap-frogging in general games (not just contests) and the inexistence of asymmetric equilibria in certain symmetric versions of the game.

not on $p(i)$), or there are no spillovers (fixed prize contest). In fact, b) and c) together reveal that in absence of direct spillovers relative profits, CB's and (technology-adjusted) cost ratios must be equal for any two agents. It follows that with an identity-independent marginal revenue function changes in the prize scheme can affect the success distribution $p(i)$ and payoff ratios $\Pi(i)/\Pi(j)$ only in presence of direct spillover effects:

Proposition 3 (Identical revenue functions without direct spillovers) *Consider a prize function $V(p,T,x) = V(T,x)$, where x is a parameter in some open parameter interval X. Suppose that $V_x(T,x) > 0$, and take assumption 1 as satisfied for any $x \in X$. Then for any $i \in [0,1)$ and any $j > i$ with $j \notin G(i)$ the equilibrium $(p(i), \Pi(i))$ satisfies*

	$dp(i)$	$d\frac{p(i)}{p(j)}$	$d\left(p(i) - p(j)\right)$	$d\frac{\Pi(i)}{\Pi(j)}$	$d\Pi(i)$	$d\left(\Pi(i) - \Pi(j)\right)$
$dx > 0$	0	0	0	0	+	+

Table 1: Comparative statics without direct spillovers

Note that proposition 3 encompasses the case of a fixed prize contest (set $V(T,x) = x > 0$). Without any direct effects the only equilibrium effect of an exogenous increase e.g. in the prize purse is that all efforts increase proportionally, payoff levels increase and the absolute payoff difference between any two different cost types increases. This follows because by proposition 2 marginal costs $c(i)p(i)^{\eta-1}T^{\eta}$ of all types must be equal in equilibrium.

If there are direct spillovers or if the prize function depends on the agent identities - both can occur in a two-stage contest - then marginal costs are no longer equated between agents, and changes in the prize structure will affect the equilibrium success and profit distribution. In particular, it may happen that an exogenous *increase* in the prize function $V(p(i),T)$ *decreases* payoff levels of certain agents – sometimes even those of the best agents! In presence of direct spillovers ($g_1 \neq 0$) the direction of the spillovers determines how CB deviates from cost ratio. With positive spillovers ($g_1 > 0$), the CB exceeds the cost ratio, because increasing own efforts does not only increase one's success changes, but also the marginal value of winning the contest.[17] A more motivated type invests relatively more effort already with a fixed prize contest, and positive spillovers reinforce these investment incentives, more for good types, thus pushing CB's over cost ratios. By the same reason negative spillovers work in the opposite way.

While propositions 2 and 3 provide us with some preliminary insight how ex-ante cost heterogeneity and equilibrium success chances or profits are related, we have not learned much about the consequences of exogenous changes e.g. in the contest design for the outcome distribution in general yet. Consider e.g. the prize function $V\left(p(i), T, x, i\right)$, where x is an exogenous parameter (e.g. a prize shifter). How do

[17]A sufficient condition for positive spillovers is that $V(\cdot, T)$ be increasing and convex.

variations of x affect the equilibrium distribution of $p(i)$ or $\Pi(i)$? Given the generality of our setting, encompassing potentially many different cost types and agents, studying this question is not trivial. In the next section we develop some analytical tools, which will be very useful for answering the above type of question.

3 Heterogeneity: Analytical tools

The main goal of this article is to study how the equilibrium success distribution $p(\cdot)$ and related distributions, such as efforts or payoffs, depend on various parameters of contests with payoffs of the form (4). As $p(i)$ is a density such comparative-static questions are not trivial, since changes in the parameters cannot just shift $p(i)$, but induce rotations or deflections of $p(i)$. In this more technical section we develop some useful ways of detecting, when changes in a contest parameter induce certain rotations of the success distribution $p(\cdot)$.

Let $X \subset \mathbb{R}$ be an open parameter interval, $A \equiv [0,1] \times X$, and consider a function $p : A \to \mathbb{R}_+$ with the properties that $\infty > p(0,x) \geq p(1,x) > 0$, and $p(\cdot,x)$ is weakly decreasing $\forall x \in X$. Let $F(i,x) \equiv \int_0^i p(i,x)di$. If $F(1,x) = 1$, then $p(\cdot,x)$ is a density, and $F(\cdot,x)$ a distribution function.[18] We consider the following two classes of density functions:

- Density p belongs to **Class I** if $p(\cdot,x)$ is decreasing, right-continuous and has the **somewhere strictly decreasing (SSD) property**, i.e. $\exists i_0 \in (0,1)$: $p(i,x) > p(j,x)$ for $i < i_0 \leq j$, $x \in X$.
- Density p belongs to **Class II** if $p(i,x)$ is both strictly decreasing and continuous in i, $x \in X$.

As we shall see, these two classes of equilibrium success distributions emerge as a consequence of our cost function classes.

Two-types case The simplest case of a class I function is the two-types case. If the fraction of good types is $\gamma \in (0,1)$, and we let $i = 0$ represent good types and $i = 1$ bad types, $p(\cdot)$ has the form

$$p(\cdot) = \begin{cases} p_0 & i \in [0,\gamma) \\ p_1 & i \in [\gamma,1] \end{cases} \qquad , p_1 = \frac{1-\gamma p_0}{1-\gamma} \qquad , p_0 \geq p_1 \tag{6}$$

3.1 Rotations

Suppose that $x' \neq x$. How do $p(\cdot,x')$ and $p(\cdot,x)$ differ? Among the simplest and most interesting movements of the (success) density $p(\cdot,x)$ as x varies is the notion of a rotation.

Definition 2 (Rotations) *Let $x \neq x' \in X$, and consider the two functions $p(\cdot,x')$ and $p(\cdot,x)$. We say that $p(\cdot,x')$ is an outward-rotation (OR) of $p(\cdot,x)$, or $p(\cdot,x)$ is an inward-rotation (IR) of $p(\cdot,x')$, if $\exists$*

[18] The equilibrium success function $p(i)$ in proposition 1 satisfies all the above properties.

$0 < i_0 \le i_1 < 1$ *such that*

$$\begin{aligned} p(i,x') &> p(i,x) \quad i \in (0,i_0) \\ p(i,x') &< p(i,x) \quad i \in (i_1,1) \\ p(i,x') &= p(i,x) \quad i \in (i_0,i_1) \end{aligned} \tag{7}$$

where the last condition only is required if $i_0 < i_1$.

Figure 1 presents some examples of rotations.

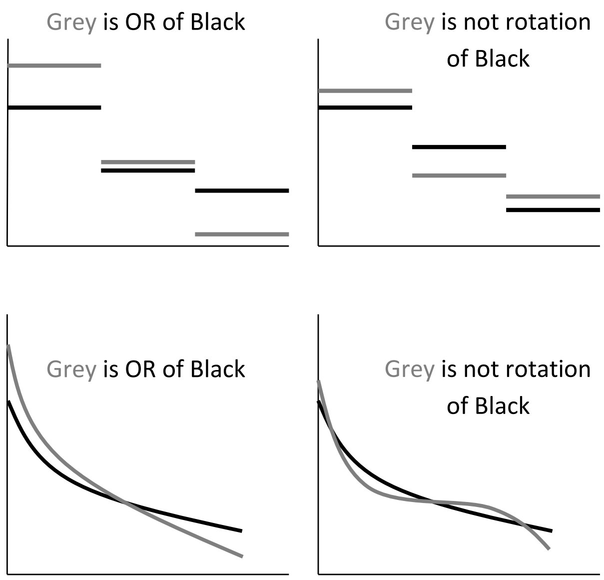

Figure 1: Class I and II rotations

Detecting rotations We now present a number of useful conditions asserting that $p(i,x')$ is an OR (or IR) of $p(i,x)$. Let $x \in X$ and $\forall i_0 \in [0,1]$ define the sets $G(i_0,x) \equiv \{i \in [0,1] : p(i,x) = p(i_0,x)\}$. In the following we assume that $G(i,x') = G(i,x)$ everywhere, so that we suppress x. In the step-function case this captures that the i-location of a step (induced by cost types) does not change e.g. if a prize component changes, while the height of the step, of course, may change. Further, $G(i,x) = G(i,x')$ is satisfied if $p(\cdot,x)$ is strictly decreasing $\forall x \in X$. Our first result shows that if $p(\cdot,x') - p(\cdot,x)$ is "setwise" strictly decreasing on[19] $\{(i,j) : j > i, j \notin G(i)\}$, then $p(\cdot,x')$ is OR of $p(\cdot,x)$.

Proposition 4 (Difference test) *Let* $x,x' \in X$ *and suppose that* $\infty > p(\cdot,x'), p(\cdot,x) > 0$ *are right-continuous, decreasing SSD densities. If for* $i \in (0,1)$

$$p(i,x') - p(i,x) > p(j,x') - p(j,x) \quad \textit{whenever } j > i, j \notin G(i) \tag{8}$$

is satisfied, then $p(\cdot,x')$ *is an OR of* $p(\cdot,x)$.

[19] In our contest setting, this means that $p(\cdot,x') - p(\cdot,x)$ must be strictly decreasing for any two different cost types.

If $p(\cdot,\cdot)$ is strictly submodular, then (8) is satisfied, but note that if $p(\cdot,x)$ is a step-function, $p(\cdot,\cdot)$ cannot be strictly submodular. In particular, if $p(\cdot,\cdot)$ is strictly submodular, then $p(\cdot,x)$ must be strictly decreasing $\forall x \in X$. See appendix A.2 for the details on how the difference and ratio test differ from standard lattice concepts, such as decreasing differences and supermodularity. We now present a further sufficient condition for the OR-property, which will be particularly useful in the later analysis.

Proposition 5 (Ratio test) *Suppose that the premise of proposition 4 is satisfied. If for $i \in (0,1)$*

$$\frac{p(i,x')}{p(j,x')} > \frac{p(i,x)}{p(j,x)} \quad \textit{whenever } j > i, j \notin G(i) \tag{9}$$

is satisfied, then $p(\cdot,x')$ is OR of $p(\cdot,x)$.

As conditions (8) and (9) both imply the OR property, one might ask how "setwise" decreasing differences (DD) and "setwise" decreasing ratios (DR) are related. If $p(i,x), p(i,x')$ are linear in i, one can show that both conditions are equivalent. In general, neither implies the other,[20] but one can trace out some relationship between the two, see section A.2 of the appendix.

3.2 Calculus criteria for rotations

The practical significance of conditions (8) and (9) is that we can derive corresponding differential tests to detect a rotation. As condition (9) turns out to be most relevant to establish rotation effects in our two-stage contest model, we present its differential version here; further results are in the appendix (section A.2).

Corollary 1 (Ratio test) *Suppose that p belongs either to class I or to class II. Further, if p belongs to class I, then p is differentiable in x except at step points, and if p belongs to class II it is everywhere differentiable in x. If $\forall j > i$ with $j \notin G(i)$ and $x_0 \in Int(X)$ we have that*

$$\frac{\partial}{\partial x}\left(\frac{p(i,x')}{p(j,x')}\right) > 0 \qquad \forall x' \geq x_0 \tag{10}$$

whenever the derivative exists, then $p(i,x')$ is OR of $p(i,x)$ whenever $x' > x_0$. If the first inequality in (10) *is reversed, then $p(i,x')$ is IR of $p(i,x)$.*

If p belongs to class II, we can use calculus to obtain a general condition[21] for verifying the OR-property, which we also make use of later. The following result says that $p(\cdot,x')$ is OR of $p(\cdot,x)$, if $p(\cdot,x')$ can intersect $p(\cdot,x')$ only from above.

[20] It is not hard to construct the corresponding counterexamples.

[21] Readers familiar with index theory will recognize the following as an index theorem result, showing that condition (11) is also necessary for the OR-property, provided that $p(i,x)$ satisfies the usual regularity conditions required by index theory (see e.g. Vives (1999)).

Corollary 2 *Suppose that p is C^2 and belongs to class II, and let $x' > x$. Then $p(i,x')$ is OR (IR) of $p(i,x)$ if*

$$p(i,x') = p(i,x) \quad \Rightarrow \quad \frac{\partial p(i,x')}{\partial i} < (>) \frac{\partial p(i,x)}{\partial i} \tag{11}$$

We conclude this section by summarizing descriptive properties of the distribution function $F(\cdot,x)$ when $p(\cdot,x)$ is a decreasing SSD density, and by applying our tools to the two-types case.

Proposition 6 (Distributional properties) *Suppose that $\forall x \in X$ the decreasing density $p(\cdot,x)$ has the SSD property. Then:*

a) $F(i,x) > i$, $i \in (0,1)$, $x \in X$

b) *If $p(i,x')$ is OR of $p(i,x)$, then $F(\cdot,x)$ strictly stochastically dominates $F(\cdot,x')$, i.e. $F(i,x') > F(i,x)$, $\forall i \in (0,1)$.*

c) *$F(i,x)$ is strictly increasing and concave in i, $x \in X$. If $p(i)$ is strictly decreasing, then $F(i)$ is strictly concave.*

Interpreting F as the distribution function pertaining to the outcome distribution $p(i)$ from last section, a) means that the chance of one of the most $i\%$ motivated agents to win the contest always exceeds $i\%$. Further, a) and c) then are consequences of heterogeneity and the no leap-frogging property of $p(i)$. Concavity originates from $p(\cdot)$ being a decreasing density, and means that adding adjacent agents to the set of the $i\%$ most motivated agents increases the joint success probability by less and less.
An intuitive result is that in the two-types case (6) the properties DD, DR, OR and stochastic dominance are equivalent:

Proposition 7 (Two-types case) *Let $x, x' \in X$ and suppose that the densities $p(\cdot,x), p(\cdot,x')$ are specified by* (6) *with distribution functions $F(\cdot,x), F(\cdot,x')$. Then properties* (7), (8), (9) *and strict stochastic dominance $F(i,x') > F(i,x)$ are equivalent.*

4 Prize structure in two-stage contests

We now turn to our main application: the case of a two-stage contest. In our baseline model there are two units A and B, with associated cost coefficient functions $c_A(\cdot), c_B(\cdot)$. Within each unit there is a separate primary contest (stage I), where a finalist is selected to compete with the finalist of the other unit in stage II. Winners of stage I obtain a **local prize** $V_A, V_B \geq 0$, and compete for a **global prize** worth $\Psi > 0$. We maintain the assumption of ATB at any stage of the contest.[22] Note that if both units are symmetric, in particular $c_A = c_B$ and $V_A = V_B$, our two-unit contest is formally equivalent to a

[22]Our results do not change if we work with the standard Nash solution at stage II instead, but the algebra naturally gets messier.

single-unit two-stage **search contest**, where two winners are selected at stage I to compete against each other at stage II.[23] The two versions are sketched in figure 2.

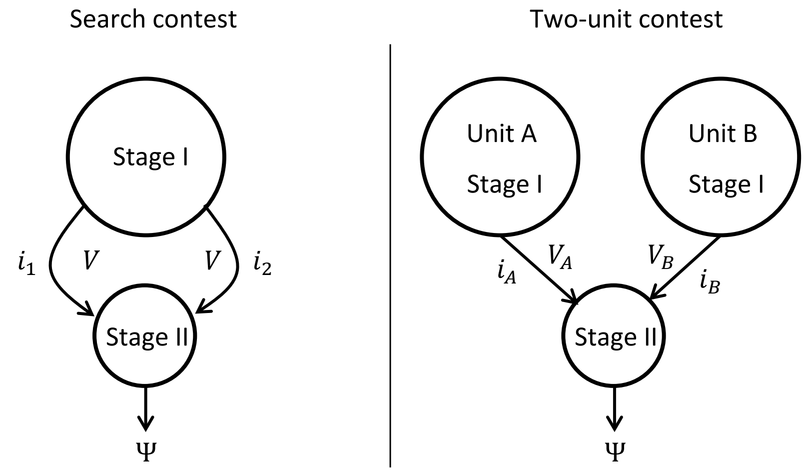

Figure 2: Search and two-unit contest

Payoff functions In the following we consider a two-unit two-stage contests from the perspective of an agent i in unit A, and derive his payoff function.[24] Contrary to the standard literature, we allow for the possibility of stage I effort **carryover** effects. With carryovers, stage II success chances, conditional on reaching stage II, are not independent of previous efforts.

Suppose that agent i of unit A exhibits stage I effort $t_1^A(i)$. Conditional on reaching stage II, agent i then competes with an agent j of B for the final prize. Let $t_2^A(i\,|j\,)$ denote i's stage I and II efforts, given that i meets j. Then, i's stage II payoff is

$$\pi_2^A(i\,|j\,) = \Psi\frac{e_A(i\,|j\,)}{T_{ij}} - \frac{1}{\eta}c_A(i)t_2^A(i\,|j\,)^\eta \qquad e_A(i\,|j\,) \equiv max\{\alpha t_1^A(i) + \beta t_2^A(i\,|j\,), 0\} \tag{12}$$

$e_A(i\,|j\,)$ is the effective effort to win stage II which, depending on carryover parameters α, β, may be composed of stage I and stage II efforts. The standard two-stage contest without carryovers satisfies $\alpha = 0$ and $\beta = 1$, and $\alpha > (<)0$ depicts the case of positive (negative) carryovers. Finally, $T_{ij} = e_A(i\,|j\,) + e_B(j\,|i\,)$ are stage II aggregate effective efforts. Let $p_A(i) \equiv t_1^A(i)/T_A$ denote agent i's stage I success chances. Agent i's ex-ante expected payoff over the entire two-stage contest then can be written

[23] This follows as we assume a unit measure of continuum agents. With atomistic agents the search and the symmetric two-unit contest differ formally with respect to an individual agent's probability of accessing the final. If $\tilde{p}(i)$ is the probability to access the final in the search contest, and $p(i)$ the corresponding probability in the two-unit contest, then $\tilde{p}(i) \neq p(i)$. Sampling with replacement implies that $\tilde{p}(i) = 2p(i)$. Sampling without replacement and sufficiently many players gives $\tilde{p}(i) \cong 2p(i)$, as the resampling probabilities become negligible. Hence the stage I success probabilities differ by the factor 2, which does not influence our later results.

[24] The payoff function for an agent j of unit B can then be obtained by appropriate index permutation.

as

$$\Pi_A(i) = p_A(i)\left(V_A + \int_0^1 p_B(j)\pi_2(i\,|j\,)dj\right) - c_A(i)p_A(i)^\eta T_A^\eta$$

$$= p_A(i)\underbrace{\left[V_A + \int_0^1 p_B(j)\left(\Psi\frac{\alpha p_A(i)T_A + \beta t_2^A(i\,|j\,)}{T_{ij}} - \frac{1}{\eta}c_A(i)t_2^A(i\,|j\,)^\eta\right)dj\right]}_{V(p_A(i),T_A,i)} - \tfrac{1}{\eta}c_A(i)p_A(i)^\eta T_A^\eta \tag{13}$$

Expression (13) shows the similarity between payoffs of a two-stage contest and our abstract model of contests with endogenous spillovers.[25] The stage I revenue function $V(\cdot)$ is generally identity-dependent if $\beta \neq 0$, because then a finalist i will exert stage II effort conditional on reaching stage II. How profitable winning stage II is for i depends, inter alia, on his type $c(i)$, which a rational agent takes into account in stage I. Therefore, the cost type of the agent will affect the stage I revenue function $V(\cdot)$ in (13). If however $\alpha = 1$ and $\beta = 0$, such that present efforts perfectly map into future efforts, then (13) corresponds exactly to the type analyzed in section 2.2.

Functional competitive balance (FCB) To analyze the effects of changes in contest parameters on $p_A(\cdot)$ and related distributions, we use the heterogeneity tools from section 3. The ratio condition (9) will provide most useful to establish certain rotation effects. While (9) generally is a stronger condition than OR, it imposes intuitively appealing structure on the predicted comparative statics. Particularly, if $x' > x$ and (9) holds, then for any two distinct cost types $i < j$, the respective competitive balance $\text{CB}_{ij} \equiv p(i)/p(j)$ must strictly decrease as $x \uparrow x'$. Hence the following definition makes sense, and generalizes the standard two-player notion of competitive balance to a property of the **function** $p(\cdot)$.

Definition 3 (Functional Competitive Balance) *If $p(i,x)$ satisfies (9) whenever $x' > x$ and $x, x' \in X$, then $p(i,x)$ has the **decreasing** FCB property. If (9) holds whenever $x' < x$, then $p(i,x)$ has the **increasing** FCB property.*

The decreasing FCB property means that for any agent $i \in (0,1)$ the relative change in his success chances compared to any less motivated type $j > i$ always increases if $x' > x$. An equivalent interpretation of the FCB-property is that if $x \uparrow x'$ the relative change in winning odds is strictly increasing in agent type.

4.1 Stage II equilibrium

Adopting the standard solution concept of backward induction, we begin by showing that there exists a unique stage II equilibrium, and derive the respective comparative statics. Let $\beta \geq 0$, $\alpha \gtreqless 0$, and suppose that i of unit A faces contestant j of unit B. A stage II equilibrium then is a triple $\left(t_2^A(i|j), t_2^B(j|i), T_{ij}\right)$

[25]Note that here $p(i)$ is the probability of reaching the final, whereas in section 2 $p(i)$ was the probability of winning the entire contest.

such that stage two effort choice $t_2(s|\neg s) \geq 0$ maximizes $\pi(s|\neg s)$ for given stage II aggregate T_{ij}. For given $T_{ij} > 0$, (12) implies the FOC:

$$t_2^A(i|j) = \left(\frac{\beta\Psi}{c_A(i)T_{ij}}\right)^{\frac{1}{\eta-1}} \tag{14}$$

Lemma 1 (Stage II equilibrium) *Suppose that i (j) is the unit A (B) finalist. Then there is a unique stage II equilibrium* $\left(t_2^A(i|j), t_2^B(j|i), T_{ij}\right)$*, and each finalist exerts positive stage II efforts iff* $\beta, \Psi > 0$*. For* $\beta, \Psi > 0$ *equilibrium effort* $t_2^A(i|j)$ *has the following comparative statics pattern:*

	α	β	Ψ	$c_A(i)$	$c_B(j)$	$p_A(i), p_B(j)$
$t_2^A(i\|j)$	$-$	$sign(\alpha)$	$+$	$-$	$+$	$-sign(\alpha)$

Table 2: Comparative statics stage II

The intuition behind table 2 closely reflects the aggregative structure of the contest, as a change in some parameter[26] may have a direct incentive effect (for given aggregate T_{ij}), and an indirect aggregate effect by changing T_{ij}.

Cost effects Effort levels $t_2^B(j|i)$ are a decreasing function of j, meaning that a weak unit B finalist invest less, which by itself implies a lower aggregate T_{ij}. Hence every unit of stage II effort of the unit A finalist i is now more effective in influencing success chances in his favor, explaining why a higher $c_B(j)$ positively affects $t_2^A(i|j)$. Conversely, a weak unit A finalist invests less, i.e. $t_2^A(i|j)$ decreases in $c_A(i)$.

Stage I efforts Stage I efforts, encoded in the success chances $p_A(i), p_B(j)$ only affect stage II efforts with non-zero carryovers ($\alpha \neq 0$). With positive carryovers ($\alpha > 0$), higher own stage I efforts also increase, ceteris paribus, the chance of a final success, and hence lowers the need to choose a high stage II effort (a “substitution” effect). Therefore $t_2^A(i|j)$ declines in $p_A(i)$, and it also declines in $p_B(j)$, because higher stage I efforts of the final opponent j reduces i’s stage II incentives to invest. $\alpha < 0$ reverses the above effects.

Carryover effects If stage I efforts carry through to stage II, then aggregate effective stage II effort is higher, ceteris paribus, which means that every unit of stage II effort is less effective in increasing stage II revenues. Therefore, stage II efforts decline in α.

Stage II reward effects A higher stage II prize Ψ increases each finalist’s incentive to compete harder, which is somewhat mitigated by the accompanying increase in aggregate efforts. The fact that the effect

[26]At stage II the stage I success chances $p_A(i), p_B(j)$ enter as exogenous parameters. A rational, backward-inducing agent will take the resulting stage II comparative statics into account, when choosing his stage I effort level.

of $d\beta > 0$ on stage II efforts depends on the type of carryovers seems surprising if looking just at (14), where β and Ψ enter the expression in the same way. If $\alpha > 0$, then an increase in β means that every unit of stage II effort is more effective in increasing, ceteris paribus, a finalist's winning chance, hence both finalists invest more. If however $\alpha < 0$, then any finalist must exert some minimal effort to at least compensate for the loss in stage II success chances due to negative carryovers from stage I. This imposes a lower bound on stage II efforts to become effective, and a higher value of β means that every unit of stage II effort is more effective in fulfilling this restriction, which then implies that stage II efforts can be relaxed (a kind of "income" effect).

4.2 Stage I equilibrium

Plugging (14) into (13) and rearranging yields

$$\Pi_A(i) = p_A(i)\left(V_A + \alpha p_A(i) T_A \Psi \int_0^1 \frac{p_B(j)}{T_{ij}} dj + \left(\frac{(\beta\Psi)^\eta}{c_A(i)}\right)^{\frac{1}{\eta-1}} \int_0^1 \frac{(\eta-1)p_B(j)}{\eta T_{ij}^{\eta/(\eta-1)}} dj\right) - \frac{1}{\eta} c_A(i) p_A(i)^\eta T_A^\eta \tag{15}$$

Taking the derivative with respect to $p_A(i)$ gives the associated FOC

$$V_A + 2\alpha p_A(i) T_A \Psi \int_0^1 \frac{p_B(j)}{T_{ij}} dj + \left(\frac{(\beta\Psi)^\eta}{c_A(i)}\right)^{\frac{1}{\eta-1}} \frac{\eta-1}{\eta} \int_0^1 \frac{p_B(j)}{T_{ij}^{\eta/(\eta-1)}} dj = c_A(i) p_A(i)^{\eta-1} T_A^\eta \tag{16}$$

We generally refer to the LHS of equation (16) as **marginal revenues**, and to the RHS as **marginal costs**. A stage I equilibrium is a tuple $((p_A(\cdot), T_A), (p_B(\cdot), T_B))$ such that $(p_A(\cdot), T_A)$ is a unit A equilibrium, given $(p_B(\cdot), T_B)$, and $(p_B(\cdot), T_B)$ is a unit B equilibrium given $(p_A(\cdot), T_A)$. Forward-looking agents take into account at stage I their prospects of obtaining the final prize Ψ, which depends, inter alia, also on unit B characteristics. We want to study how differences e.g. in prize or carryover parameters affect the distribution of stage I success chances, expected stage I payoffs, the between-unit distribution of winning chances and other interesting variables. There are two dimensions of heterogeneity. First, there can be heterogeneity among the agents **within** each unit as specified by the cost coefficient function $c_A(\cdot), c_B(\cdot)$. Second, there can also be heterogeneity **between** the units, e.g. because the units feature different cost functions $c_A(\cdot) \neq c_B(\cdot)$, or because of different interim prize values V_A, V_B. While it is fairly simple to solve for the equilibrium in case of a perfectly symmetric world, a glance at (16) makes clear that analyzing the model in complete generality, i.e. without imposing some restriction on the two heterogeneity dimensions is futile. Moreover, at such a level of generality we could not expect to learn much about how heterogeneity interferes with changes in contest parameters.

Therefore we proceed by analyzing interesting "insections" of the model. We first concentrate on how exogenous changes to the contest affect the equilibrium success distribution within unit A in the classical two-stage no-carryovers contest by assuming a symmetric unit B. We then present a number of variations in carryover parameters, which lead to sufficiently tractable variations of the contest, and confirm our essential results in these variations.

4.3 The canonical contest: No carryovers $(\alpha, \beta) = (0, 1)$

We now analyze the standard case of a two-stage contest, where only the efforts within a certain stage determine the agents' respective success chances at that stage. Then (16) reduces to[27]

$$V_A + \Psi \frac{\eta - 1}{\eta} \underbrace{\int_0^1 p_B(s) \frac{c_B(s)^{\frac{1}{\eta-1}}}{c_A(i)^{\frac{1}{\eta-1}} + c_B(s)^{\frac{1}{\eta-1}}} ds}_{=E\left[p_2^A(i|s)\right]} = c_A(i) p_A(i)^{\eta-1} T_A^{\eta} \tag{17}$$

$p_2^A(i|s)$ is the probability that agent i from unit A wins stage II conditional on facing agent s from unit B. For given coefficient functions $c_A(\cdot), c_B(\cdot)$ and a given density $p_B(\cdot)$, the functional equation (17) together with $\int p_A(i)di = 1$ characterizes the stage I success probabilities of unit A.

Hence a change e.g. in V_A, V_B or Ψ may trigger a direct effect in unit A (for fixed $p_B(\cdot), T_B$), as well as a feedback effect (as $p_B(\cdot), T_B$) also change). To limit the complications caused by the feedback effect, we assume symmetric agents within unit B, i.e. we set $c_B(\cdot) = c \geq 1$. This implies that $p_B(i) = 1$, $i \in [0, 1]$, in any equilibrium.[28] We impose no restrictions on $c_A(\cdot)$. Then (17) and unit A equilibrium stage I payoffs become

$$\begin{aligned} & V_A + \Psi k(i) = c_A(i) p_A(i)^{\eta-1} T_A^{\eta} \qquad\qquad k(i) \equiv \tfrac{\eta-1}{\eta} \frac{c^{\frac{1}{\eta-1}}}{c_A(i)^{\frac{1}{\eta-1}} + c^{\frac{1}{\eta-1}}} \\ & \Pi_A(i) = p_A(i) \left(V_A + \Psi k(i)\right) \tfrac{\eta-1}{\eta} \end{aligned} \tag{18}$$

Let $\hat{p}_A(i) = p_A(i) \int p_B(j) \frac{t_2^A(i|j)}{T_{ij}} dj$ denote agent i's overall success chance to win the entire two-stage contest. We refer to two agents $j > i$ with $i \in [0, 1)$ and $j \notin G_A(i)$ as two different cost types. Equation (18) implies the following expressions for the competitive balance CB_{ij}^A, the payoff ratio $\frac{\Pi_A(i)}{\Pi_A(j)}$ the ratio of overall success chances for any two different cost types $\hat{p}_A(i) = p_A(i) \int p_B(j) \frac{t_2^A(i|j)}{T_{ij}} dj$:

$$\text{CB}_{ij}^A = \left(\frac{c_A(j)}{c_A(i)}\right)^{\frac{1}{\eta-1}} \left(\frac{V_A + \Psi k(i)}{V_A + \Psi k(j)}\right)^{\frac{1}{\eta-1}} \qquad \frac{\Pi_A(i)}{\Pi_A(j)} = \frac{p_A(i)}{p_A(j)} \frac{V_A + \Psi k(i)}{V_A + \Psi k(j)} \qquad \frac{\hat{p}_A(i)}{\hat{p}_A(j)} = \frac{p_A(i) k(i)}{p_A(j) k(j)} \tag{19}$$

These three expressions reveal a monotonic relationship between relative stage I success chances, relative profits and relative overall success chances:[29]

Lemma 2 *Let $x \in \{V_A, \Psi\}$. For any two different cost types $j > i$ we have:*

[27] By changing the unit index we obtain the equation characterizing $p_B(\cdot)$.

[28] We argue below that the feedback effect with heterogeneity in unit B is likely to be of second order compared to the direct effect, but frequently works in the same direction as the direct effect.

[29] We omit the obvious proof.

$$sign\,(d_x CB_{ij}) = sign\left(d_x \frac{\Pi_A(i)}{\Pi_A(j)}\right) = sign\left(d_x \frac{\hat{p}_A(i)}{\hat{p}_A(j)}\right)$$

We now state the main result of this section, namely that changes in the local or global prize components have **diametrically opposed effects** on the equilibrium outcome distribution.

Theorem 1 *Suppose that* $V_A \geq 0$, $\Psi > 0$ *and* $c_B(\cdot) = c \geq 1$. *A unique equilibrium* $(p_A(\cdot), p_B(\cdot), T_A, T_B) > 0$ *exists. There is no equilibrium leap-frogging, and* $\Pi_A(\cdot), \Pi_B(\cdot) > 0$. *For* $V_A, \Psi > 0$, $i \in [0, 1)$ *and* $j > i$, $j \notin G_A(i)$, *a change in a stage prize has the following equilibrium rotation effects in unit A:*

	$d(\mathrm{CB}^A_{ij})$	$d\frac{\Pi_A(i)}{\Pi_A(j)}$	$d\frac{\hat{p}_A(i)}{\hat{p}_A(j)}$
$dV_A > 0$	$-$	$-$	$-$
$d\Psi > 0$	$+$	$+$	$+$

Table 3: Canonical two-stage contest: Comparative statics

Thus $p_A(\cdot)$ *gas the decreasing (increasing) FCB property with respect to* Ψ *(to* V_A*).*

It follows from table 3 that for any two different cost types relative efforts $t(i)/t(j)$ decrease in V_A but increase in Ψ. Comparing the quantitative implications of changes in V or Ψ between relative profits and relative success chances (CB_{ij}) we note that the effects on profits are stronger: If CB_{ij} increases by $x\%$, then $\frac{\Pi(i)}{\Pi(j)}$ increases by more than $x\%$.

Theorem 1: Intuition Why is it that an increase in the global prize Ψ induces an OR of $p_A(\cdot)$, whereas an increase in the local prize V_A leads to an IR? To understand the rotation effect, it is helpful to decompose the total effect of $dV_A, d\Psi > 0$ into a **direct incentive effect** and an **indirect aggregate effect**. The former pertains to how stage I efforts respond to a prize change if the aggregate T_A would remain fixed. The indirect aggregate effect captures how a ceteris paribus change of aggregate effort T_A would affect stage I effort provision.

An increase in either prize motivates all agents to increase their stage I efforts. For fixed T_A, good types would increase their efforts comparably more because of their lower cost coefficients. Taken by itself, this direct incentive effect would suggest an OR of $p_A(\cdot)$. Yet, the analysis is incomplete as it ignores changes in T_A. A ceteris paribus increase of T_A means that maintaining a certain success chance has become more expensive. Optimization requires all agents to balance their marginal costs of efforts against marginal benefits. For fixed marginal benefits a ceteris paribus increase of T_A would require good types to decrease their efforts (their success chances) by comparably more in absolute terms, which suggests an IR of $p_A(\cdot)$. In the benchmark case of a fixed prize one-stage contest, direct and indirect effects cancel (proposition 3), because in equilibrium marginal cost $c(i)p(i)^{\eta-1}T^{\eta}$ must be equal among all agents. This is not so in the general case, because heterogeneity drives a wedge between marginal costs. In fact, it follows from (18) and no leap-frogging that good types have higher (marginal) equilibrium costs than bad types. This extra expenditure needs compensation, which implies that for relative success chances to increase in a prize, it must be the case that marginal revenues increase more elastically in the prize for good types. A unit increase in the local component V_A has a type-independent incentive effect for all agents, which means that marginal revenues of good types increase by a smaller proportion than those of a bad type. This means that the indirect aggregate effect is dominant, hence an increase in V_A tends towards equating success chances and expected profits in the contest. A change in the global prize Ψ has a type-dependent incentive effect, and marginal revenues of good types increase by a larger proportion if $V_A > 0$, because their expected success chance in stage II is higher. Hence $d\Psi > 0$ intensifies the pre-existing relative inequality in success chances and payoffs. Note that if $V_A = 0$ the equilibrium inequality is maximal given $c_A(\cdot)$ and independent of Ψ (see figure 3 below). In fact, (19) shows that only changes in the relative prize V_A/Ψ matter for the success distribution and related measures. It follows readily from the first row of table 3 that $sign(d\text{CB}_{ij}) = sign(-d\frac{V_A}{\Psi})$.

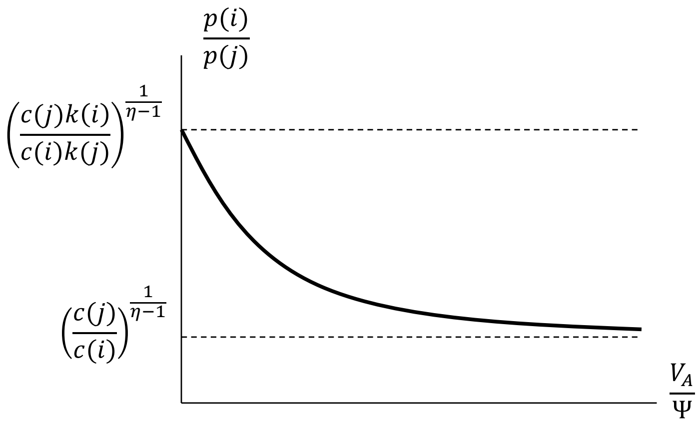

Figure 3: The increasing FCB property of the relative prize

Marginal revenues and the FCB property The above result shows the type-wise responsiveness of marginal revenues to exogenous variations in a prize to be crucial for the resulting rotation. This fact also holds for non-linear marginal revenues. Suppose that marginal revenues in (18) are given by a general, positive function $R(x, c(i))$, where x is a parameter, and $R_x(x, c(i)) > 0$.

Corollary 3 *Let $R(x, c) > 0$ be continuously differentiable in x with $R_x(x, c) > 0$, and suppose that $p_A(\cdot, x)$ is the unique equilibrium success function. Then $p(\cdot, x)$ has the decreasing (increasing) FCB property in x if $R(x, c)$ has strictly decreasing (increasing) log-differences.*

Prize effects on payoff levels While the comparative statics of **relative payoffs** are monotonic in V_A and Ψ, payoff **levels** may increase for certain agents but decrease for others. Put differently, the equilibrium payoff function $\Pi_A(\cdot)$, may shift and rotate simultaneously.

$$\begin{array}{lll} \frac{\partial}{\partial V_A}\Pi_A(i) > 0 & \Leftrightarrow & \int_0^1 \left(\frac{V_A+\Psi k(s)}{c_A(s)}\right)^{\frac{1}{\eta-1}} \left(\eta - \frac{V_A+\Psi k(i)}{V_A+\Psi k(s)}\right) ds > 0 \\ \frac{\partial}{\partial \Psi}\Pi_A(i) > 0 & \Leftrightarrow & \int_0^1 \left(\frac{V_A+\Psi k(s)}{c_A(s)}\right)^{\frac{1}{\eta-1}} \left(k(i)\eta - \frac{(V_A+\Psi k(i))k(s)}{V_A+\Psi k(s)}\right) ds > 0 \end{array} \tag{20}$$

(20) shows that each $i \in [0,1]$ has a higher (expected) payoff if either V_A or Ψ increases provided that heterogeneity is sufficiently weak.[30] Further, it follows from (20) that all top types in some interval $[0,x]$, $x \in (0,1]$ have a higher expected payoff level if Ψ increases. Similarly, all bottom types in some interval $[y,1]$, $y \in [0,1)$ have a higher expected payoff level if V_A increases. With strong heterogeneity, it may happen that payoffs of the **best** types **decrease** as $dV_A > 0$, and vice-versa payoffs of the **worst** types **decrease** as $d\Psi > 0$.[31]

Prize effects on stage I aggregate effort The first equation of (18) implies that

$$T_A^{\frac{\eta}{\eta-1}} = \int \left(\frac{V_A + \Psi k(s)}{c_A(s)}\right)^{\frac{1}{\eta-1}} ds$$

Hence aggregate stage I efforts T_A increases steeper in V_A than in Ψ as $0 < k(s) < 1 \; \forall \; s \in [0,1]$. This is intuitive, as a change in V_A incentivizes all agents equally whereas a change in Ψ affects the agents disproportionally. Hence if the goal is to increase aggregate stage I efforts, then increasing V_A is more effective, which may reduce the overall chance of unit A to win the contest because the best agents are less likely to reach the final.

Unit B cost effects and multiple finalists In this paragraph we begin to explore how changes in unit B might affect the outcome in unit A. We maintain symmetry in unit B, but ask how the unit A equilibrium depends on unit B cost efficiency c. A related question is: How does an increase of the units supplying a finalist affects the outcome in unit A? Both questions can be addressed by the same extension of the above model. Generally, a change in unit B efficiency or an increase/decrease of the number of competing units has more subtle distributional consequences for unit A than a change in contest prizes. Suppose that there are $M \geq 1$ symmetric units $B_1, ..., B_M$, each with a constant cost coefficient function $c \geq 1$. Each unit plays a separate contest to select a single stage II finalist. Proceeding as in the derivation of (18) gives as equilibrium equations for unit A:

$$\begin{array}{ll} V_A + \Psi k(i) = c_A(i) p_A(i)^{\eta-1} T_A^{\eta} & k(i) \equiv \frac{\eta-1}{\eta} \frac{1}{1+c_A(i)^{1/(\eta-1)} \frac{M}{c^{1/(\eta-1)}}} \\ \Pi_A(i) = p_A(i) \left(V_A + \Psi k(i)\right) \frac{\eta-1}{\eta} & \end{array} \tag{21}$$

[30] For $c_A(\cdot) = 1$ all inequalities in (20) hold. It then follows from continuity that if $c_A(1) > c_A(0) = 1$ is sufficiently close to 1, these inequalities remain valid.

[31] Parametric conditions for when this happens can be obtained, e.g. in the two-types model. For example, bad types are more likely to end up with lower payoffs as $d\Psi > 0$, if c_B is large or the fraction of bad types is small.

An increase in $R \equiv \frac{M}{c^{1/(\eta-1)}}$ means that it becomes harder for a unit A finalist to succeed. As R does not effect the sign of $k(i) - k(j)$, a change in Ψ or V_A has the same distributional effects in unit A as in theorem 1. For given $V_A, \Psi > 0$, differentiation of (21) immediately yields the following result:

$$\frac{\partial}{\partial R}\mathrm{CB}_{ij} > 0 \quad \Leftrightarrow \quad \frac{\Psi}{V_A}\frac{(\eta-1)}{\eta} > R^2 c_A(i)^{\frac{1}{\eta-1}} c_A(j)^{\frac{1}{\eta-1}} - 1 \tag{22}$$

Hence it depends on further details of the parameters whether $p_A(\cdot)$ has the global or rather a "piecewise" FCB property with respect to changes in R. If unit A agents are not less efficient than unit B agents, such that $c_A(1) \leq \frac{c}{M^{\eta-1}}$ holds, then $dR > 0$ causes an OR of $p_A(i)$. However, if there are several other units $R \geq 1$ becomes more likely. In this case the rotations effects caused by $dR > 0$ are unambiguous provided that $\frac{\Psi}{V}$ is sufficiently large/small, as summarized by the following table.[32]

	$d(\mathrm{CB}^A_{ij})$	$d\frac{\Pi_A(i)}{\Pi_A(j)}$	$d\frac{\hat{p}_A(i)}{\hat{p}_A(j)}$
$\frac{\Psi}{V_A}$ small	$-$	$-$	$-$
$\frac{\Psi}{V_A}$ large	$+$	$+$	$+$

Table 4: Comparative statics: Unit B effects

The intuition for these results also builds on the direct incentive and indirect aggregate effect. First, $dR > 0$ reduces earnings outlook, and thus decreases stage I marginal benefits of effort provision. In response unit A agents will reduce their efforts, but the current level of R affects which cost type reduces effort by how much. If $R \geq 1$, e.g. M large or unit B much more efficient, then marginal benefits of a good type decreases by more because worse types know that they are highly unlikely to win the final, and therefore their effort level is very low already. For fixed T_A this by itself implies in IR of $p_A(i)$. But as all unit A agents reduce their stage I efforts, also T_A decreases (indirect effect), which by itself would suggest an OR of $p_A(\cdot)$. Now, if the local prize V_A is much higher than Ψ, then unit A agents do not worry much about changes in stage II success chances, and therefore aggregate efforts T_A do not change by much. Thus the direct effect (IR) of $dR > 0$ dominates. In contrast, if the global prize Ψ is much higher than V_A, then changes in T_A become the dominant source of rotation.
Overall, we see that an increase of inter-unit competition, which makes it harder for each unit to win the final, may have equalizing or inequalizing effects within a unit depending on the relative importance of winning stage II.

Inter-unit effects Theorem 1 established the equilibrium rotation effects for changes in contest prizes in case of a symmetric unit B. If there is asymmetry also within unit B, then changes in prizes induce inter-unit heterogeneity-dependent feedback effects. For a given arbitrary success distribution $p_B(\cdot) > 0$

[32]The first column of table 4 follows from (22). Calculating the respective derivative of $d\frac{\Pi(i)}{\Pi(j)}$ and $d\frac{\hat{p}(i)}{\hat{p}(j)}$ gives exactly the same sign condition as in (22).

the unit A success ratio is

$$\frac{p_A(i)}{p_A(j)} = \left(\frac{c_A(j)}{c_A(i)} \frac{V_A + \Psi \frac{\eta-1}{\eta} E\left[p_2^A(i|s)\right]}{V_A + \Psi \frac{\eta-1}{\eta} E\left[p_2^A(j|s)\right]} \right)^{\frac{1}{\eta-1}} \tag{23}$$

Because $E\left[p_2^A(i|s)\right] > E\left[p_2^A(j|s)\right]$ for two different cost types $j > i$, it is a simple corollary to the proof of theorem 1 that an exogenous prize change of V_A or Ψ induces the same (rotation) effects on $p_A(\cdot)$ and $\Pi_A(\cdot)$ as in table 3. That is, the direct effect of a change in V_A or Ψ on $p_A(\cdot)$ and $\Pi_A(\cdot)$ is qualitatively the same as the equilibrium effect with a symmetric unit B. Overall, one would expect the equilibrium indirect feedback effects not to dominate the direct rotation effects induced by a prize change, but we were not able to prove or disprove this claim in general.
A related question is how the unit A equilibrium is distorted by an exogenous rotation of $p_B(\cdot)$. In particular, we can show the following result: If unit A is cost-dominant, i.e. $c_A(1) < c_B(0)$ then unit A "imports" the changes in in success chances of unit B.

Proposition 8 *If unit A cost-dominates unit B, then an exogenous OR (IR) of $p_B(\cdot)$ causes an OR (IR) of $p_A(\cdot)$, and relative payoffs $\frac{\Pi_A(i)}{\Pi_A(j)}$ increase (decrease) for any two different cost types $j > i$.*

If $c_B(\cdot)$ is not constant and A cost-dominates B, an increase (decrease) of V_B by itself causes an IR (OR) of $p_B(\cdot)$, which then tends towards triggering a similar rotation in unit A. Similarly, an increase in the global prize Ψ by itself causes an OR in both units, which tends to be reinforced in the cost-dominant unit.

5 Two-stage contests with carryovers

In this section we ask, whether theorem 1 on prize-induced rotations extends to contests with positive carryovers. In full generality, expression (16) is not tractable enough for a non-numerical assessment of this question. One important variant of the model is if stage I efforts perfectly determine stage II success chances $((\alpha, \beta) = (1, 0))$, i.e. if there are complete carryovers and no reinvestment opportunities at stage II. In this case there fortunately are some interesting scenarios which can be handled analytically. We later use one of these scenarios to explore how carryovers themselves affect the equilibrium distribution.

5.1 Perfect carryovers without reinvestments

If $(\alpha, \beta) = (1, 0)$, then (15) reduces to

$$\Pi_A(i) = p_A(i) \left(V_A + p_A(i) T_A \Psi \int_0^1 \frac{p_B(s)}{p_A(i)T_A + p_B(s)T_B} ds \right) - \frac{1}{\eta} c_A(i) p_A(i)^\eta T_A^\eta \tag{24}$$

Despite the contest having two stages, there only is one effort choice to make, which determines success chances in both subcontests. Analyzing (24) in general is hopeless as now T_B also affects effort choices in unit A: With carryovers stage I effort choices of B agents are decisive for stage II success chances of A finalists. In particular, assuming symmetry in unit B as in section 4.3 does not simplify the problem. We now consider two essential simplifications of (24).

Almost symmetric scenario Suppose that unit A and unit B have the same cost function, $c_A(i) = c_B(i))$ and $V_A = V_B$, hence also $T_A = T_B$. As mentioned earlier we can interpret this situation also as a single unit **search contest**. (24) now reads

$$\Pi_A(i) = p_A(i)\left(V + p_A(i)\Psi \int_0^1 \left(\frac{p_B(s)}{p_A(i) + p_B(s)}\right) ds\right) - \frac{1}{\eta} c_A(i) p_A(i)^\eta T_A^\eta \tag{25}$$

(25) is a complicated integral equation which without a further simplification we can only analyze in the quadratic two-types case (see below). If heterogeneity is small, i.e. $c_A(i)$ close to 1, it is reasonable to approximate[33] the integral in (25) by 1/2. Then:

$$\Pi_A(i) = p_A(i)\left(V + \frac{p_A(i)\Psi}{2}\right) - \frac{1}{\eta} c_A(i) p_A(i)^\eta T_A^\eta \tag{26}$$

We refer to (26) as the **almost symmetric scenario**. An equilibrium then is $(p_A(i), T_A)$ such that $p_A(i) = arg\,max\,\Pi_A(i)$ for each $i \in [0,1]$ and $\int p_A(i)di = 1$.

Small impact scenario Suppose that unit A faces many other symmetric units $B_1, ..., B_M$ and let $\tilde{T} = MT_B \gg T_A = T$. Then it is reasonable to approximate[34] the integral in (24) by $1/\tilde{T}$. Suppressing the index A (24) becomes

$$\Pi_A(i) = p_A(i)\left(V + \frac{p_A(i)T_A\Psi}{\tilde{T}}\right) - \frac{1}{\eta} c_A(i) p_A(i)^\eta T_A^\eta \tag{27}$$

We refer to (27) as the **small impact scenario**. A similar approximation makes sense if a single unit B had a much higher aggregate effort level compared to unit A, e.g. because of lower costs. In the following we will treat $\tilde{T}$ as a parameter rather than as an equilibrium variable, i.e. we concentrate on the direct rotation effects implied by different incentives in unit A and ignore inter-unit feedback effects. We refer to $(p_A(i), T_A)$ as a (unit A) equilibrium if $p_A(i) = arg\,max\,\Pi_A(i)$ for each $i \in [0,1]$ and $\int p_A(i)di = 1$.

Equilibrium: Existence and rotations in the two scenarios For simplicity, we drop the unit A index. Marginal benefits of both scenarios are of the general form [35] $g(p(i), T)$, i.e. the two-stage contest

[33] To justify this approximation, one can note that $|p(i) - 1| < \varepsilon \Rightarrow \left|\int \frac{p(s)}{p(i)+p(s)} ds - \frac{1}{2}\right| < \frac{\varepsilon}{2}$, i.e. the integral approaches 1/2 quicker than $p(i)$ approaches 1.

[34] Note that $\frac{1}{p_A(i)T_A + \tilde{T}}$ monotonically approaches $\frac{1}{\tilde{T}}$ if $\tilde{T}$ becomes large, or $p_A(i)T_A$ small.

[35] Strictly spoken, g depends only on $p(i)$ in the almost symmetric scenario.

with perfect carryovers without reinvestments is formally equivalent to a one-stage contest with direct, and in case of the small impact scenario also indirect, spillovers.

Because spillovers are positive, we know from proposition 2 that $\frac{p(i)}{p(j)} > \left(\frac{c(j)}{c(i)}\right)^{\frac{1}{\eta-1}}$ for any two cost types $c(j) > c(i)$, thus type $c(i)$ faces *higher* equilibrium (marginal) costs than type $c(j)$. Moreover, we see that in both scenarios $g(p(i),T) = V + \Psi k(i)$, where $k(i) > k(j)$ iff $c(i) < c(j)$. Hence marginal revenues resemble those of the canonical model; see (18). The additional formal difficulty is, that now $k(i), k(j)$ are not constant, but may depend on $p(i)$ and T. As before, an increase in Ψ has a stronger incentive effect on a good types because of their lower effort costs today, while an increase in V has the same incentive effect for all agents. We would therefore expect changes in V or Ψ to have similar rotation effects as in theorem 1. We could never disprove this claim, and the next proposition summarizes the cases, where we actually could verify the conjecture.[36]

Proposition 9 *Let $V, \Psi > 0$, $\eta \geq 2$ and suppose that $\int_0^1 \frac{1}{c(i)-1} di = \infty$. Both the small impact and the almost symmetric scenario have a unique equilibrium with $(p(i), T) > 0$. There is no equilibrium leap-frogging and all equilibrium payoffs are positive. For $\eta = 2$, $p(i)$ has the FCB property, $\frac{\Pi(i)}{\Pi(j)} = CB_{ij}$ and*

	$d(\mathrm{CB}_{ij})$	$d\frac{\Pi(i)}{\Pi(j)}$
$dV > 0$	$-$	$-$
$d\Psi > 0$	$+$	$+$

Table 5: FCB property of $p(i)$ in V, Ψ ($\eta = 2$)

If $c(i)$ is of type II and $\eta \geq 2$, then $dV > 0$ induces an IR of $p(i)$, while $d\Psi > 0$ causes an OR.

We demonstrate that the rotations of table 5 also hold for non-extreme (α, β)-combinations at the end of section 5.2 (proposition 13) in case of the small impact scenario.

Two-type search contest with arbitrary costs We now show that proposition 9 essentially extends to the case of a two-types search contest with arbitrary costs. There is a fraction $\gamma \in (0,1)$ of good types, each with a unit cost coefficient; a fraction $1-\gamma$ have cost coefficient $c > 1$. Then (25) implies the type-specific FOC's

$$\begin{aligned} V + 2p_0\Psi\left(\tfrac{1}{2}\gamma + \tfrac{p_1}{p_0+p_1}(1-\gamma)\right) &= p_0^{\eta-1}T^{\eta} \\ V + 2p_1\Psi\left(\tfrac{p_0}{p_0+p_1}\gamma + \tfrac{1}{2}(1-\gamma)\right) &= cp_1^{\eta-1}T^{\eta} \end{aligned} \tag{28}$$

[36]The requirement $\int_0^1 \frac{1}{c(i)-1} di = \infty$ in the proposition is not very restrictive. In particular, it holds with any type I function $c(i)$, and e.g. functions of the form $c(i) = (1+ri)^p$, with $r > 0$ and $p > 0$ satisfy the condition. Moreover, the proof of proposition 9 shows that the condition is only required for the knife-edge case where $\eta = 2$.

We first derive some properties of an equilibrium provided that it exists, and then prove existence, uniqueness and rotation effects of V and Ψ.

Lemma 3 *Suppose that $V, \Psi > 0$, $\eta \geq 2$ and $(p(i), T) > 0$ is an equilibrium pertaining to (28). Then for any two different cost types $j > i$: $p(i) > p(j)$, $\Pi(i) > \Pi(j) > 0$, and*

1. $CB_{ij} > \left(\frac{c(j)}{c(i)}\right)^{\frac{1}{\eta-1}}$
2. $\frac{\Pi(i)}{\Pi(j)} = CB_{ij}$ *for* $\eta = 2$, *and* $\frac{\Pi(i)}{\Pi(j)} > CB_{ij}$ *for* $\eta > 2$.

Proposition 10 *Consider the two-type case with $V, \Psi > 0$, $0 < \gamma < 1$ and $\eta \geq 2$. A unique equilibrium $(p_0, p_1, T) > 0$ exists, and $p_0 > 1 > p_1$. Then $dV > 0$ induces an IR of $p(i)$, while $d\Psi > 0$ causes an OR of $p(i)$. If $\eta = 2$, then $\frac{d}{dV}\frac{\Pi_0}{\Pi_1} < 0$ and $\frac{d}{d\Psi}\frac{\Pi_0}{\Pi_1} > 0$ are satisfied.*

5.2 Incentives and carryovers

In this section we are interested how unit A stage I effort incentives and finalist selection depend on carryovers, i.e. on the weights that past and current effort levels receive at stage II for a fixed prize structure $V_A, \Psi 0$. As before we simplify (15) by considering the small impact scenario, i.e. $p_B(\cdot) = 1$ and $T_{ij} = \tilde{T} \gg 0$, and assume that $\eta = 2$. Hence (dropping the A-index)

$$\Pi(i) = p(i)\left(V + \frac{\alpha p(i) T \Psi}{\tilde{T}} + \frac{(\beta\Psi)^2}{2c(i)\tilde{T}^2}\right) - \frac{1}{2}c(i)p(i)^2T^2$$

Throughout this section we generally consider parameters such that $\beta \in [0,1]$, $\alpha \in [-1,1]$, $V \geq 0$ and $\Psi > 0$. Solving the corresponding FOC for $p(i)$ on can show that a unique unit A equilibrium exists[37] if either $V > 0$, or $V = 0$ and $\beta, \Psi > 0$. We call a parameter vector (V, Ψ, β) **admissible** if these requirements are satisfied. For admissible parameters, the unit A equilibrium $(p(i), T)$ is characterized by

$$p(i) = \frac{2c(i)\tilde{T}^2 V + \beta^2\Psi^2}{2c(i)\tilde{T}\left(c(i)T^2\tilde{T} - 2\alpha\Psi T\right)} \qquad \int_0^1 \frac{2c(i)\tilde{T}^2 V + \beta^2\Psi^2}{2c(i)\tilde{T}\left(c(i)T^2\tilde{T} - 2\alpha\Psi T\right)} di = 1 \tag{29}$$

The next proposition summarizes parameter constellations for which we could unambiguously show a rotation effect to occur.

Proposition 11 (Carryover effects) *Consider an admissible (V, Ψ, β). Then:*

i) *$p(i)$ has the increasing FCB property in α: $CB'_{ij}(\alpha) > 0$, and also $\frac{\partial}{\partial\alpha}\frac{\Pi(i)}{\Pi(j)} > 0$.*

ii) *If either $\alpha < 0$, or $\alpha = 0$ and $V, \Psi, \beta > 0$, then $p(i)$ has the increasing FCB property in β: $CB'_{ij}(\beta) > 0$, and also $\frac{\partial}{\partial\beta}\frac{\Pi(i)}{\Pi(j)} > 0$. If $\alpha > 0$ and $V = 0$, then however $CB'_{ij}(\beta), \frac{\partial}{\partial\beta}\frac{\Pi(i)}{\Pi(j)} < 0$*

[37]It is straightforward to establish existence and uniqueness along the same lines as in the proof of proposition 9; therefore we omit the proof.

Proposition 11 shows that if past efforts' positive influence for future success increase, or past efforts become less punishing for future winning chances in case of negative carryovers, then relative payoffs and relative chances to advance in the contest become less equal. The proposition also shows that moving from a canonical two-stage contest without carryovers (section 4.3) to a contest with perfect carryovers and no reinvestment (section 5.1) necessarily increases success and payoff inequality if V is sufficiently small.

Intuition As before the interplay between the direct incentive effect and the indirect aggregate effect determines the total distributional effect of a change in carryover parameters. If $d\beta > 0$, then every unit of stage II effort is more effective at increasing the possibility of a final victory, which incentives all agents to compete harder in stage I. The direct effect of $d\beta > 0$ unambiguously induces an OR of $p(i)$ by itself[38] independent of $sign(\alpha)$. As before, $dT > 0$ by itself triggers a larger reduction in stage I efforts by good types to adjust their increased marginal costs to a constant marginal revenue. For $\alpha \neq 0$ marginal revenues also depend on T, but for $\alpha \leq 0$ the indirect effect unambiguously suggests an IR of $p(i)$.[39] The direct effect of $d\beta > 0$ is unambiguously dominant with negative carryovers, but not necessarily so with positive carryovers.

For $d\alpha > 0$ the direct effect always suggests an OR of $p(i)$ as good types invest more in stage I, and thus benefit more if either every stage I unit of effort transfers into more effective stage II effort, or the punishment of stage I efforts at stage II becomes less severe. Moreover, other than with $d\beta > 0$, the direct effect always is dominant.

Carryovers and adverse candidate selection Proposition 11 has some implications to candidate selection processes, such as pre-selection of a board member by a company presented to the shareholds for election, or a party primary candidate selection for a final political election. Especially in the latter case the existence of carry-overs e.g. because of advertising efforts is obvious, but not necessarily their direction. If, for example, winning the primaries requires to convince the extreme of one's party, the accompanying exposure by the media might postmark the candidate of being an extremist or of a fickle attitude, and generally blur the perception of a candidate's profile to final voters. Proposition 11 tells us that more negative repercussions of efforts exerted to win the primary selection on final success chances imply an increasingly equalized chance for all candidates to obtain the ticket to the final, while the worst candidates profit most of such a change.

It is interesting that heterogeneity in conjunction with negative carryovers impose a negative externality on their hosting unit, as unit A's overall chance to win the contest declines with more negative carryovers.

[38] For a fixed $T > 0$: $\frac{\partial^2 p(i)}{\partial\beta\partial c(i)} < 0$ iff $(c(i)T\tilde{T} - \Psi\alpha)\beta > 0$, and for $\beta > 0$ the second inequality is satisfied in equilibrium.

[39] If $\alpha \leq 0$, then $\frac{\partial^2 p(i)}{\partial T\partial c(i)} > 0$.

Unit A's chance of a final victory is

$$P(A,\alpha) = \frac{\alpha T(\alpha)}{\tilde{T}} \int_0^1 p(i,\alpha)^2 di + \frac{\beta^2 \Psi}{\tilde{T}^2} \int_0^1 \frac{p(i,\alpha)}{c(i)} di \tag{30}$$

Proposition 12 *For admissible parameters unit A becomes more likely to win the overall contest if $d\alpha > 0$.*

With symmetric agents $P(A,\alpha)$ increases in α because all agents are equally motivated to compete harder. Hence T_A increases which, ceteris paribus, makes a final victory of unit A more likely.
From (30) we see that besides this aggregative effect there are additional positive selection effects with heterogeneous agents. If $d\alpha > 0$ then good types increase their stage I efforts by more, which makes their participation in the final more likely. Hence chances of a total victory increase. The opposite conclusion holds if $d\alpha < 0$. In this case it becomes harder to predict the unit A finalist ex ante (by the "uniformization" of $p(i)$), and at the same time unit A is less likely to win the grand contest. This shows that the worst types would prefer a low (possibly negative) value of α, and suggests that especially weak candidates, e.g. in a two-stage political election, have an incentive to induce negative carryovers, e.g. by triggering a medial muckraking. If it cannot be avoided, this hurts strong candidates more. One could even imagine that a lower α might encourage more dubious (less capable) candidates to participate in the contest. Generally, this suggests that endogenizing participation in multi-stage contests with different carryovers and heterogeneous agents could be an interesting extension.

Prize-induced rotations for general non-negative carryovers The small impact scenario with quadratic costs allows us to verify whether our previous results on prize-induced rotations are also present for general non-negative carryover parameters. With negative carryovers, the rotations can be reverted, or $p(i)$ might depend in a more complicated way on the prizes.

Proposition 13 *Consider an admissible (V, Ψ, β). Then:*

i) If either $\alpha > 0$, or $\alpha = V = 0$, then CB_{ij} and $\frac{\Pi(i)}{\Pi(j)}$ are strictly decreasing in V.

ii) If either $\alpha > 0$, or $\alpha = 0$ and $V, \Psi, \beta > 0$, then $p(i)$ has the increasing FCB property in Ψ: $CB'_{ij}(\Psi) > 0$, and also $\frac{\partial}{\partial \Psi}\frac{\Pi(i)}{\Pi(j)} > 0$. If $\alpha < 0$ and $V = 0$, then however $CB'_{ij}(\Psi), \frac{\partial}{\partial \Psi}\frac{\Pi(i)}{\Pi(j)} < 0$.

The fact that $d\Psi > 0$ might cause an IR of $p(i)$ with negative carryovers is also intuitive. If $V = 0$ then the only reason to exert stage I efforts is to access stage II. If Ψ increases, then the agents have to balance the incentives to invest more stage I efforts to access the final against the fact that higher stage I efforts reduce their ultimate success chances with negative carryovers. As good agents always exert comparably higher efforts, the punishment of high stage I efforts in stage II weighs comparably more, which implies that good types increase their stage I efforts by less than bad types, hence an IR of $p(i)$

results. This is an interesting difference to how β affects $p(i)$. If $\alpha < 0$ and β increases, this means that the restriction imposed by negative carryovers to invest high efforts at stage II is relaxed, but more so for good types, causing an OR of $p(i)$. While Ψ and β both positively affect expected stage II revenues the last result indicates that changes in prizes may have different implications than changes in the possibilities to influence the outcome of the game at stage II, depending on the nature of the carryovers.

6 Across-unit effects of contest prizes

Until now we have mostly focused on the distributional effects of exogenous changes in the contest within a given unit. In this section we explore the the across-units effects of such changes. That is, we ask whether a change in a contest parameter tends to increase or decrease a pre-existing inequality between two units.

To isolate the across-unit effects from within-unit effects, we assume that each unit is symmetric in itself, but there is asymmetry between the units. Formally, we set $c_A(\cdot) = c_A \geq 1$, $c_B(\cdot) = c_B$. As before V_A, V_B denote local prizes and Ψ denotes the global prize. In a **quasi-symmetric equilibrium** stage I success chances are equalized among all agents, i.e. $p_A(\cdot) = p_B(\cdot) = 1$, thus also $t_i^A(i) = T_A$ and $t_i^B(i) = T_B$. Individual stage I and II efforts, aggregate efforts T_A, T_B and payoff levels Π_A, Π_B may differ *between* units. By symmetry, aggregate stage II effort T_{ij} is independent of agent identity, i.e. $T_{ij} = \hat{T}$. It then follows from (14) and (16) that in a quasi-symmetric equilibrium $(T_A, T_B, \hat{T})$ are determined by

$$\begin{array}{ll} V_A + \frac{2\alpha T_A \Psi}{\hat{T}} + \left(\frac{(\beta\Psi)^\eta}{c_A \hat{T}^\eta}\right)^{\frac{1}{\eta-1}} \frac{\eta-1}{\eta} = c_A T_A^\eta & V_B + \frac{2\alpha T_B \Psi}{\hat{T}} + \left(\frac{(\beta\Psi)^\eta}{c_B \hat{T}^\eta}\right)^{\frac{1}{\eta-1}} \frac{\eta-1}{\eta} = c_B T_B^\eta \\ \hat{T} = \alpha(T_A + T_B) + \beta\left(\frac{\beta\Psi}{\hat{T}}\right)^{\frac{1}{\eta-1}} \left(c_A^{\frac{-1}{\eta-1}} + c_B^{\frac{-1}{\eta-1}}\right) & \end{array} \quad (31)$$

In the following we present an analysis for the benchmark case without carryovers ($\alpha = 0$, $\beta = 1$). In the online appendix we confirm our main insights also in the more intricate case of perfect carryovers ($\alpha = 1$, $\beta = 0$). We now show that the **cause** of heterogeneity between the units matters crucially for how changes in the prize structure affect the payoff distribution. In particular, an increase in the importance of the global prize component Ψ benefits the relatively efficient unit more, and leads to a more unequal payoff distribution if heterogeneity originates mainly from efficiency differences. Put differently, an increase in global competition can never lead the less efficient unit to “catch up” (in absolute and relative terms) in such a case. The intuition roughly is that an increase in Ψ has a stronger incentive effect in stage II for the relatively more efficient unit, which translates into relatively higher expected stage I payoffs, provided that the efficiency difference is sufficiently strong. The opposite occurs if heterogeneity originates mainly from differences in local prizes: Then an increase in Ψ tends to equate payoffs.

In the case without carryovers, (31) becomes

$$V_A + \left(\frac{\Psi^\eta}{c_A \hat{T}^\eta}\right)^{\frac{1}{\eta-1}} \frac{\eta - 1}{\eta} = c_A T_A^\eta \qquad V_B + \left(\frac{\Psi^\eta}{c_B \hat{T}^\eta}\right)^{\frac{1}{\eta-1}} \frac{\eta - 1}{\eta} = c_B T_B^\eta \tag{32}$$

with

$$\hat{T} = \left(\frac{\Psi}{\hat{T}}\right)^{\frac{1}{\eta-1}} \left(c_A^{\frac{-1}{\eta-1}} + c_B^{\frac{-1}{\eta-1}}\right).$$

Let $r_U \equiv Pr(U\text{wins})$ denote the probability, that a member of unit $U \in \{A, B\}$ wins the final stage.

Proposition 14 *Let* $x_U \equiv \frac{V_U}{\Psi}$*,* $U \in \{A, B\}$*, and* $\Psi > 0$*. System* (32) *has a unique quasi-symmetric equilibrium, and*

$$r_A = \frac{c_B^{\frac{1}{\eta-1}}}{c_A^{\frac{1}{\eta-1}} + c_B^{\frac{1}{\eta-1}}} \qquad r_B = \frac{c_A^{\frac{1}{\eta-1}}}{c_A^{\frac{1}{\eta-1}} + c_B^{\frac{1}{\eta-1}}} \qquad \frac{\Pi_A}{\Pi_B} = \frac{x_A + \frac{\eta-1}{\eta} r_A}{x_B + \frac{\eta-1}{\eta} r_B} \tag{33}$$

Proof: The expressions for r_A, r_B follow from (14) and $r_A = \frac{t_2^A}{t_2^A + t_2^B}$. Expected equilibrium stage I payoffs for a member of unit A are $\Pi_A = V_A + \left(\frac{\Psi^\eta}{c_A \hat{T}^\eta}\right)^{\frac{1}{\eta-1}} - \frac{1}{\eta} c_A T_A^\eta$. Plugging in (32) and rearranging yields $\Pi_A = \frac{\eta-1}{\eta}\left(V_A + \frac{\eta-1}{\eta}\Psi r_A\right)$, and similarly for payoffs Π_B. Taking the ratio Π_A/Π_B and rearranging gives the last expression of (33). Existence and uniqueness are trivial. ■

In a quasi-symmetric equilibrium without carryovers a unit's overall chance to win the grand contest is determined by relative efficiency c_A/c_B, where $r_A > r_B$ iff $c_A < c_B$. Thus the prize structure $\{V_A, V_B, \Psi\}$ matters for a unit's overall success chance only if carryovers exists or there is heterogeneity within a unit. Nevertheless, the prize structure matters in a non-trivial way for the distribution of payoffs between the units. In the following we assume parameters such that initially $\Pi_A > \Pi_B$, for example $(c_A \leq c_B, V_A > V_B)$, or $(c_A < c_B, V_A = V_B)$.[40] A change in the prize structure, in general, leads to a change in both x_A and x_B. Given efficiency levels c_A, c_B, (33) implies the following relation to hold:

$$d\frac{\Pi_A}{\Pi_B} > 0 \qquad \Leftrightarrow \qquad dx_A > \frac{\Pi_A}{\Pi_B} dx_B \tag{34}$$

(34) shows that the **cause** of heterogeneity between the units matters crucially for how changes in the prize structure affect the payoff distribution. Suppose that heterogeneity originates mainly from efficiency differences, such that $\frac{V_A}{V_B} < \frac{r_A}{r_B}$. This inequality is in fact necessary and sufficient for $\frac{\Pi_A}{\Pi_B}$ to increase in Ψ. For example, in the special case, where $V_A = V_B = V$ and $r_A > r_B$, $d\Psi > 0$ (or $dV < 0$) implies payoffs to become relatively more unequal. The opposite holds if heterogeneity is mainly caused by different local prizes, i.e. $\frac{V_A}{V_B} > \frac{r_A}{r_B}$. Then $d\Psi > 0$ reduces $\frac{\Pi_A}{\Pi_B}$. It follows that only relative efficiency determines which unit earns more whenever Ψ becomes sufficiently high, possibly inverting the initial payoff distribution: If

[40] These two parameter constellations also imply that $T_A > T_B$.

$V_A > V_B$ but $c_B < c_A$, then $\Pi_A > \Pi_B$ for $\Psi = 0$. Because the first two inequalities imply that $\frac{V_A}{V_B} > \frac{r_A}{r_B}$, relative payoffs decrease in Ψ, and $\lim_{\Psi \to \infty} \frac{\Pi_A}{\Pi_B} = \frac{r_A}{r_B} < 1$. A joint increase of local prizes and the global prize may well lead to more unequal payoffs: If the global prize increases more quickly than the local prizes ($\frac{dV_U}{V_U} < \frac{d\Psi}{\Psi}$) but the difference in local prizes $\Delta V \equiv V_A - V_B$ increases at least as quickly as Ψ ($\frac{d(\Delta V)}{\Delta V} \geq \frac{d\Psi}{\Psi}$), then $\frac{\Pi_A}{\Pi_B}$ becomes larger.[41]

Finally, we consider how redistribution pari passu from local prizes to the global prize affects payoff inequality. Consider prize and efficiency parameters such that $\Pi_A > \Pi_B$, and suppose thta $dV_A = dV_B = -\frac{1}{2}$, and $d\Psi = k > 0$, where $k < 1$ means that there is some friction loss of redistribution, e.g. because of a costly overhead. Plugging these values into (34) shows that redistribution from local to global implies a more unequal payoff distribution iff the following inequality is satisfied:

$$V_A - V_B + \Psi \frac{\eta - 1}{\eta}(r_A - r_B) > 2k \frac{\eta - 1}{\eta}(r_B V_A - r_A V_B) \tag{35}$$

It follows that whenever a pre-existing payoff inequality is mainly caused by efficiency differences ($\frac{r_A}{r_B} \geq \frac{V_A}{V_B}$), then redistribution unambiguously increases payoff inequality for any $k > 0$.[42] This may also be the case if the payoff inequality is caused only by heterogeneous local prizes. If $V_A > V_B$ and $r_A = r_B$, then (35) is satisfied iff $\frac{1}{k} > \frac{\eta - 1}{\eta}$, which holds if $k \leq 1$.

7 Conclusion

This paper develops a two-stage, imperfectly discriminating elimination contest with an arbitrary number of heterogeneous contestants. It shows that contest prizes on different stages can have opposite effects on contestants' behavior: While higher first-stage prices increase competitive balance (with respect to efforts and profits), higher second-stage prizes decrease competitive balance. The qualitative pattern of results appears to be robust to the inter-temporal connection between the two stages. The intuition behind these results is as follows. Higher prizes - either on the first or the second stage - always increases contestants' effort incentive (positive incentive effect) and therefore aggregate effort increases. However, a change in marginal benefits affects the highly talented contestant more due to the smaller marginal effort costs. As the aggregate effort increases, the highly talented contestant reacts stronger than the low talented contestant (negative incentive effect). Thus, two opposite effects are present. As a consequence, contest prizes have opposite effects on different stages because altering a first-stage prize affects all contestants similarly, but highly talented contestants are more affected by altering a second-stage prize because they have a higher likelihood to reach the second stage. This mechanism implies that a contest organizer, who worries about distribution and inequality, should carefully evaluate how to set prizes on different stages. Our result regarding the effect of second-stage prizes can explain the following empirical finding by

[41] Formally, the stated assumptions imply that $0 > dx_A \geq dx_B$, and the result then follows from (34).

[42] This also holds for absolute payoff differences, because $\Pi_A - \Pi_B$ increases by redistribution iff $r_A > r_B$.

Pawlowski et al. (2010) who examine the effect of an increase in the prize for the UEFA Champions League on competitive balance. After a major modification in the 1999-2000 season, the payments to the participating teams in the UEFA Champions League have experienced a large increase. Simultaneous to the increased value of Champions League participation, Pawlowski et al. (2010) find empirical evidence that competitive balance in the Big Five European football leagues has experienced a "persistent decline". This result of a decline in competitive balance and a significant increase in the Champions League payments to the participating teams is consistent with our findings.

References

Alos-Ferrer, C. and Ania, A. B. (2005). The evolutionary stability of perfectly competitive behavior. *Economic Theory*, 26:497–516.

Bognanno, M. L. (2001). Corporate tournaments. *Journal of Labor Economics*, 19(2):290–315.

Corchon, L. C. (1994). Comparative statics for aggregative games the strong concavity case. *Mathematical Social Sciences*, 28(3):151–165.

Fu, Q. and Lu, J. (2012). The optimal multi-stage contest. *Economic Theory*, forthcoming.

Glazer, A. and Gradstein, M. (2005). Elections with contribution-maximizing candidates. *Public Choice*, 122(3):467–482.

Gradstein, M. and Konrad, K. A. (1999). Orchestrating rent seeking contests. *Economic Journal*, 109:536–545.

Groh, C., Moldovanu, B., Sela, A., and Sunde, U. (2012). Optimal seedings in elimination tournaments. *Economic Theory*, 49(1):59.

Grossmann, M., Lang, M., and Dietl, H. (2011). Transitional dynamics in a tullock contest with a general cost function. *BE Journal of Theoretical Economics*, 11(1):Article 17.

Harbaugh, R. and Klumpp, T. (2005). Early round upsets and championship blowouts. *Economic Inquiry*, 43(2):316–329.

Hefti, A. (2014a). Equilibria in symmetric games: Theory and applications. *ECON working paper.* (Web: http://www.econ.uzh.ch/faculty/hefti/publications.html).

Hefti, A. (2014b). On the relationship between stability and uniqueness. *ECON working paper series.*

Jensen, M. K. (2010). Aggregative games and best-reply potentials. *Economic theory*, 43(1):45–66.

Klumpp, T. and Polborn, M. K. (2006). Primaries and the new hampshire effect. *Journal of Public Economics*, 90(6):1073–1114.

Loury, G. C. (1979). Market structure and innovation. *Quarterly Journal of Economics*, 13:395–410.

Moldovanu, B. and Sela, A. (2006). Contest architecture. *Journal of Economic Theory*, 126(1):70–96.

Pawlowski, T., Breuer, C., and Hovemann, A. (2010). Top clubs performance and the competitive situation in european domestic football competitions. *Journal of sports economics*, 11(2):186–202.

Rosen, S. (1986). Prizes and incentives in elimination tournaments. *American Economic Review*, 76:701–715.

Schmitt, P., Shupp, R., Swope, K., and Cadigan, J. (2004). Multi-period rent-seeking contests with carryover: Theory and experimental evidence. *Economics of Governance*, 5(3):187–211.

Stein, W. E. and Rapoport, A. (2004). Asymmetric two-stage group rent-seeking: Comparison of two contest-structures. *Public Choice*, 118:151–167.

Stracke, R. (2013). Contest design and heterogeneity. *Economics Letters*, 121(1):4–7.

Szymanski, S. (2003). The economic design of sporting contests. *Journal of economic literature*, 41(4):1137–1187.

Taylor, C. R. (1995). Digging for golden carrots: An analysis of research tournaments. *American Economic Review*, 85:872–890.

Topkis, D. M. (1998). *Supermodularity and Complementarity.* Princeton: University Press.

Vives, X. (1999). *Oligopoly pricing.* Cambridge MIT Press.

A Appendix

A.1 Continuum-agent representation

Consider a population consisting of $n \in \mathbb{N}$ atomistic (or "discrete") agents, indexed by $\{1/n, 2/n, ..., 1\}$. Suppose that the agents differ in their cost coefficient $c(i)$, $i \in \{1/n, 2/n, ..., 1\}$. Then, the agents can be partitioned into $1 \leq K \leq n$ groups of identical agents, with group size n_k, $\sum_k n_k = n$. This partition gives $1 \leq K \leq n$ equivalence classes (groups) of sizes $n_1, ..., n_K$, $\sum_k n_k = n$. We identify each group by a "representative" agent i_k. In a discrete ATB equilibrium every agent (i/n) chooses $p^d(i/n)$ (d for "discrete") to solve (4), where $p^d(i/n)$ must satisfy $\sum_{i=1}^n p^d(i/n) = 1$. Let $p(i)$ denote the (step) density function that characterizes our (continuum) ATB equilibrium from definition 1 with the corresponding cost step function $c(i) = c(i_k)$ on $[i_k, i_{k+1})$, and group measures $\gamma_1, ..., \gamma_K$ satisfying $\gamma_k = n_k/n$. We now establish the formal equivalence between the discrete equilibrium probability distribution $\left\{p^d(1/n), ..., p^d(1)\right\}$ and the equilibrium step density $p(i)$.

Theorem 2 (Representation-theorem) *Let $n \in \mathbb{N}$ and suppose that agents are partitioned in K cost groups. If $\left\{p^d(i/n)\right\}$ corresponds to the discrete ATB equilibrium and $p(i)$ is the equilibrium ATB (step) density of the corresponding continuum problem, then $p^d(i/n) = \frac{1}{n}p(i/n)$ is satisfied for all $i \in \{1, ..., n\}$*

Proof: First, note that in the continuum case we only have to solve problem (4) for the representative agents i_k. In equilibrium $1 = \sum_{i=1}^n p^d(i/n) = \sum_{k=1}^K p^d(i_k)n_k$. The claim now is that $\frac{1}{n}p(i_k) = p^d(i_k)$ for $k = 1, ..., K$. But because in the continuum equilibrium we must have

$$1 = \int_0^1 p(i)di = \sum\nolimits_{k=1}^K p(i_k)\gamma_k = \sum\nolimits_{k=1}^K \left(\frac{1}{n}p(i_k)\right) n_k$$

the claim follows from the uniqueness of the ATB equilibrium. ■

Hence the continuum step-function case and the atomistic case are equivalent up to the multiplicative constant $1/n$ (independent of group composition), which means that we can work with either model, and justifies our procedure of the main text. It then also follows that $p(i_k)\gamma_k = p^d(i_k)n_k$ corresponds to the probability that a member of group k wins the contest, illustrating why we used the notion of a "representative" agent.

Theorem 2 remains valid as n grows arbitrarily large. This provides a justification for using strictly increasing and continuous cost coefficient functions (class II) as an approximation for the case of many different agents. To see this let $c(i)$ be a class II function defined on $[0, 1]$ and for a given n consider the atomistic distribution induced by $c(1/n), c(2/n), ..., c(1)$. Let $\left\{p_n^d(i/n)\right\}$ be the corresponding atomistic ATB equilibrium. By theorem 2 we can identify this equilibrium by the corresponding (n)-step continuum equilibrium, which gives

$$\sum_{i=1}^{n} p_n^d(i/n) = \sum_{i=1}^{n} \frac{1}{n} p(i/n) = 1 \qquad n \in \mathbb{N}$$

But as this statement is correct for any $n \in \mathbb{N}$ this implies by definition of the (Lebesgue) integral that

$$\lim_{n\to\infty} \sum_{i=1}^{n} p_n^d(i/n) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} p(i/n) = \int_0^1 p(i)di = 1$$

As n grows larger and larger and players are squeezed together more tightly on $[0,1]$, the distribution $p(i/n)$ approaches the continuous density $p(i)$, which in turn corresponds to the ATB solution pertaining to the continuous cost coefficient function $c(i)$.

A.2 Rotations

We state the differential version of proposition 4:

Corollary 4 *Suppose that the presumptions of corollary 1 hold. If* $\forall i < j$ *with* $j \notin G(i)$ *and* $x_0 \in Int(X)$ *we have that*

$$\frac{\partial}{\partial x}\left(p(i,x') - p(j,x')\right) > 0 \qquad \forall x' \geq x_0 \tag{36}$$

whenever the derivative exists, then $p(i,x')$ *is OR of* $p(i,x)$ *whenever* $x' > x_0$*. If the inequality in* (10) *is reversed, then* $p(i,x')$ *is IR of* $p(i,x)$*.*

As the proof is logically similar to the proof of corollary 1 we omit it here. We include the following calculus result for the sake of completeness (we omit the obvious proof.)

Corollary 5 *Suppose that* p *is* C^2 *and belongs to class II, let* X *be an open interval and* $x' > x$*. If* $\frac{\partial^2}{\partial i \partial x} p(i,x) < 0$, $(i,x) \in Int(A)$*, then* $p(i,x)$ *satisfies* (8) *on* A*. If* $\frac{\partial^2}{\partial i \partial x} Ln\left(p(i,x)\right) < 0$, $(i,x) \in Int(A)$*, then* $p(i,x)$ *satisfies* (9) *on* A*.*

Relation between "setwise" DD and DR While the ratio test and the difference test can both be used to establish an OR or IR of $p(\cdot)$, they are not equivalent, and we explore their relation in this section.

Proposition 15 (DR and DD) *Suppose that the premise of proposition 4 is satisfied. If* (9) *is satisfied, then* (8) *holds for all* i *such that* $p(i,x') \geq p(i,x)$*. If* (8) *is satisfied, then* (9) *holds for all* $j > i$ *such that* $j \notin G(i)$ *and* $p(i,x') \leq p(i,x)$*.*

Proof: If (9) is satisfied, then equivalently $\frac{p(i,x')-p(i,x)}{p(i,x)} > \frac{p(j,x')-p(j,x)}{p(j,x)}$ whenever $j > i$, and $j \notin G(i)$. Suppose that $p(i,x') \geq p(i,x)$, and take $j > i$, $j \notin G(i)$. Hence $p(i,x) > p(j,x)$, and the first claim follows from $\frac{p(i,x')-p(i,x)}{p(j,x)} > \frac{p(j,x')-p(j,x)}{p(j,x)}$.

If (8) is satisfied, then:

$$\frac{p(i,x')}{p(i,x)} > \frac{p(j,x') - p(j,x)}{p(i,x)} + 1 \qquad j > i,\, j \notin G(i) \tag{37}$$

Now, because of (37) condition (9) is satisfied if $\frac{p(j,x')-p(j,x)}{p(i,x)} + 1 \geq \frac{p(j,x')}{p(j,x)}$, or equivalently, if $\frac{p(j,x')-p(j,x)}{p(i,x)} \geq \frac{p(j,x')-p(j,x)}{p(j,x)}$. But this inequality must hold, because as $p(i,x') \leq p(i,x)$ (37) implies that $p(j,x') - p(j,x) < 0$. ■

If p is the (stage I) success chance, then proposition 15 says that if p satisfies setwise decreasing differences, the setwise decreasing competitive balance property is satisfied at least for the "loosing" range, where $p(i,x') \leq p(i,x)$. Conversely, if the decreasing competitive balance property is satisfied, then the "winning" range, where $p(i,x') \geq p(i,x)$, satisfies setwise decreasing differences.

To place the last result in the relevant theoretical context, note that $p(i,x)$ is setwise strictly submodular on $A \equiv [0,1] \times X$ if and only if $p(i,x)$ has setwise strictly decreasing differences on its domain, i.e. (8) is satisfied on A. Similarly, $p(i,x)$ has setwise strictly decreasing ratios on A if and only if $p(i,x)$ is setwise strictly log-submodular on A, i.e. if and only if $Ln(p(i,x))$ is setwise strictly submodular on A. It is a known result that if $p(\cdot,\cdot)$ were monotonic on A, then log-supermodularity would imply supermodularity, and submodularity would imply log-submodularity (see Topkis (1998), p. 65). However, because in our context $p(i,x)$ generically cannot be montonic in both arguments[43], the result does not apply to our setting, but proposition 15 can be viewed as an extension of the result to the case of a partially monotonic function.

A.3 Proofs

Proof of proposition 1

The proof consists of two steps. i) Consider an arbitrary agent i. It follows from continuity and assumption 1 that for $T > 0$ the FOC $g\,(p(i),T) = c(i)p(i)^{\eta-1}T^{\eta}$ has a unique solution $p(i;T) > 0$. Now, consider the function $p(i,T) \equiv p(i;T)$, noting that $p(i,\cdot)$ is C^1 on $(0,\infty)$. Further, $p(\cdot,T)$ "inherits" the class property (with reversed sign) of the cost coefficient function $c(\cdot)$. Specifically, if c is a (finite) step function, then so is $p(\cdot,T)$, and if $c(i)$ is a (bounded) C^1-function, so is[44] $p(\cdot,T)$. In any case, $p(\cdot,T)$ is decreasing and thus integrable over $[0,1]$, so let $G(T) \equiv \int_0^1 p(i,T)di$, noting that G also is differentiable.
ii) We now show $\exists!\ T > 0$: $G(T) = 1$. Because $g(1,0) > 0$, $g(1,\cdot)$ continuous and bounded from above $\exists T' > 0$: $p(0;T') = 1$. Because $p(i,\cdot)$ strictly decreasing, it follows that $p(0;T) < 1$ for $T > T'$. Hence also $p(i,T) < 1$ for any $i \in [0,1]$ and $T > T'$, which implies that $\lim_{T\to\infty} G(T) < 1$. Similarly, it follows

[43] E.g. in the continuous case the integral condition implies the existence of at least one intersection of $p(i,x)$ and $p(i,x')$.
[44] A consequence of the Implicit Function Theorem.

that $\exists T'' > 0$ such that $p(1;T'') = 1$. Thus $p(i,T'') > 1$ for $i \in [0,1]$ and $T < T''$, hence $\lim_{T\to 0} p(i;T) > 1$. As $G(\cdot)$ continuous, $\exists T > 0$ such that $G(T) = 1$, and uniqueness holds because $\mu_T((p(i);T),T) < \eta$ implies $\frac{\partial p(i,T)}{\partial T} < 0$, hence $G'(T) < 0$. Finally, $\Pi(i) > 0$ in the equilibrium, because $p(i) = p(i;T) > 0$ is the unique maximizer and $\Pi(i)|_{p(i)=0} = 0$. ■

Proof of proposition 2

Let $(p(i),T) > 0$ denote the unique equilibrium, and suppose that $j > i$, $j \notin G(i)$, and hence $c(j) > c(i)$.

a) $c(i) < c(j) \Rightarrow \Pi(i) > \Pi(j)$ follows readily from $pV(p,T) - \frac{1}{\eta}c(i)p^\eta T^\eta > pV(p,T) - \frac{1}{\eta}c(j)p^\eta T^\eta$ for $p > 0$, and thus also $c(i) \geq c(j) \Rightarrow \Pi(i) \geq \Pi(j)$. Next, define $h(p,c) \equiv g(p,T) - cp^{\eta-1}T^\eta$, and note that the equilibrium $p(s) > 0$, $s \in [0,1]$, satisfies $h(p(s),c(s)) = 0$. As with profits, $c(i) < c(j) \Leftrightarrow p(i) > p(j)$ now follows from the fact that $h(p,c(i)) > h(p,c(j))$ if $p > 0$. As by definition $p(i) = t(i)/T$, the proof is complete.

b) We only show the claim for $g_1 > 0$. If $g_1 > 0$, then as $p(i) > p(j)$ the FOC imply that $c(i)p(i)^{\eta-1}T^\eta > c(j)p(j)^{\eta-1}T^\eta$, which proves the claim.

c) Again, we only prove ">". Using FOC the equilibrium payoff ratio satisfies

$$\frac{\Pi(i)}{\Pi(j)} = \frac{p(i)}{p(j)} \underbrace{\frac{V(p(i),T)(\eta-1) - p(i)V_1(p(i),T)}{V(p(j),T)(\eta-1) - p(j)V_1(p(j),T)}}_{\equiv A}$$

Hence $\frac{\Pi(i)}{\Pi(j)} > \frac{p(i)}{p(j)}$ iff $A > 1$, and the claim follows because $p(i) > p(j)$ and

$$\frac{\partial}{\partial p}\left(V(p,T)(\eta-1) - pV_1(p,T)\right) = V_1(p,T)(\eta-2) - pV_{11}(p,T) > 0$$

■

Proof of proposition 3

$g(p(i),T) = V(t,x)$. FOC and $\int_0^1 p(i)di = 1$ imply that the equilibrium $(p(i),T,\Pi(i))$ is determined by

$$p(i) = \frac{c(i)^{\frac{-1}{\eta-1}}}{\int_0^1 c(s)^{\frac{-1}{\eta-1}}ds} \qquad 1 = \left(\frac{V(T,x)}{T^\eta}\right)^{\frac{1}{\eta-1}} \int_0^1 c(s)^{\frac{-1}{\eta-1}}ds \qquad \Pi(i) = p(i)V(T,x)\frac{\eta-1}{\eta}$$

The first four entries of table 1 are clear. $\frac{\partial}{\partial x}\Pi(i) > 0$ iff $\Lambda \equiv V_T T'(x) + V_x(T,x) > 0$. As $T'(x) = \frac{TV_x}{\eta V - TV_t}$, $\Lambda = \frac{\eta V}{\eta V - TV_T}V_x$, and because $V_x > 0$ and $\eta > \mu_T = \frac{V_T T}{V}$ by assumption 1, it follows that $\Lambda > 0$. The last entry is a direct consequence of the fourth and fifth entry. ■

Proof of proposition 6

a) Consider $i_0 \in (0,1)$. If $p(i_0) \geq 1$, then by SSD and because $p(\cdot, x)$ is a decreasing density $\exists\ \hat{i} < 1$: $p(i) < 1$ on $[\hat{i}, 1]$. Thus there must also be $\tilde{i} \in (0, i_0]$ such that $p(i) > 1$ on $[0, \tilde{i})$. Hence $F(i_0) > i_0$.
If $p(i_0) < 1$ then, by SSD and the decreasing density property, $\exists i \in (0, i_0]$:

$$F(i_0) = \int_0^i \underbrace{p(s)}_{\geq 1} + \int_i^{i_0} \underbrace{p(s)}_{<1}$$

Then $\int_0^i (p(s) - 1) = \int_i^1 (1 - p(s)) > \int_i^{i_0} (1 - p(s))$, where the inequality follows from the fact that $1 - p(s) > 0$ for $s \in [i, 1]$. Hence $F(i_0) > i_0$.
b) Follows from a similar type of argument as in a). If $i \in (0, i_0]$, then $p(i, x') > p(i, x) > 0$ for $i \in (0, i_0)$, and hence $F(i, x') > F(i, x)$. If $i \in (i_0, 1)$, then

$$\int_0^{i_0} (p(s, x') - p(s, x)) = \int_{i_0}^1 (p(s, x) - p(s, x')) > \int_{i_0}^i (p(s, x) - p(s, x'))$$

where the inequality follows from $p(s, x') = p(s, x)$ on (i_0, i_1) and $p(s, x) > p(s, x')$ on $(i_1, 1]$. Hence again $F(i, x') > F(i, x)$.
c) The first claim follows directly from $p(\cdot, x) > 0$. Let $i, j \in [0, 1]$ and $i < j$. Let $t \in [0, 1]$. First, note that $tF(i) + (1-t)F(j) = F(j) - t\int_i^j p(s)$, and $F(ti + (1-t)j) = \int_0^j p(s) - \int_{i'}^j p(s)$ where $i' = ti + (1-t)j$. From these two expressions it follows that F is concave iff $t\int_i^j p(s) \geq \int_{i'}^j p(s)$, or equivalently if

$$t\int_i^{i'} p(s) - (1-t)\int_{i'}^j p(s) \geq 0 \qquad t \in (0, 1), i < i' < j \tag{38}$$

Because $p(i)$ is decreasing, we must have $p(i) \geq p(i') \geq p(j)$ (with strict inequalities in the strictly decreasing case). Hence (38) is satisfied (strictly so in the strictly decreasing case) if

$$t\int_i^{i'} p(i') - (1-t)\int_{i'}^j p(i') \geq 0 \qquad t \in (0, 1)$$

This inequality reduces to $t(i' - i) - (1-t)(j - i') \geq 0$, which, by construction of i', is satisfied.

■

Proof of proposition 4

Define $g(i) \equiv p(i, x') - p(i, x)$, and note that $\int g(s)ds = 0$. Suppose that $g(0) \leq 0$ By presupposition, g is decreasing, right-continuous and, by SSD, $\exists i_0 \in (0, 1)$: $0 \geq g(0) > g(i)$, $\forall i \geq i_0$. Hence $\int g(s)ds < 0$, a contradiction. Therefore $g(0) > 0$, and a similar argument shows that $g(1) < 0$. Because g is decreasing, right-continuous and $g(0) > 0$, the set $\{i : g(i) > 0, i > 0\}$ is non-empty, and we let $i_0 = sup\{i : g(i) > 0, i > 0\}$, noting that $i_0 < 1$. It follows that $p(i, x') > p(i, x)$ on $(0, i_0)$, and also $\int_0^{i_0} g(s)ds > 0$. Because g decreases and $\int g(s)ds = 0$, the set $\{i : g(i) < 0, i \geq i_0\}$ is non-empty, and we set $i_1 = inf\{i : g(i) < 0, i \geq i_0\}$. If $i_0 < i_1$ then $g(i) = 0$ on (i_0, i_1), as g is decreasing and right-continuous. These facts together imply that $p(\cdot, x')$ is OR of $p(\cdot, x)$. ■

Proof of proposition 5

Proceed exactly as in the proof of proposition 4.

Proof of proposition 7

The claim follows by noting that all alluded properties are equivalent to $p_0(x') > p_0(x)$. ■

Proof of corollary 1

We only show the OR case. First, note that both classes satisfy the presumptions of proposition 5. Define $f(x; i, j) \equiv \frac{p(i,x)}{p(j,x)}$. If p is of class II and (10) is satisfied, we may conclude that $f(x'; i, j) > f(x; i, j)$ whenever $x' > x$, and the claim follows from proposition 5. Suppose now that p belongs to class I. As p then is piecewise constant in i for any x with a finite number of downward jumps, it suffices to show that condition (9) holds for all non-jump points of p that satisfy $i < j$ and $j \notin G(i)$. Note that for given $j > i$ that are not jump points, $p(i, x)$ and $p(j, x)$ are both differentiable in x by presumption. Therefore, if (10) is satisfied for any two $j > i$ whenever the derivative exists (no jump point), condition (9) must be satisfied, proving the claim. ■

Proof of corollary 2

Define $f(i) \equiv p(i, x') - p(i, x)$. By the density condition and continuity there must be $i \in (0, 1)$ such that $f(i) = 0$. Hence if $f'(i) < 0$ whenever $f(i) = 0$ there is a unique $i_0 \in (0, 1)$ such that $f(i_0) = 0$, and $f(i) > 0$ on $(0, i_0)$ and $f(i) < 0$ on $(i_0, 1)$. ■

Proof of lemma 1

If $\beta = 0$ or $\Psi = 0$ then clearly $t_2 = 0$, so let $\beta, \Psi > 0$. From (14) and the definition of T_{ij} it follows that the equilibrium value T_{ij} solves

$$T_{ij} = \alpha\left(p(i)T_A + p(j)T_B\right) + \beta\left(\frac{\beta\Psi}{T_{ij}}\right)^{\frac{1}{\eta-1}}\left(c(i)^{\frac{-1}{\eta-1}} + c(j)^{\frac{-1}{\eta-1}}\right) \tag{39}$$

Because $c(i), c(j) > 0$ and $\eta > 1$ there is a unique $T_{ij} > 0$ solving this equation, which proves existence and uniqueness of stage II equilibrium as claimed. Table 2 then is a straightforward Implicit Function Theorem result derived from the two-equation system (14), (39). ■

Proof of theorem 1

Existence and uniqueness follow as a straightforward corollary to the proof of proposition 1. The no leap-frogging property ($p(i) > p(j)$ and $\Pi(i) > \Pi(j)$) follows from $k(i) > k(j)$. Further, the claimed FCB properties follow from the facts that $sign\text{CB}'_{ij}(V) = sign[k(j)-k(i)]$ and $sign\text{CB}'_{ij}(\Psi) = sign[k(i)-k(j)]$ and corollary 1. The rest of table 3 follows from lemma 2. ■

Proof of corollary 3

We only show the claim for the decreasing FCB property. From $\text{CB}^A_{ij} = \left(\frac{c_A(j)}{c_A(i)}\right)^{\frac{1}{\eta-1}} \left(\frac{R(x,c(i))}{R(x,c(j))}\right)^{\frac{1}{\eta-1}}$ it follows that $\text{CB}^A_{ij}{}'(x) > 0$ iff $\frac{R_x(x,c(i))}{R(x,c(i))}$ decreases over cost types, which is implied if $R(x,c)$ has strictly decreasing log-differences. ■

Proof of proposition 8

Let $k(z) \equiv \int p_B(s) \frac{c^s_B}{c^s_B + c^z_A} ds$, $z \in [0,1]$, where $c^s_B = c_B(s)^{\frac{1}{\eta-1}}$, $c^z_A = c_A(z)^{\frac{1}{\eta-1}}$. Let $j > i$ be any two different cost types, and note that $k(i) > k(j)$. If $\tilde{p}_B(\cdot)$ is OR of $p_B(\cdot)$, then $\tilde{k}(z) < k(z)$. If $\tilde{p}_B(\cdot)$ is IR of $p_B(\cdot)$, then $\tilde{k}(z) > k(z)$. Let $q_{ij} \equiv \frac{x+k(i)}{x+k(j)}$ and note that, by (23), if q_{ij} increases (decreases) for a given $x \geq 0$, then $p_A(\cdot)$ makes an OR (IR). Let $dk(z) \equiv \tilde{k}(z) - k(z)$. Hence $dk(z) < (>) 0$ if $\tilde{p}_B$ is OR (IR) of p_B. Then $dq_{ij} > 0$ iff $dk(i)(x + k(j)) > dk(j)(x + k(i))$. Further, it holds that

$$sign\,[dk(i) - dk(j)] = sign \int_0^1 (\hat{p}_B(s) - p_B(s)) \underbrace{\frac{c^s_B}{(c^s_B + c^i_A)(c^s_B + c^j_A)}}_{=x(s)} ds \tag{40}$$

If A cost-dominates B, then $x(s)$ is decreasing. If $\tilde{p}_B$ is OR of p_B, then $0 > dk(i) > dk(j)$ by (40), hence an OR of p_A results. If $\tilde{p}_B$ is IR of p_B, then $0 < dk(i) < dk(j)$ by (40), hence an IR of p_A results. ■

Proof of proposition 9

We only prove the claim for the almost symmetric scenario – the proof of the small impact scenario follows the same lines. To prove existence and uniqueness we can proceed as in the general proof of proposition 1. In the almost symmetric scenario we have for any $i \in [0,1]$:

$$V + p(i)\Psi = c(i)p(i)^{\eta-1}T^{\eta} \qquad (FOC)$$
$$\Psi - (\eta-1)c(i)p(i)^{\eta-2}T^{\eta} < 0 \qquad (SOC)$$

Recalling assumption 1 we see that, for $\eta > 2$, $g(p(i),T) = V + p(i)\Psi$ only fails to satisfy assumption (A1) in such that $g(\cdot,T)$ is unbounded. This particular assumption was required to establish the existence of a $p(i;T)$ that solves FOC for a given $T > 0$ in the proof of proposition 1. If $\eta > 2$ it is clear that such a solution $p(i;T)$ exists, and existence/uniqueness of the equilibrium $(p(i),T)$ then follows from the proof of proposition 1. If $\eta = 2$, then $p(i,T) = \frac{V}{c(i)T^2 - \Psi}$. Presuming $T^2 > \Psi$ (required by SOC) implies that $p(\cdot,T) > 0$. For $G(T) \equiv \int p(i,T)di$ we then have $G'(T) < 0$, $\lim_{T\to\infty} G(T) = 0$, and $\lim_{T^2 \downarrow \Psi} G(T) =$

$\frac{V}{\Psi}\int_0^1 \frac{1}{c(i)-1}di = \infty$. Hence $\exists!\ T > 0$ such that $T^2 > \Psi$ and $G(T) = 1$, which proves the claim also for $\eta = 2$. The no leap-frogging property is an immediate consequence of FOC and SOC, and the equilibrium payoff level is $\Pi(i) = p(i)V\frac{\eta-1}{\eta} + p(i)^2\Psi\frac{\eta-2}{2\eta} > 0$.

To prove table 5, note first that $G(T;\Psi,V) = 1$ implies $T'(V), T'(\Psi) > 0$. Let $i < j$ and $j \notin G(i)$. Then

$$\text{CB}_{ij} = \frac{c(j)K - \Psi}{c(i)K - \Psi} \qquad K \equiv T^2$$

where we know that $K > \Psi$. Because $K'(V) > 0$ it follows that $\frac{\partial}{\partial V}\text{CB}_{ij} < 0$ iff $c(i) < c(j)$, proving the claim for $dV > 0$. Further, $\frac{\partial}{\partial \Psi}\text{CB}_{ij} < 0$ iff $K > \Psi K'(\Psi)$. From $\int_0^1 \frac{V}{c(i)K-\Psi} = 1$, we obtain

$$K'(\Psi) = \frac{\int_0^1 (c(i)K - \Psi)^{-1}di}{\int_0^1 c(i)(c(i)K - \Psi)^{-1}di} < 1$$

and the claim follows from $K > \Psi$. Finally, $\frac{\Pi(i)}{\Pi(j)} = \text{CB}_{ij}$ follows from proposition 2 c) by noting that here we have $V(p(i),T) = V + p(i)\Psi$.

To verify the claimed rotations if $c(i)$ (and thus $p(i)$) is type II, we use corollary 2. We show the result for $\Psi' > \Psi$; the case $V' > V$ is proved identically. Consider the two equilibria $(\hat{p}(i;\Psi'), T'(\Psi'))$, $(p(i;\Psi), T(\Psi))$, and suppose that $\hat{p}(i_0) \equiv \hat{p}(i_0,\Psi') = p(i_0,\Psi) \equiv p(i_0)$. Then, FOC imply that

$$\frac{K}{K'} = \frac{V + p(i_0)\Psi}{V + p(i_0)\Psi'} \qquad K, K' \equiv T^\eta, (T')^\eta$$

According to corollary 2 we now need to show that $\hat{p}'(i_0) < p'(i_0)$. The respective FOC implies

$$p'(i_0) = \frac{c'(i_0)p(i_0)^{\eta-1}T^\eta}{\Psi - (\eta-1)c(i_0)p(i_0)^{\eta-2}T^\eta}$$

and a similar expression for $\hat{p}'(i_0)$. It follows that $\hat{p}'(i_0) < p'(i_0)$ holds if

$$\frac{V + \Psi p(i_0)\frac{\eta-2}{\eta-1}}{V + \Psi' p(i_0)\frac{\eta-2}{\eta-1}} > \frac{K}{K'} = \frac{V + \Psi p(i_0)}{V + \Psi' p(i_0)}$$

which is satisfied as $\frac{V+\Psi x}{V+\Psi' x}$ is a strictly decreasing function of x (for $x \geq 0$).

■

Proof of lemma 3

No leap-frogging follows from dividing (25) by $p(i)^{\eta-1}$, and noting that then the LHS of this equation satisfies $\text{LHS}'(p(i)) < 0$. Further, a) is an immediate consequence of no leap-frogging. Turning to payoffs we note that the payoff function is

$$\Pi(i) = p(i)V\left(p(i), T\int\cdot\right) - \frac{1}{\eta}c(i)p(i)^\eta T^\eta \qquad V\left(p(i), T\int\cdot\right) = V + p(i)\Psi T\int\cdot$$

with $V_1(p(i), T\int\cdot) = \Psi T\int\cdot > 0$, $V_{11}(p(i), T\int\cdot) = 0$, and b) then follows from proposition 2. Finally, using (25) gives the equilibrium payoff level

$$\Pi(i) = p(i)\left(V\frac{\eta-1}{\eta} + p(i)\Psi\frac{\eta-2}{\eta}\int_0^1 \frac{p(j)}{p(i)+p(j)}dj\right) > 0$$

■

<u>Proof of proposition 10</u>

If an equilibrium exists as claimed, then $p_1 = \frac{1-p_0\gamma}{1-\gamma}$, and $p_0 > 1 > p_1$ follows from no leap-frogging (lemma 3). Moreover, $p_1 > 0$ requires that $p_0 < 1/\gamma$. Use p_1 in (28), let $h(x) \equiv x^{\eta-1}$, and rewrite (28) by eliminating T^η as

$$\frac{V}{h(p_0)} + \frac{\Psi p_0}{h(p_0)}\left(\gamma + \frac{2(1-\gamma p_0)(1-\gamma)}{1+p_0(1-2\gamma)}\right) = \frac{V}{ch\left(\frac{1-p_0\gamma}{1-\gamma}\right)} + \frac{\Psi\left(\frac{1-p_0\gamma}{1-\gamma}\right)}{ch\left(\frac{1-p_0\gamma}{1-\gamma}\right)}\left(1+\gamma\frac{2p_0}{1+p_0(1-2\gamma)}\right) \quad (41)$$

Considering separately the two sides of (41) we note the following facts: $LHS(p_0) \overset{p_0\to 1}{\longrightarrow} V+\Psi$, $LHS(p_0) \overset{p_0\to 1/\gamma}{\longrightarrow} \gamma^{\eta-1}(V+\Psi)$, $LHS'(p_0) < 0$ on $(1, 1/\gamma)$, as well as $RHS(p_0) \overset{p_0\to 1}{\longrightarrow} \frac{V+\Psi}{c}$, $RHS(p_0) \overset{p_0\to 1/\gamma}{\longrightarrow} \infty$, $RHS'(p_0) > 0$ on $(1, 1/\gamma)$. Taken together, these facts imply existence and uniqueness of equilibrium as claimed. To see the claimed comparative statics, rewrite (41) as $F(p_0, V, \Psi) \equiv LHS(p_0) - RHS(p_0) = 0$, noting that the previous facts imply $F_1 < 0$ for $p_0 \in (1, 1/\gamma)$, and also note that F is linear and additive separable in V and Ψ. Then $sign(p_0'(V)) = sign(F_V)$. Lemma 3 a) and the fact that

$$F_V = \frac{1}{(p_0)^{\eta-1}} - \frac{1}{c(p_1)^{\eta-1}} < 0 \quad \Leftrightarrow \quad \frac{p_0}{p_1} > \left(\frac{c}{1}\right)^{\frac{1}{\eta-1}}$$

imply that $dV > 0$ causes an IR of $p(\cdot)$. This result, together with the linearity of F in Ψ and the requirement that $F = 0$, then immediately implies that $p_0'(\Psi) > 0$, i.e. $d\Psi > 0$ causes an OR. Finally, the remaining claims about the equilibrium payoffs follow from these comparative statics and lemma 3 b). ■

<u>Proof of proposition 11</u>

The equilibrium T must satisfy $\xi(i) \equiv T\tilde{T}c(i) - 2\alpha\Psi > 0$. Letting $\zeta(i) \equiv \beta^2\Psi^2 + 2\tilde{T}^2Vc(i)$ we obtain from (29):

$$T'(\alpha) = \frac{\Psi\int_0^1 \frac{\zeta(i)}{c(i)\xi(i)^2}\mathrm{d}i}{\frac{1}{T}\int_0^1 \frac{(\xi(i)+\alpha\Psi)\zeta(i)}{c(i)\xi(i)^2}\mathrm{d}i} > 0 \qquad T'(\beta) = \frac{\beta\Psi^2\int_0^1 \frac{1}{c(i)\xi(i)}\mathrm{d}i}{\frac{1}{T}\int_0^1 \frac{(\xi(i)+\alpha\Psi)\zeta(i)}{c(i)\xi(i)^2}\mathrm{d}i} > 0 \quad (42)$$

Note that

$$\mathrm{CB}_{ij} = \frac{2c(i)\tilde{T}^2V + \beta^2\Psi^2}{2c(j)\tilde{T}^2V + \beta^2\Psi^2} * \frac{c(j)T\tilde{T} - 2\alpha\Psi}{c(i)T\tilde{T} - 2\alpha\Psi} \equiv A * B \qquad \frac{\Pi(i)}{\Pi(j)} = \mathrm{CB}_{ij}A\frac{c(j)}{c(i)} \quad (43)$$

Differentiation yields that $\mathrm{CB}'_{ij}(\alpha) > 0$ iff $\frac{\alpha T'(\alpha)}{T(\alpha)} < 1$, hence for $\alpha \leq 0$ we must have $\mathrm{CB}'_{ij}(\alpha) > 0$. This also holds for $\alpha > 0$, because $\frac{\alpha T'(\alpha)}{T(\alpha)} < 1$ if $\xi(i) > 0$, which is satisfied in equilibrium. It follows that also $\frac{\Pi(i)}{\Pi(j)}$ increases in α.

For $d\beta > 0$ we have $A'(\beta) > 0$ if $V, \Psi, \beta > 0$, and $A'(\beta) = 0$ if one of these parameters is zero. Further, $sign\, B'(\beta) = sign\,(-\alpha)$. Hence if $\alpha < 0$ we always have $\mathrm{CB}'_{ij}(\beta) > 0$, and this also holds if $\alpha = 0$ and $V, \Psi, \beta > 0$. The same conditions obviously assert that also $\frac{\Pi(i)}{\Pi(j)}$ increases in β. Finally, if $V = 0$ and $\alpha > 0$ it follows that $A'(\beta) = 0$ and $B'(\beta) < 0$, hence CB_{ij} and relative payoffs decrease in β in this case.

■

Proof of proposition 12

We first show that for admissible parameters: $\alpha T'(\alpha) + T(\alpha) > 0$. If $\alpha \geq 0$ this follows from (42). For $\alpha < 0$ using (42) shows that the inequality $\alpha T'(\alpha) > -T(\alpha)$ is equivalent to $\int_0^1 \frac{\zeta(s)(2\alpha\Psi+\xi(s))}{c(s)\xi(s)^2}\mathrm{ds} > 0$. But the last inequality is satisfied because $2\alpha\Psi + \xi(s) = T\tilde{T}c(s) > 0$ holds for any $s \in [0,1]$. Hence for the case of symmetric players, i.e. $c(i) = c$ and $p(i,\alpha) = 1$, it immediately follows that $\frac{\partial P(A,\alpha)}{\partial\alpha} > 0$. We next claim that with heterogeneous agents $\frac{\partial}{\partial\alpha}\int p(i,a)^2 di > 0$ and $\frac{\partial}{\partial\alpha}\int \frac{p(i,\alpha)}{c(i)} di > 0$, which the proves the proposition. We will only verify the first inequality; the second can be proven by the same type of argument. From proposition 11 we know that $d\alpha > 0$ causes an OR of $p(i)$. This means that we can decompose

$$\int_0^1 P_\alpha(i,\alpha)di = \int_0^{i_0} \underbrace{P_\alpha(i,\alpha)}_{>0}di + \int_{i_1}^1 \underbrace{P_\alpha(i,\alpha)}_{<0}di = 0$$

To show the first inequality, we need to verify that $\int_0^1 P(i,\alpha)P_\alpha(i,\alpha)di > 0$, which by decomposition means that $\int_0^{i_0} P(i,\alpha)P_\alpha(i,\alpha)di > -\int_{i_1}^1 P(i,\alpha)P_\alpha(i,\alpha)di$. This inequality is correct as the following argument shows:

$$\begin{aligned}\int_0^{i_0} P(i,\alpha)P_\alpha(i,\alpha)di &> \int_0^{i_0} P(i_0,\alpha)P_\alpha(i,\alpha)di \\ &= -\int_{i_1}^1 P(i_0,\alpha)P_\alpha(i,\alpha)di > -\int_{i_1}^1 P(i,\alpha)P_\alpha(i,\alpha)di\end{aligned}$$

■

Proof of proposition 13

From (29) one obtains $T'(V) > 0$ by implicit differentiation.

Using the multiplicative decomposition in (43) we have $A'(V) < 0$ if $V = 0$ and $\beta, \Psi > 0$, and $A'(V) = 0$ if $V > 0$ and $\beta = 0$. Further $sign\, B'(V) = sign(-\alpha)$.

Then, using differentiation is is straightforward to verify that $\mathrm{CB}'_{ij}(V) < 0$, given the parameter constraints in the proposition, and because $\frac{\Pi(i)}{\Pi(j)} = \frac{p(i)}{p(j)}\frac{\zeta(i)}{\zeta(j)}\frac{c(j)}{c(i)}$ and $\frac{\partial}{\partial V}\frac{\zeta(i)}{\zeta(j)} < 0$ the first part of the claim is clear.

Implicit differentiation yields

$$T'(\Psi)=\frac{\int_0^1 \frac{\tilde{T}(2\tilde{T}V\alpha+T\beta^2\Psi)c(s)-\alpha\beta^2\Psi^2}{c(s)\xi(s)^2}\mathrm{d}s}{\frac{1}{T}\int_0^1 \frac{(\xi(s)+\alpha\Psi)\zeta(s)}{c(i)\xi(s)^2}\mathrm{d}s}>0$$

From this expression it can be shown that $\frac{T'(\Psi)\Psi}{T(\Psi)}<1$ iff $-\int \frac{1}{c(s)\xi(s)}ds<0$, hence we have $\frac{T'(\Psi)\Psi}{T(\Psi)}<1$ for admissible parameters. One can verify that $A'(\Psi)>0$ if $V,\Psi,\beta>0$, and $A'(\Psi)=0$ if one of these parameters is zero. Further, $sign\,B'(\Psi)=sign\,(\alpha(T-T'(\Psi)\Psi))$, hence $sign\,B'(\Psi)=sign\,(\alpha)$. These results together with (43) imply ii) in the proposition. ■

B Quasi-symmetric equilibria with perfect carryovers

In this section we show that our main insights from section 6 on how the prize structure $\{V_A,V_B,\Psi\}$ and efficiency parameters $\{c_A,c_B\}$ determine relative payoffs $\frac{\Pi_A}{\Pi_B}$ extend to the case of perfect carryovers. This case is much more difficult to analyze, because each unit's chance to win the grand contest depends on stage I efforts, and thus on the prize structure as well: $Pr(A\,\text{wins})=\frac{T_A}{T_A+T_B}$, $Pr(B\,\text{wins})=\frac{T_B}{T_A+T_B}$. In the following we assume that $c_A,c_B\geq 1$, $V_A,V_B>0$, $\Psi\geq 0$ and $\eta>2$. For $\alpha=1$ and $\beta=0$ (31) becomes

$$V_A+\frac{2T_A\Psi}{T_A+T_B}=c_AT_A^\eta \qquad V_B+\frac{2T_B\Psi}{T_A+T_B}=c_BT_B^\eta. \tag{44}$$

These conditions determine (T_A,T_B). We first show that a unique, regular quasi-symmetric equilibrium exists.[45]

Proposition 16 *System* (44) *has a unique, regular solution* $(T_A,T_B)>0$.

Proof: We show existence and uniqueness by an application of index theory (see e.g. Vives (1999)). First, note that e.g. $t_A=0$ is never optimal for any agent in unit A, hence $T_A=0$ (or $T_B=0$) cannot be a part of an equilibrium. Next, for $T_A,T_B>0$ the two equilibrium expressions

$$\begin{aligned}F_A\left(T_A,T_B\right)&=V_A+2\Psi\tfrac{T_A}{T_A+T_B}-c_AT_A^\eta\\ F_B\left(T_A,T_B\right)&=V_B+2\Psi\tfrac{T_B}{T_A+T_B}-c_BT_B^\eta\end{aligned} \tag{45}$$

induce a vector field $F:(0,\infty)^2\to\mathbb{R}^2$, $F(T_A,T_B)=(F_A,F_B)$. Because $V_A,V_B,c_A,c_B>0$ $\exists\varepsilon>0$, sufficiently close to zero, such that F points inwards at all "lower" boundary points (e.g. (T_A,ε)). Similarly, $\exists E>0$, sufficiently large, such that no agent would optimally choose $T_A,T_B\geq E$. Thus F points inwards on the set $(\varepsilon,E)^2$. Now, suppose that $(T_A,T_B)>0$ is an equilibrium, i.e. a zero of F.

[45] An equilibrium is regular if the Jacobian matrix J associated with (44) has a non-zero determinant at the equilibrium point.

The determinant of the Jacobian associated with (44) is

$$Det(J) = \frac{\eta\left(c_A T_A^\eta c_B T_B^\eta (T_A+T_B)^2 \eta - 2\Psi T_A T_B \left(c_A T_A^\eta + c_B T_B^\eta\right)\right)}{T_A T_B (T_A+T_B)^2} \tag{46}$$

Hence $Det(J) > 0$ iff $c_A T_A^\eta c_B T_B^\eta (T_A+T_B)^2 \eta > 2\Psi T_A T_B \left(c_A T_A^\eta + c_B T_B^\eta\right)$. Using (44) this condition can be rephrased as

$$c_A T_A^{\eta+1} c_B T_B^\eta \eta + c_B T_B^{\eta+1} c_A T_A^\eta \eta > c_A T_A^{\eta+1}\left(c_B T_B^\eta - V_B\right) + c_B T_B^{\eta+1}\left(c_A T_A^\eta - V_A\right)$$

which is satisfied because $V_A, V_B > 0$ and $\eta > 1$. As $Det(J) > 0$ at critical points, existence and uniqueness of the equilibrium follows from the Index theorem. ■

Remark: It follows from a result in Hefti (2014b), that the unique equilibrium (T_A, T_B) is also stable with respect to several standard adjustment processes, including gradient stability or a discrete tatonnement.

Equilibrium payoffs are

$$\Pi_A = \frac{\eta-1}{\eta}\left(V_A + \Psi\frac{T_A}{T_A+T_B}\right) \qquad \Pi_B = \frac{\eta-1}{\eta}\left(V_B + \Psi\frac{T_B}{T_A+T_B}\right) \tag{47}$$

We first verify that $(T_A, \Pi_A) > (T_B, \Pi_B)$ holds for the same sufficient conditions as in section 6.

Lemma 4 *$(T_A, \Pi_A) > (T_B, \Pi_B)$ if one of the following conditions holds: 1) $(V_A > V_B, c_A \leq c_B)$ or 2) $(V_A = V_B, c_A < c_B)$*

Proof: The claim on T_A, T_B can be shown using a result in Hefti (2014a). To see this, suppose perfect symmetry, i.e. set $V_A = V_B = V$ and $c_A = c_B = c$. Because then $T'(V) > 0$ and $T'(c) < 0$, this game is monotonic in V and c. By the quoted proposition, it suffices to verify that this symmetric game has no asymmetric equilibrium, which holds if $T_A'(T_B) > -1$ (see again Hefti (2014a) for details). Hence we must show

$$T_A'(T_B) = \frac{2\Psi\frac{T_A}{(T_A+T_B)^2}}{2\Psi\frac{T_B}{(T_A+T_B)^2} - \eta c_A T_A^{\eta-1}} > -1$$

By (44) the denominator is negative, and we need to verify that

$$2\Psi\frac{T_A}{(T_A+T_B)^2} < \eta c_A T_A^{\eta-1} - 2\Psi\frac{T_B}{(T_A+T_B)^2}$$

which is equivalent to (use (44)) $2\Psi\frac{T_A}{T_A+T_B} < \eta c_A T_A^\eta$. Using (44) again shows, that this condition is equivalent to $-V_A < c_A T_A^\eta(\eta-1)$, which is satisfied. Hence, by the quoted proposition, starting from $V \equiv V_B$ or $c \equiv c_B$ deviations such as 1) and 2) imply that $T_A > T_B$ must hold in any equilibrium. It is now a direct consequence of $T_A > T_B$ and (47) that $\Pi_A > \Pi_B$ under 1) and 2). ■

Proposition 17 *An increase of Ψ has the following effect on relative payoffs:*

(i) if $V_A > V_B$, $c_A \geq c_B$ and $T_A > T_B$ then Π_A/Π_B decreases

(ii) if $V_A \leq V_B$, $c_A < c_B$ and $T_A > T_B$ then Π_A/Π_B increases

Proof First, note that (44) implies that

$$T_A V_B < T_B V_A \qquad \Leftrightarrow \qquad c_B T_B^{\eta-1} < c_A T_A^{\eta-1} \tag{48}$$

The Implicit Function Theorem yields

$$T_A{}'(\Psi) = \frac{2T_A}{Det(H)(T_A+T_B)^2 T_B}\left(c_B {T_B}^{\eta}\eta(T_A+T_B) - 2\Psi T_B\right) \tag{49}$$

Plugging in (46) gives

$$T_A{}'(\Psi) = \frac{2T_A^2\left(c_B {T_B}^{\eta}\eta(T_A+T_B) - 2\Psi T_B\right)}{\eta\left(c_A T_A^{\eta} c_B T_B^{\eta}(T_A+T_B)^2\eta - 2\Psi T_A T_B\left(c_A T_A^{\eta} + c_B T_B^{\eta}\right)\right)}$$

Using (44) it can be verified that the nominator and the denominator of this expression are positive. We may therefore conclude that $T_A'(\Psi), T_B'(\Psi) > 0$. Write relative payoffs as

$$\frac{\Pi_A}{\Pi_B} = \frac{V_A + \frac{\eta-2}{\eta} c_A T_A^{\eta}}{V_B + \frac{\eta-2}{\eta} c_B T_B^{\eta}} \tag{50}$$

$\frac{\Pi_A}{\Pi_B}$ increases in Ψ iff

$$(\eta-2)\, c_A T_A^{\eta-1}\left(V_B + \frac{\eta-2}{\eta} c_B T_B^{\eta}\right) T_A{}'(\Psi) > (\eta-2)\, c_B T_B^{\eta-1}\left(V_A + \frac{\eta-2}{\eta} c_A T_A^{\eta}\right) T_B{}'(\Psi)$$

or equivalently if

$$\frac{c_B {T_B}^{\eta}\eta(T_A+T_B) - 2\Psi T_B}{c_A {T_A}^{\eta}\eta(T_A+T_B) - 2\Psi T_A} < \frac{c_B T_B^{\eta+1}\left(V_A + \frac{\eta-2}{\eta} c_A T_A^{\eta}\right)}{c_A T_A^{\eta+1}\left(V_B + \frac{\eta-2}{\eta} c_B T_B^{\eta}\right)} \tag{51}$$

If $c_B T_B^{\eta-1} \leq c_A T_A^{\eta-1}$ (implied by (i)) then a sufficient condition for (51) to hold is that

$$\frac{c_B {T_B}^{\eta}\eta(T_A+T_B)}{c_A {T_A}^{\eta}\eta(T_A+T_B)} < \frac{c_B T_B^{\eta+1}\left(V_A + \frac{\eta-2}{\eta} c_A T_A^{\eta}\right)}{c_A T_A^{\eta+1}\left(V_B + \frac{\eta-2}{\eta} c_B T_B^{\eta}\right)}$$

which, using (44), can be reduced to $T_A V_B < T_B V_A$. This inequality holds by (48) and (i). To see the second claim, proceed analogously. The claim holds if (51) holds with reversed sign. A similar line of arguments establishes that this new inequality is true if $c_B T_B^{\eta-1} > c_A T_A^{\eta-1}$, which holds under (48) (with reversed signs) and (ii). ■

Proposition 17 resembles our earlier result, because (i) implies $\frac{V_A}{V_B} < \frac{r_A}{r_B}$ and (ii) implies $\frac{V_A}{V_B} > \frac{r_A}{r_B}$ in the model section 6. The difficulty with proposition 17 is that conditions (i) and (ii) additionally require that $T_A > T_B$, which is endogenous to the model. However, together with lemma 4 an unambiguous result can be obtained:

Corollary 6 *If $V_A > V_B$ and $c_A = c_B$ then Π_A/Π_B decreases in Ψ. If $V_A = V_B$ and $c_A < c_B$ then Π_A/Π_B increases in Ψ.*

Finally, we show that as in section 6 a sufficiently high value of Ψ crowds out any initial advantage of a less efficient unit with a higher local prize.

Proposition 18 *Suppose that $c_A > c_B$, $V_A > V_B$, $V_A/c_A > V_B/c_B$, and $\eta > 2$. Then $\Psi = 0$ implies that $T_A > T_B$ and $\Pi_A > \Pi_B$ but both T_A/T_B and Π_A/Π_B are decreasing functions of Ψ. There exists a $\Psi' > 0$ such that $T_A < T_B$ for all $\Psi > \Psi'$. Moreover, there exists $\Psi'' > \Psi'$ such that $\Pi_A < \Pi_B$ for all $\Psi > \Psi''$.*

Proof: For $\Psi = 0$ we get $T_A > T_B$ from (44) and $\Pi_A > \Pi_B$ follows. T_A/T_B increases in Ψ iff $T'_A(\Psi)/T'_B(\Psi) > T_A/T_B$. Using (49) this holds if $c_B T_B^{\eta-1} > c_A T_A^{\eta-1}$. Hence if $T_A \geq T_B$, then $c_A > c_B$ directly implies that T_A/T_B decreases in Ψ. If $T_A < T_B$ then using (48) shows that T_A/T_B decreases in Ψ also in this case. We now show that $\frac{\Pi_A}{\Pi_B}$ is decreasing in Ψ. By proposition 17 Π_A/Π_B decreases in Ψ if $T_A > T_B$. If $T_A \leq T_B$, then also $V_A T_B > V_B T_A$, which by the proof of proposition 17 implies Π_A/Π_B to decrease in Ψ. Let $\Psi' > 0$ solve $\frac{V_A+\Psi'}{c_A} = \frac{V_B+\Psi'}{c_B}$. Then, by uniqueness of the equilibrium, it follows from (44) that $T_A(\Psi') = T_B(\Psi')$. As T_A/T_B is decreasing in Ψ this implies that $T_A/T_B < 1$ for all $\Psi > \Psi'$. We now show: $\exists \Psi'' > \Psi'$: $\Pi_A(\Psi) < \Pi_B(\Psi)\ \forall \Psi > \Psi''$. By (47): $\Pi_B > \Pi_A$ iff $\Psi \frac{T_B - T_A}{T_A + T_B} > V_A - V_B$. Hence $\Pi_B < \Pi_A$ if $\Psi \leq \Psi'$, but $\Pi_B > \Pi_A$ for any $\Psi \geq V_A - V_B$. Because $\frac{\Pi_A}{\Pi_B}$ is strictly decreasing in Ψ and continuous $\exists \Psi'' > \Psi'$: $\Pi_B(\Psi) > \Pi_A(\Psi)$, $\Psi > \Psi''$. ■