Maximum public debt in the Diamond OLG model:

Three overlapping generations are better than two[1]

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Abstract:

While the public debt has an interior maximum in the Diamond OLG model, due to the concavity of the wage-saving relationship [Rankin and Roffia (2003)], Braeuninger (2005) shows this property extends to a linear, AK model when combined with a backward-looking adjustment process for public debt. We show that the maximum values, however, will be of an implausibly low order of magnitude, unless households save over at least two periods in order to permit a crucial distinction to be made between the *stocks* and *flows* of public debt. Flows crowd-out other investment flows, but these are synonymous with debt stocks in a model with only two non-altruistic overlapping generations. Removing this restriction allows empirically plausible maximum values, while also implying that a low rate of economic growth may be a cause as well as a consequence of a high public debt. This extension to Diamond (1965) makes it the appropriate model for considering also extreme cases of the phenomenon it was originally formulated to address.

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1. Introduction

There is current concern with very high levels of public debt and whether these are sustainable. Fiscal sustainability has traditionally focussed on the deficit rather than on the debt with an emphasis on the stability rather the existence of equilibrium. For example, there are papers by Nielsen (1992), Bohn (1995) and Chalk (2000). A main question has been the convergence of public deficits or surpluses and thence of the stock of debt. In an infinite-horizon model or from the perspective of an altruistic and dynastically-minded individual, indifferent to the time-profile of taxation, in the absence of distortionary taxes, the size of any convergent stock of debt does not matter [Barro (1974)]. Thus the notion of a maximum debt is meaningless, and this model is unable to address present concerns with very high levels.

More recently, Rankin and Roffia (2003) analysed public debt in the very different setting of the Diamond (1965) overlapping generations model where non-altruistic households have finite lives. In this model, where debt clearly does matter, even as one that was originally formulated to analyse this issue, they found that it has maximum value due to the nonlinearity of the model, namely, that the capital stock is an increasing and concave function of its own past value through the wage-saving relationship. They designate this a *bifurcation* maximum, because from a reversed perspective there are two steady states that could meet at a single interior point. These authors also point out the alternative possibility of a corner or *degenerate* maximum, where the maximum debt would eliminate any economic activity, which is also a result in Rankin's (2012) application of this analysis to the Blanchard-Yaari model of perpetual youth [Yaari (1965) and Blanchard (1985)].

There have been a number of offshoots from the original paper. Braeuninger (2005) applies the analysis to a model of growth. Farmer and Zotti (2010) also obtain similar results in an open-economy extension. Roberts (2013) returns to the earlier closed-economy form to consider various other fiscal policy rules.[3] The paper most relevant to

[3] These are found to be important both for the nature and for the size of the maximum debt. A bifurcation maximum also occurs where tax revenue instead is treated as exogenous, implying a higher level of a now

our present concern is Braeuninger (2005). Bifurcation maxima naturally depend on the nonlinearity of the model, so *prima facie* would appear to be precluded from the linear class of AK, endogenous growth models, as expounded by Romer (1986) and exemplified by Lucas (1988). Braeuninger, however, shows that the combination of a standard AK model with a backward-looking dynamic process for the public debt generates two steady state solutions on either side of another bifurcation point where the debt-GDP ratio and the economic growth rate are, respectively, at a maximum and a minimum.

We extend Braeuninger's analysis in two ways by considering an alternative form of debt dynamics and then by generalising the underlying OLG model. More hypothetically, we initially consider debt from an alternative, Ricardian, forward-looking perspective as the present value of future budget surpluses. As this also entails a reversal of the dynamic stability condition, the debt-growth relationship becomes monotonic, admitting instead a *degenerate* maximum. However, for either of these dynamic forms, the computed numerical values turn out to be implausibly low. Furthermore, there is little improvement in this respect in an attempt to convert figures that are constructed on the basis of OLG half-life-periods into more familiar annualised measures. It seems unfortunate that a model that was originally geared to demonstrate the benefits of modest levels of public debt is unable to realistically address possible problems when they are very high.

This prompts us to make the second and, as we find, substantive extension of the model to three overlapping generations, allowing households to save in the two periods when they are young and middle-aged. This seemingly innocuous generalisation may cause a quantum change in the model in allowing a distinction to be made between debt *stocks* and *flows*, such that maximum debt-GDP ratios may rise to plausibly much higher orders of magnitude. As debt *flows* crowd-out other, investment flows, the size of the stock

endogenous debt, while exogenous income tax rates lead to a degenerate maximum, provided that the first *Inada condition* holds.

may be extremely high without affecting anything other than the amount of expenditure required for servicing it. Moreover, this may be low where debt constitutes the accumulation of past deficits. A corollary is that a low rate of economic growth may also support a large stock of debt – at least in the steady state – because of the implication of low flows of public debt and, hence, little asset crowding-out. Thus, a minor extension to the Diamond model enables it to address concerns about very high levels, which elude models where households either live for ever or where their finite lives are divided into only two periods.

The paper is organised as follows. In *Section 2* the standard form of the model with two overlapping generations is considered. This is basically a re-run of Braeuninger (2005) but with a quantitative assessment. It also briefly discusses a case more hypothetical to the finite horizon case, where public debt is forward-looking. *Sections 3* generalises the model to three overlapping generations and contains the main analysis. The concluding *Section 4* provides a brief summary.

2. The basic 2-OLG model

2.1 Production

Firms produce subject to a Cobb-Douglas production technology with constant returns to scale with respect to both to the firm's own factors of capital and labour and to the general capital stock, private and social. It may thus be presented in per capita terms as,

$$y_t(i) = Ak_t^{1-\alpha} k_t(i)^{\alpha}, \tag{1}$$

where $y_t(i)$ and $k_t(i)$ are the output and the capital of the single firm, while k_t is aggregate capital. Marginal cost pricing implies

$$w_t(i) = Ak_t^{1-\alpha} k_t(i)^{\alpha}, \qquad R_{K,t} = \alpha A k_t^{1-\alpha} k_t(i)^{\alpha-1}\, k_t(i) = k_t, \ \ \forall i,$$

where $w_t(i)$ is the wage in firm i, where and $R_{K,t}$ is the interest factor at which firms borrow. Symmetric equilibrium gives

$$y_t = Ak_t, \qquad w_t = (1-\alpha)Ak_t, \qquad R_{K,t} = \alpha A. \tag{2}$$

There is a standard lag from saving to investment, while the assumptions of full depreciation within the period and of constant population give

$$k_{t+1} = s_t^K \tag{3}$$

Household saving, s_t, also comprises the acquisition of public debt, d_t,

$$s_t = s_t^K + d_t \tag{4}$$

2.2 Households

Households live for two periods and derive utility from consumption in each.

$$U_t = \ln c_t^Y + \lambda \ln c_{t+1}^O, \tag{5.1}$$

There is a standard logarithmic form, and λ is a time-preference factor. Households supply a fixed unit of labour when young for which they receive a wage, w_t, which is then taxed at the rate τ. The amount they save accumulates by the gross interest factor of return R, $R \equiv 1 + r \leq R_K$, where r is the net-of-tax interest rate. It is assumed both that only interest is taxed and that there is no inflationary erosion of the principal, so the post-tax, real return factor is $1 + (1-\tau)r$. The household budget constraints for young and old households are $c_t^Y = (1-\tau)w_t - s_t$ and $c_{t+1}^O = \left(1 + (1-\tau)r\right)s_t$. As the income and substitution effects of the rate of interest rate exactly cancel, because of a constant elasticity of intertemporal substitution under logarithmic preferences, and as there is no future, earned income to be discounted, saving depends only on the post-tax wage,

$$s_t = \left(\lambda/(1+\lambda)\right)(1-\tau)w_t \tag{6}$$

Capital accumulation is determined from equations (3), (4) and (6) as

$$k_{t+1} = \left(\lambda/(1+\lambda)\right)(1-\tau)(1-\alpha)Ak_t - d_t \tag{7}$$

Defining the growth factor and the debt-GDP ratio as

$$G_{t+1} \equiv 1 + g \equiv k_{t+1}/k_t, \qquad \delta_t \equiv d_t/y_t = d_t/Ak_t, \tag{8}$$

enables equation (7) to be presented in scale-free terms as as

$$G_{t+1} = (1-\tau)G^* - A\delta_t \qquad \text{where} \qquad G^* \equiv \left(\lambda/(1+\lambda)\right)(1-\alpha)A \tag{9.1}$$

The composite term $G^*, G^* > 1$, is the benchmark level of the growth factor in the absence of both public debt and taxes. A steady state of positive output naturally requires that the levels of debt and taxes are never so high that $G < 1$ ($g < 0$), while a *degenerate* maximum in the present context is defined where $G = 1$ ($g = 0$), below which there can be no long-run economic activity.

2.3 Backward-looking public debt dynamics in the 2-OLG model

The government financing requirement is given by

$$d_t = E_t - T_t + \left(1+(1-\tau)r\right)d_{t-1}, \tag{10.B}$$

The size of the debt depends on primary public expenditure, E_t, less the revenue raised from taxing the factors of production, T_t, plus the amount of net-of-tax servicing required to service its previous level, d_{t-1}. This public debt in this sense is "backward-looking" as the accumulation of past primary deficits,

$$d_t = \sum_{i=0}^{\infty} \left(1+(1-\tau)r\right)^{-i}(E_{t-i} - T_{t-i}) \tag{11.B}$$

Using additional terms to denote the share of government expenditure and the tax take,

$$\gamma_t \equiv E_t/Y_t, \qquad \tau_t \equiv T_t/Y_t, \tag{12}$$

allows us to rewrite equation (10.B) in a scale-free, ratio form,

$$\delta_t = \gamma_t - \tau_t + \left(\frac{1+(1-\tau)r}{G_t}\right)\delta_{t-1}. \tag{13.B}$$

A necessary, but not sufficient, condition for a backwardly-stable steady state with public debt is that $G > 1+(1-\tau)r$.[4] The requirement that rates of return on public debt are substantially lower than those on capital, since these are believed to be well in excess of

[4] Bohn (1998) found for the US a stabilizing positive feedback rule from the debt to the surplus. Mauro (2013) *et al* investigated this possibility for an extensive panel of countries and time periods. We note, however, that a *linear* rule will have limited use for stabilization, if the model, as the present one, is *nonlinear*.

economic growth rates, finds support in Dutta, Kapur and Orszag (2000).[5] This spread may reflect large premia on risky capital or else either financial repression or imperfectly competitive rates of return from a segmented market to which bond yields are linked through arbitrage.[6] For present purposes, this condition is just assumed in order to consider first the backward-looking case. Stability ensures convergence to a debt ratio of

$$\delta = \frac{G}{G-\left(1-(1-\tau)r\right)}(\gamma-\tau)\,. \tag{14.B}$$

Notably, an equilibrium ($\delta > 0$) requires a steady state of *primary deficits (* $\gamma > \tau$).

The present concern is rather with the endogeneity of economic growth in equation (9.1), which with equation (13.B) gives rise to a nonlinear difference equation for the debt ratio,

$$\delta_t = \gamma_t - \tau_t + \left(\frac{1+(1-\tau)r}{(1-\tau)G^* - A\delta_{t-1}}\right)\delta_{t-1}, \qquad \frac{\partial \delta_t}{\partial \delta_{t-1}} = \frac{\left(1+(1-\tau)r\right)(1-\tau)G^*}{\left((1-\tau)G^* - A\delta_{t-1}\right)^2} > 0,$$

$$\frac{\partial^2 \delta_t}{\partial {\delta_{t-1}}^2} = \frac{2\left(1+(1-\tau)r\right)(1-\tau)G^* A}{\left((1-\tau)G^* - A\delta_{t-1}\right)^3} > 0\,, \tag{15.B1}$$

with a corresponding equation for its *dual*, economic growth,

$$G_{t+1} = \left((1-\tau)G^* + 1 + (1-\tau)r - A(\gamma-\tau)\right) - \frac{\left(1+(1-\tau)r\right)(1-\tau)G^*}{G_t}$$ [7]

$$\frac{\partial G_{t+1}}{\partial G_t} = \frac{\left(1+(1-\tau)r\right)(1-\tau)G^*}{{G_t}^2} > 0\,,\ \frac{\partial^2 G_{t+1}}{\partial {G_t}^2} = \frac{2\left(1+(1-\tau)r\right)(1-\tau)G^*}{{G_t}^3} < 0 \tag{16.B1}$$

The prospect of stability may be more tenuous in a nonlinear model, since high levels of public debts will in themselves be destabilizing by reducing the rate of economic growth.[8] We can now present the first of the two main *propositions*.

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[5] Averaging across the five economies, the USA, the UK, France and Germany and Japan, for the period 1900-1989, these authors find average bond returns, growth rates and equity returns are, respectively, are 0.06%, 2.45% and 5.97% per annum, while bond returns were even negative for three of the five countries.

[6] Reinhart and Sbrancia (2011) consider the relevance of financial repression for debt stabilization.

[7] A backward-looking debt is clearly a source of a monotonic adjustment for economic growth and, thus, of a second-order dynamic process for the level of activity.

Proposition One:* *If the public debt dynamics are backward looking with two overlapping generations, (a) there is a non-monotonic relationship between economic growth and public debt, (b) there are various possibilities concerning the existence of a steady state.

Part (a) Substituting (15b) into (9) gives the implicit function,

$$F \equiv G-(1-\tau)G^{*}+\frac{AG}{G-(1+(1-\tau)r)}(\gamma-\tau)=0, \text{ where}$$

$$\frac{\partial(\gamma-\tau)}{\partial G}=-\frac{1-\left(A/(G-(1+(1-\tau)r))^{2}\right)(1+(1-\tau)r)(\gamma-\tau)}{AG/(G-(1+(1-\tau)r))}\ . \tag{17.B1}$$

The derivative sign reversal points to a non-monotonic relationship between growth and the primary deficit, $\gamma-\tau$, which implies a non-monotonic relationship between economic growth and the debt ratio, as deficits and debts are positively related,

$$\frac{\partial\delta}{\partial(\gamma-\tau)}=\frac{G}{G-(1+(1-\tau)r)}\left(1-\frac{A(\gamma-\tau)((1+(1-\tau)r)}{(G-(1+(1-\tau)r))^{2}}\right)^{-1}>0$$

Thus, there is the possibility of two steady state solutions and a quadratic solution both for growth,

$$G_1, G_2=\frac{(1-\tau)G^{*}+(1+(1-\tau)r)-A(\gamma-\tau)}{2}\pm\sqrt{(..)^{2}+(1+(1-\tau)r)(1-\tau)G^{*}} \tag{18.B1}$$

and for the debt-GDP ratio,

$$\delta_1, \delta_2=\frac{(1-\tau)G^{*}-(1+(1-\tau)r)+A(\gamma-\tau)}{2A}\pm\sqrt{(..)^{2}+\frac{(1-\tau)G^{*}(\gamma-\tau)}{A}} \tag{19.B1}$$

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[8] This same issue may arise, if growth is exogenous, since the crowding-out of capital by of public debt will instead raise the return on capital, which may then feed through to the return on public debt.

Part (b) There is a critical value for the primary public expenditure ratio, $\tilde{\gamma}$:

$$\tilde{\gamma}-\tau \equiv A^{-1}\left((1-(1-\tau)r)+(1-\tau)G^{*}-2\sqrt{(1+(1-\tau)r)(1-\tau)G^{*}}\right)>0,$$

which according to the backward-stability condition, $1+(1-\tau)r<(1-\tau)G^{*}$, is strictly positive. There are three possibilities.

(i) If $\gamma=\tilde{\gamma}$, there are unique solutions for economic growth and the debt ratio,

$$\tilde{G}=G_1=G_2=\sqrt{(1+(1-\tau)r)(1-\tau)G^{*}},$$

$$\tilde{\delta}=\delta_1=\delta_2=A^{-1}\left((1-\tau)G^{*}-\sqrt{(1+(1-\tau)r)(1-\tau)G^{*}}\right),$$[9]

which are borderline stable as $\partial G_t/\partial G_{t-1}\big|_{\tilde{G}}=1$ and $\partial\delta_t/\partial\delta_{t-1}\big|_{\tilde{\delta}}=1$.

(ii) If $\gamma<\tilde{\gamma}$, there are two steady states, $\delta_1<\tilde{\delta}<\delta_2$ and $G_1<\tilde{G}<G_2$. The monotonicity and convexity of the debt adjustment process implies that the lower valued debt steady state alone is locally stable: $\partial\delta_t/\partial\delta_{t-1}\big|_{\delta_t=\delta_{t-1}=\delta_1}<1$, while, correspondingly, the higher valued growth outcome is uniquely so, $\partial G_t/\partial G_{t-1}\big|_{G_t=G_{t-1}=G_1}>1$. A *correspondence principle* applies, since the locally stable equilibria have regular comparative static properties, $\partial\delta_1/\delta(\gamma-\tau)>0$ and $\partial G_1/\delta(\gamma-\tau)<0$, while the locally unstable alternatives give rise to perverse responses.

(iii) If $\gamma>\tilde{\gamma}$, no steady state exists, and the adjustment properties of the model point to exploding debt and to imploding growth.

2.4 The solution for the maximum backward-looking debt

The first case $\gamma=\tilde{\gamma}$ implies a maximum steady state for the debt ratio given by $\delta=A^{-1}\left((1-\tau)G^{*}-\sqrt{(1+(1-\tau)r)(1-\tau)G^{*}}\right)$ and a corresponding, minimum for the

[9] The bifurcation value of $\tilde{\gamma}$ can be solved from either the debt or the growth equation, since the latter is a linear function of the former through equation (9).

steady state growth factor as $G = \sqrt{\left((1-\tau)G^*\right)\left(1+(1-\tau)r\right)}$. This maximum will be at its highest where taxes are at their lowest, since this raises the values both of the primary deficit and of economic growth. If $\tau = 0$, the respective values are $\delta = A^{-1}\left(G^* - \sqrt{RG^*}\right)$ and $G = \sqrt{G^* R}$. We note that the bifurcation occurs where the economic growth factor is an unweighted, geometric average of its debt/tax-free level and the untaxed interest factor.[10]

2.5 A valuation of the maximum backward-looking debt

To quantify the results, we consider some possible parameter values. Throughout the analysis we assume that the debt/tax-free annual growth rate is 2.5%, while we set $\tau = 0$ in order to obtain the largest maximum value for this backward-looking case. For this two-period version of the model, the OLG periods are assumed to last 36 years, so that $G^* = 2.4325$. We then assign three possible values for the annualised interest rate of 0.25%, 1.25% and 2.25%, which give rise to separate bifurcation values of G. After setting $\lambda = 1$ and $\alpha = 1/3$, a consistent value for the productivity parameter A may be determined, which in turn allows us to calculate maximum values for the expenditure and debt ratios, δ and γ.

2.6 An annualised measure of the debt-GDP ratio

It is of some interest also to obtain more familiar *annualised* measures of the debt-GDP ratio rather than those that would be generated by a 36 year OLG period. We propose the following procedure for converting the OLG figures into annualised measures. First, we expand the identity for the deficit ratio, $\delta \equiv d/y$, to $\delta \equiv (d/Gk)G(k/y)$. Secondly, on the basis of equations (3) and (4), the term d/Gk, may be regarded as a *de facto* ratio of portfolio shares that should in principle not depend on the length of the periods. This *constant* is initially set at an arbitrary value ρ, $\rho = d/Gk$. Thirdly, the

[10] Note that the requirement $G > 1$ limits the extent of possible in financial repression to $R > 1/G^*$.

Cobb-Douglas technology gives rise to constant factor income shares, such that $R_K k = \alpha y$, where α is another but exogenous *constant*, also unrelated to the time-dimension of the model. Thus, the debt-ratio is always $\delta = \alpha\rho G / R_K$, whether in terms of an OLG period, $\delta^{OLG} = \alpha\rho G^{OLG} \big/ R_K^{OLG}$, or of annual data, $\delta^{pa} = \alpha\rho G^{pa} \big/ R_K^{pa}$, implies $\delta^{pa} = \left(G^{pa} / R_K^{pa}\right) \big/ \left(G^{OLG} / R_K^{OLG}\right) \delta^{OLG}$. Finally, using the compound equations for the interest and growth factors, $R^{OLG} = R^{pa^L}$ and $G^{OLG} = G^{pa^L}$, where L is the number of years in an OLG period, gives

$$\delta^{pa} = \left(G^{OLG} / R^{OLG}\right)^{1/L-1} \delta^{OLG} = \left(R^{pa} / G^{pa}\right)^{L-1} \delta^{OLG}. \tag{20}$$

The following *Table* of values may now be computed.

Table 1B: Values at the bifurcation of the maximum for backward-looking debt with two overlapping generations: $g^{pa*} = 2.5\%$

	$r^{pa} = 0.25\%$	$r^{pa} = 1.25\%$	$r^{pa} = 2.25\%$
g^{pa}	1.36%	1.87%	2.37%
γ	6.04%	2.36%	0.1206%
δ^{2OLG}	10.98%	6.61%	1.43%
δ^{pa}	26.06%	15.68%	3.40%

The main conclusion is that under the present specification the model is unable to deliver plausible values for the maximum debt-GDP ratio, whether in terms of OLG periods or single years, and even when the extreme assumption of zero taxes is made in favour of higher numbers. The best case is where interest rates are at their lowest, which generates a rather meagre annualized measure of 26% for the debt-GDP ratio. This figure would surely be regarded as insignificant in practice, but within the world of this model it defines a point of catastrophe.[11]

2.7 A forward-looking debt

A debt may be defined as *forward-looking* in the Ricardian sense as the present value of all future primary surpluses.[12] This possibility is straightforward for models of infinitely-lived households or dynasties, but is more hypothetical within the basic form of the OLG model. We suggest it may also have an application here, if the debt instrument is specified to compensate for the finiteness of households' planning horizons by mimicking two features of financial equity. The first is that it takes the form of perpetuity bonds, thus allowing an infinite horizon of coupons. The second is that these payments, instead of being fixed cash amounts, are deemed to comprise the entire primary surplus.[13] The analytics of this particular case, which is more speculative than practical, are relegated to *Appendix 1*, but the main result is discussed as follows.

The equation for a forward-looking debt is given by

$$\delta = \frac{G}{\left(1-(1-\tau)r\right)-G}(\tau-\gamma)\,. \tag{14.F}$$

This is equivalent to equation (14.B) except for sign reversals to both the numerator and denominator. The first reversal is because the debt now relates to future *surpluses* instead of past *deficits*; the second is because the forward-stability condition, which is the reverse of the one for backward-stability, is now required. Combining equation (14.F) instead of (14.D) with the growth equation in (9.1), however, fundamentally changes the

[11] With respect to emerging market *external* debt, Reinhart, Rogoff and Savatano ((2003) claim that a critical ratio could be as low as 15%. Their analysis, however, is based on countries having a reputation and probability of default, which is not considered here, rather than the *internal* crowding out-effect, which is.

[12] A forward-looking *debt* is to distinct forward-looking *debt instruments*, since bond prices may reflect expected *future* coupons, while bonds may be issued in response to *past* deficits.

[13] One cannot escape the notion that increasing the public debt here amounts to a policy of raising transfers in favour of bond-holders at the expense of other tax-payers and the beneficiaries of primary government expenditure. Ferguson (2002) argues that *de facto* something quite similar occurred in nineteenth century Britain, where a politically well-represented 1-2% of the population held a large national debt, which was serviced through regressive, indirect taxes paid by an unenfranchised majority.

structure of the model to give a *monotonically*, negative relationship between public debt – or the primary surplus – and economic growth. Thus, the maximum debt is exists at a point where the economic growth factor is at a feasible minimum, the point of *degeneracy* where $G = 1$. Despite a marked change in the qualitative nature of the model, the quantitative implications are of little consequence, since some computed values, shown in *Table 1F* of *Appendix 1*, remain at the same low order of magnitude.[14] This prompts a further and final generalization of the model.

3. A 3-OLG model

3.1 Modifications

The utility function in equation (5.1) is first extended to three periods,

$$U_t = \ln c_t^Y + \lambda \ln c_{t+1}^M + \lambda^2 \ln c_{t+1}^O, \tag{5.2}$$

depending on consumption when households are young (*Y*), middle-aged (*M*) and old (*O*) and under geometric discounting. The household is assumed to work only in the first two of life, receiving pre-tax wages of w_t^Y and w_{t+1}^M, while facing the post-tax budget constraint,

$$(1-\tau)w_t^Y + \frac{1}{R^N}(1-\tau)w_t^M = c_t^Y + \frac{1}{R^N}c_{t+1}^M + \frac{1}{{R^N}^2}c_{t+2}^O, \text{ where } R^N \equiv 1+(1-\tau)r .$$

3.2 The asset flow equations

The households' asset accumulation equations are generalized to

$$\begin{aligned} k_{t+1} - (1-\delta)k_t &= (A_{K,t}^Y - 0) + (A_{K,t}^M - A_{K,t-1}^Y) \\ b_t - b_{t-1} &= (A_{B,t}^Y - 0) + (A_{B,t}^M - A_{B,t-1}^Y) \end{aligned} \tag{21}$$

The left-hand-sides represent the *flow supplies*, respectively, of capital issued by firms – whether in the form of equities, loans, etc – and of public debt issued by the government

[14] Some difference also arises, as the new stability condition requires higher values for the interest factor, which have been chosen so as not to bias the results.

– as bonds and treasury bills. To clarify this point, we indicate the rate of depreciation as δ rather than assigning it the value of unity in order to avoid the illusion that stocks and flows may be synonymous also for capital. The right-hand-sides represent the *flow demands* from young and middle-aged households.

The *flow* demands of the young, however, are the same as their *stock* demands, $A^Y_{K,t}$ and $A^Y_{B,t}$, since they arrive on the scene without having either acquired or inherited asset stocks from a previous period. The asset flow demands of the middle-aged comprise their current stocks less those previously acquired when they were young, $A^M_{K,t} - A^Y_{K,t-1}$ and $A^M_{B,t} - A^Y_{B,t-1}$. Thus, an additional generation of savers is required to make a distinction between the stocks and flows of assets. Finally, the old, as before, just consume their entire asset holdings without leaving a bequest to their children or, as the model now permits, directly to their grandchildren.

Aggregating equations (21), where $A^Y_t = A^Y_{K,t} + A^Y_{B,t}$ and $A^M_t = A^M_{K,t} + A^M_{B,t}$, gives

$$k_{t+1} - (1-\delta)k_t = A^Y_t + A^M_t - A^Y_{t-1} - (d_t - d_{t-1}) \tag{22}$$

This makes explicit the fact that public debt *flows,* $d_t - d_{t-1}$, crowd-out investment flows, $k_{t+1} - (1-\delta)k_t$. Having made this point, we may return to the earlier assumption of 100% depreciation, $\delta = 1$, for the shorter but still extended time-periods of 24 year duration,

$$k_{t+1} = A^Y_t + A^M_t - A^Y_{t-1} - (d_t - d_{t-1})\,. \tag{23}$$

This equation may be compared with $k_{t+1} = A^Y_t - d_t$ [from equation (3)], where $A^Y_t = s_t$ and where d_t is also both a stock and flow.

Certain parameter restrictions are needed for an equilibrium where both working generations accumulate assets. For example, the young must not borrow against their future – and, with economic growth, higher – wage income. Financial frictions might lead not only to an interest spread, $R^S < R^L$, but to the possibility that young households with intermediate degrees of time-preference may choose neither to save nor to borrow. In the event that the young do not save, the present generalization of the model would be redundant by reverting to its earlier form with only one generation saving. This problem could potentially be overcome by successively increasing the number of generations and periods until the latter are of a sufficiently short duration to facilitate saving over two of them. For present analytical purposes, it is more tractable merely to assume parameter values that are consistent with saving over two periods. Thus, the asset stocks must be strictly positive, $A^Y_{B,t} > 0$, $A^M_{B,t} > 0$, as are net flows of capital in order to prevent firms from borrowing from the government.

The details are given in an *Appendix 2* while the solution is as follows. First, the condition for the young to save is a sufficiently high value of time-preference (patience),

$$\lambda + \lambda^2 > GR_N{}^{-1} \tag{24}$$

The solution for economic growth is

$$G_{t+1} = B_0 + B_1 G_t{}^{-1} - B_2\left(\delta_t - G_t{}^{-1}\delta_{t-1}\right), \quad \text{where}$$

$$B_0 \equiv \frac{(1-\tau)(1-\alpha)A\left(\lambda + 2\lambda^2 + R_N{}^{-1}\right)}{1 + \lambda + \lambda^2 + (1-\tau)(1-\alpha)AR_N{}^{-1}} > 0$$

$$B_1 \equiv \frac{(1-\tau)(1-\alpha)A\left(\lambda^2(R_N - 1) - \lambda\right)}{1 + \lambda + \lambda^2 + (1-\tau)(1-\alpha)AR_N{}^{-1}} > (<)0 \qquad \text{if} \quad \lambda(R_N - 1) > (<)0$$

$$B_2 \equiv \frac{(1 + \lambda + \lambda^2)A}{1 + \lambda + \lambda^2 + (1-\tau)(1-\alpha)AR_N{}^{-1}} > 0 \tag{9.2}$$

3.3 Backward-looking public debt in the 3-OLG model

The necessary stability condition for a backward-looking alongside the condition that the young are also savers in equation (24) are together satisfied by an intermediate value of the growth factor, $1+(1-\tau)r < G < (\lambda+\lambda^2)(1+(1-\tau)r)$. Thus, a judicious choice of parameter values is required to incorporate both these features.

3.4 Steady state

The final main *Proposition* is now given.

Proposition Two:* *If the public debt dynamics are backward-looking with two generations saving, (a) if* $(1-\tau)r > 0$*, the maximum public debt is at a bifurcation, (b) if* $(1-\tau)r \le 0$*, it is at a degeneracy and unbounded.

Substituting (14.B) into (9.2) gives the implicit function,

$$F = G - B_0 - B_1 G^{-1} + B_2 \frac{G-1}{G-\left(1+(1-\tau)r\right)}(\gamma-\tau), \qquad \text{where}$$

$$\frac{\partial(\gamma-\tau)}{\partial G} = -\left(\frac{1+B_1G^2 - B_2(\gamma-\tau)\big/\left(G-\left(1+(1-\tau)r\right)\right)^2(1-\tau)r}{B_2\,(G-1)\big/\left(G-\left(1+(1-\tau)r\right)\right)}\right) \tag{17.B2}$$

Critically, the numerator changes sign *only* if $(1-\tau)r > 0$, noting also that

$$\frac{\partial\delta}{\partial(\gamma-\tau)} = \frac{\left((1-B_1G^{-2})(G-R)^2\right)G + B_2(G-1)\gamma}{\left(1-B_1G^{-2}\right)(G-R)^2 - B_2(R-1)\gamma} > 0$$

Technically, *part (a)* merely replicates *Proposition One*, but there are significant quantitative differences, as shown in *Table 2B*. *Part (b)*, however, states that non-positive real interest rates lead not only to a degenerate but to an unbounded steady state maximum for the public debt. We first report the quantitative results, before giving the intuition.

3.5 Numerical values

We maintain the same values for the annual rates of the debt/tax-free growth rate and the return on saving as in *Section 2*.

Table 2B: Values at the bifurcation of the maximum for a backward-looking debt

in the 3-OLG model where two generations save $g^{pa*} = 2.5\%$			
	$r^{pa} = 0.25\%$	$r^{pa} = 1.25\%$	$r^{pa} = 2.25\%$
g^{pa}	0.82%	1.78%	2.37%
γ	27.21%	5.35%	0.17%
δ^{3OLG}	212.15%	45.12%	5.97%
δ^{pa}	374.36%	79.62%	10.53%

There is extreme variation the size of the maximum in response to changes in the exogenous interest rate. If the difference between the interest rate and the growth rate is small, there is virtually no change from the previous specification with two overlapping generations. Within the steady state, reductions in the real interest rate lead to exponential increases in the debt-GDP ratio. Another interpretation of these results is that if, for a given interest rates, current deficits may point to an unsustainable debt, say, if $r^{pa} = 1.25\%$ and $\gamma > 5.35\%$. If there is an unwillingness or inability to cut expenditure or to raise taxes, the only recourse would be to reduce interest rates, say, to where $r^{pa} = 0.25\%$, provided $\gamma < 27.21\%$. This possibility of financial repression is discussed by Reinhart and Sbrancia (2011).

3.6 Intuition

The maximum *stock* of public debt in the earlier version of the model was constrained to be very low, because of the default feature of its equality with the debt *flow*, which is the source of crowding-out. Alternatively, with an additional generation saving, the model may combine low flows of public debt, with the consequence of little crowding-out, in conjunction with large stocks of public debt. This is configuration arises where economic growth and thus asset flows are low, implying that low growth may be a cause as well as a consequence of a high public debt.[15] Thus, moving from two to three overlapping

[15] This is quite distinct from the same effect within demand-determined models, where public debt may increase because of low growth with counter-cyclical deficits.

generations may cause a *quantum change* by allowing an extra variable or a "degree of freedom" to enter the model.

The strongest result is of an unbounded maximum value for the steady state public debt, if its real rate of return is non-positive. This is due to the fact that economic growth then becomes a monotonic function of public debt with a degenerate maximum at $G = 1$, which is analogous to the effect of switching from the backward- to forward-looking debt in the previous *Section*. As the economy is then static, there will be growth neither in the stock of public debt, implying zero flows and thus precluding asset crowding-out. The only constraint on capital accumulation would be debt-servicing taxes, but these may be even set at zero, if public debt constitutes the accumulation of past deficits. Thus, in the steady state, an unbounded public debt does not require that households live for ever, but only that their finite lives are divided into at least three periods in order to facilitate saving in at least two of them.

Although we find that a low rate of economic growth supports a large steady-state public debt, we stress that this analysis considers only steady-states and overlooks possible problem of transition. Since the economic adjustment process in the presence of public debt is cyclical, as evident from equation (23), there will be periods where the economy shrinks, unless the stock is constrained to increase at a negligible rate.

3.7 Forward-looking debt again

We return to the hypothetical case of a forward-looking debt with the details of the solution relegated to *Appendix 3*. Again we find a degenerate maximum, as in the 2-OLG case, but with much larger values, since the same principle of stock-flow differentiation works again. However, the computed values are lower than in backward-looking case, because public the debt is now the present value of future primary surpluses. A larger steady state stock of debt thus requires commensurately higher taxes, but these lead to lower disposable income, acting as a brake on saving and capital accumulation. Furthermore, an unbounded steady state public debt is here precluded, because the forward-stability condition requires the real interest rate must be strictly positive in order to exceed the economic growth rate at its degeneracy value of zero.

4. Summary

The question *how large can the public debt get?* is both of theoretical interest and of practical concern. While the Diamond model is versatile in its ability to address a wide range of issues both in public finance and in finite-horizon macroeconomics, a version with three overlapping generations equips it for the task of also analysing economies with very large public debts. This is through allowing an important distinction to be made between asset stocks and flows, effectively giving the model an extra variable or "degree of freedom". The benefit does not depend on the nature of the production technology – whether economic growth is, as here, endogenous or exogenous, but only on the household accumulation equations. The Diamond model can then accommodate extreme cases of the phenomenon it was originally formulated to address.

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Appendix 1: Forward-looking public debt in the 2-OLG model

A1.1 Solution with forward-looking debt dynamics

The analysis so far has in essence been a reworking of Braeuninger (2005). We now how it changes qualitatively, when public debt is forward-looking according to,

$$d_t = -(E_t - T_t) + \left(1 + (1-\tau)r\right)^{-1} d_{t+1} \tag{10.F}$$

The forward solution for debt is solved as the expected present value of future surpluses,

$$d_t = -\sum_{i=0}^{\infty} (1 + (1-\tau)r)^{-i} (E_{t+i} - T_{t+i}) \tag{11.F}$$

Re-using the definitions in (8), gives a debt-GDP ratio of

$$\delta_t = -\frac{G_{t+1}}{1+(1-\tau)r}\left((\gamma_{t+1} - \tau_{t+1}) + \frac{G_{t+2}}{1+(1-\tau)r}(\gamma_{t+2} - \tau_{t+2}) + \frac{G_{t+2}G_{t+3}}{\left(1+(1-\tau)r\right)^2}(\gamma_{t+3} - \tau_{t+3}) + .. \right)$$

The size of the debt now depends on expectations of the trajectories of fiscal policy along with the anticipated effects of economic growth. A solution exists if the infinite sum of present value *primary surpluses* is bounded, which requires that primary surpluses are ultimately discounted by a greater factor than the one by which they grow. Thus there is a reversal of the previous stability condition, since the condition $G < 1 + (1-\tau)r$ or $g < (1-\tau)r$ is now required for adding up over an infinite future. *Financial efficiency* is most favourable to this case, defined as the situation where households receive the highest possible return on the saving. In a steady state, the debt ratio given by

$$\delta = \frac{G}{1+(1-\tau)r - G}(\tau - \gamma), \tag{14.F}$$

which is noteworthy in being the same as equation (14.B) for a backward-looking debt, but with sign reversals for both the numerator and denominator.[16]

[16] The dynamics are notably different because the size of the debt is determined instantaneously by beliefs of the future well before the capital stock has been given time to adjust. The model outside the steady state will thus demonstrate saddle-path properties with jumps in debt followed by responses in growth that are distributed over time.

Lemma Two: There is a unique equilibrium for growth and for public debt, if the debt dynamics are forward-looking.

Substituting (14.F1) into (9) gives the implicit function,

$$F \equiv G-(1-\tau)G^{*}+\frac{AG}{1+(1-\tau)r-G}(\tau-\gamma)=0,$$

$$\frac{\partial(\tau-\gamma)}{\partial G}=-\frac{1+\left(A/(1+(1-\tau)r-G)^{2}\right)(1+(1-\tau)r)(\tau-\gamma)}{AG/(1+(1-\tau)r-G)}<0. \qquad (17.F1)$$

The derivative indicates that growth is decreasing in the primary surplus, as $1+(1-\tau)r>G$ and $\tau>\gamma$. Growth is also negatively related to the debt ratio, as

$$\frac{\partial\delta}{\partial(\tau-\gamma)}=\frac{G}{(1+(1-\tau)r)-G}\left(1+\frac{A(1+(1-\tau)\alpha R)(\tau-\gamma)}{(1+(1-\tau)r-G)^{2}}\right)^{-1}>0$$

Proposition Three: In the 2-OLG model with forward-looking public debt, there is a degenerate steady state maximum ($G=1$) at $\delta^{\max}=(\tau^{\max}-\gamma)/(1-\tau^{\max})r$, where

$$\tau^{\max}=1+\frac{1}{2G^{*}}\left(\frac{A}{r}-1\right)-\sqrt{\frac{1}{4G^{*2}}\left(\frac{A}{r}-1\right)^{2}+\frac{1}{G^{*}}\frac{A}{r}(1-\gamma)}$$

The monotonically negative relationship between the debt ratio and economic growth implies that in the steady state the former is highest where the latter is at its steady state minimum of $G=1$. This implies $\delta^{\max}=(\tau^{\max}-\gamma)/(1-\tau^{\max})r$ according to equation (14.F) and $1=(1-\tau)G^{*}-A\delta$, according to equation (9.1). These are solved simultaneously to give the maximum tax rate as above.

The key result, so far, is that while there is a bifurcation maximum for the backward-looking debt, there is a degenerate maximum for the forward-looking one. This is because the two distinct dynamic cases have different stability conditions, which imply opposing signs for the partial response of public debt to economic growth, $\partial\delta/\partial G$. The crowding-out effect of debt on growth, $\partial G/\partial\delta<0$, in being common to both cases, thus causes a non-monotonic relationship as the source of a bifurcation in the first case, but a monotonically decreasing response, determining the degeneracy, in the second case.

A1.2 A valuation of the steady state maximum when debt is forward-looking

Equations (2) and (9.1) give $R_K = (\alpha/(1-\alpha))((1+\lambda)/\lambda)G^*$. If we fix the debt/tax-free growth factor at is previous level, we may consider various possibilities of the interest rate in this financial efficiency by varying the value of α. We now consider higher annualised returns on capital, r_K^{pa} of 3%, 4% and 5%, and find that setting $\lambda = 0.8$ allows a more set of plausible vales for capital income share of 0.34621, 0.42851 and 0.51414. The debt is increasing in the primary surpluses, so will be highest where public expenditure is minimized at $\gamma = 0$. The results are presented as follows.

Table 1F: Values at the degeneracy of the maximum for forward-looking debt with two overlapping generations: $g^{pa*} = 2.5\%$

	$r^{pa} = 3\%$	$r^{pa} = 4\%$	$r^{pa} = 5\%$
g	0	0	0
τ	18.30%	22.36%	25.61%
δ^{2OLG}	11.80%	9.28%	7.19%
δ^{pa}	33.19%	36.61%	39.09%

We cannot sensibly make a quantitative comparison of the two cases, since they are predicated on the different parameter values underlying the separate stability conditions. In qualitative terms, if the dynamics are forward-looking case, the debt is potentially larger, because the responsive fall in growth is potentially further to a corner point of degeneracy rather than to an interior point of bifurcation, but this effect is countered by the fact that the forward-looking maximum is supported by high and contractionary taxes. However, an unambiguous conclusion is that these two qualitatively separate cases each deliver debt maxima that are far below what might be imagined empirically.

Appendix 2: Solution for economic growth in the 3-OLG model

Defining $R_N \equiv 1+(1-\tau)r$, the planned (and actual) consumption demands are

$$c_t^Y = \frac{(1-\tau)\left(w_t + R_N{}^{-1} w_{t+1}\right)}{1+\lambda+\lambda^2}, \qquad c_{t+1}^M = \lambda R_N c_t^Y, \qquad c_{t+2}^O = (\lambda R_N)^2 c_t^Y \tag{A1}$$

This implies that the young household's asset holding is

$$A_t^Y = (1-\tau)w_t^Y - c_t^Y = \frac{(1-\tau)\left((\lambda+\lambda^2)w_t^Y - R_N{}^{-1} w_{t+1}^M\right)}{1+\lambda+\lambda^2} \tag{A2}$$

The condition that the young save, $A_t^Y > 0$, is $\lambda + \lambda^2 > GR_N{}^{-1}$. Clearly, the forward-looking dynamic case where $GR_N{}^{-1} < 1$ is more conducive to this possibility, while if $GR_N{}^{-1} > 1$ in the backward-looking case, the condition $\lambda+\lambda^2 > G$ will never hold unless $-1/2 + \sqrt{1/4 + GR_N{}^{-1}} < \lambda \le 1$, for which it is necessary that $G < 2R_N$.

The asset position of the middle-aged is

$$A_{t+1}^M = \frac{\lambda^2(1-\tau)\left(R_N w_t^Y + w_{t+1}^M\right)}{1+\lambda+\lambda^2} \tag{A3}$$

which is strictly positive as there is no earned income when old. Subtracting (A2) from (A3) gives the asset flow demand of the *future* middle-aged as

$$A_{t+1}^M - A_t^Y = \frac{(1-\tau)\left(\left(\lambda^2(R_N-1)-\lambda\right)w_t^Y + \left(\lambda^2 + R_N{}^{-1}\right)w_{t+1}^M\right)}{1+\lambda+\lambda^2}$$

or for the *current* middle-aged as

$$A_t^M - A_{t-1}^Y = \frac{(1-\tau)\left(\left(\lambda^2(R_N-1)-\lambda\right)w_{t-1}^Y + \left(\lambda^2 + R_N{}^{-1}\right)w_t^M\right)}{1+\lambda+\lambda^2} \tag{A4}$$

Where there are no seniority or productivity effects, $w_t^Y = w_t^M = w_t$, adding (A2) and (A4) gives

$$A_t - A_{t-1} = \frac{(1-\tau)\Big(\big(\lambda^2(R_N-1)-\lambda\big)w_{t-1} + \big(\lambda + 2\lambda^2 + R_N{}^{-1}\big)w_t - R_N{}^{-1}w_{t+1}\Big)}{1+\lambda+\lambda^2} \qquad \text{(A5)}$$

Equation (20) then implies

$$k_{t+1} = \frac{(1-\tau)\Big(\big(\lambda^2(R_N-1)-\lambda\big)w_{t-1} + \big(\lambda + 2\lambda^2 + R_N{}^{-1}\big)w_t - R_N{}^{-1}w_{t+1}\Big)}{1+\lambda+\lambda^2} - (d_t - d_{t-1})$$

which with (2), after rearranging gives

$$k_{t+1} = \frac{(1-\tau)(1-\alpha)A\Big(\big(\lambda^2(R_N-1)-\lambda\big)k_{t-1} + \big(\lambda + 2\lambda^2 + R_N{}^{-1}\big)k_t\Big) - (1+\lambda+\lambda^2)\big(d_t - d_{t-1}\big)}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N{}^{-1}}$$

and, using the definitions in (8)

$$G_{t+1} = \frac{(1-\tau)(1-\alpha)A\Big(\big(\lambda^2(R_N-1)-\lambda\big)G_t{}^{-1} + \big(\lambda + 2\lambda^2 + R_N{}^{-1}\big)\Big) - (1+\lambda+\lambda^2)A\big(\delta_t - G_t{}^{-1}\delta_{t-1}\big)}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N{}^{-1}}$$

$$G_{t+1} = B_0 + B_1 G_t{}^{-1} - B_2\big(\delta_t - G_{t-1}\delta_{t-1}\big),$$

$$B_0 = \frac{(1-\tau)(1-\alpha)A\big(\lambda + 2\lambda^2 + R_N{}^{-1}\big)}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N{}^{-1}} > 0$$

$$B_1 = \frac{(1-\tau)(1-\alpha)A\big(\lambda^2(R_N-1)-\lambda\big)}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N{}^{-1}} > (<)0 \qquad \text{if} \quad \lambda(R_N-1) > (<)0$$

$$B_2 = \frac{(1+\lambda+\lambda^2)A}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N{}^{-1}} > 0$$

Appendix 3: Forward-looking public debt in the 3-OLG model

A3.1 The steady state

Lemma Four: There is a unique steady state for growth and for public debt, if the debt dynamics are forward-looking with two generations saving.

Substituting (14.F) into (9.2) gives the implicit function,

$$F = G - B_0 - B_1 G^{-1} + B_2 \frac{G-1}{\left(1+(1-\tau)r\right) - G}(\tau - \gamma),$$

$$\frac{\partial(\tau-\gamma)}{\partial G} = -\left(\frac{1 + B_1 G^2 + B_2 (\tau-\gamma) \big/ \left(\left(1+(1-\tau)r\right)-G\right)^2 (1-\tau)r}{B_2 (G-1) \big/ \left(\left(1+(1-\tau)r\right)-G\right)} \right) < 0 \qquad \text{(17F2)}$$

Proposition Four: In the 3-OLG model, if public debt is forward-looking, its maximum level is at a degeneracy ($G = 1$) where $\delta^{\max} = (\tau-\gamma)/(1-\tau)r$ and it is clearly highest for this previous case where $\gamma = 0$ and $\tau = \tau^{\max}$, and where $\tau = \tau^{\max}$ satisfies $1 - B_0 - B_1 = 0$ from equation (9.2). This replicates the previous case with a degeneracy at a zero rate of economic growth, except the asset crowding-out effect is now precluded because of zero stock changes, leading to potentially higher values.

A3.2 Numerical values

We choose the same values for the interest rate as for the three OLG version of the model in *Section 4*, except $\lambda = 0.6$ is chosen to obtain reasonable values for α, which is solved as residual.

Table 2F: Values at the degeneracy for the maximum forward-looking public debt ratio in a 3-OLG model where two generations save

	$r^{pa} = 3\%$	$r^{pa} = 4\%$	$r^{pa} = 5\%$
g^{pa}	0%	0%	0%
τ	30.71%	32.98%	32.88%
δ^{3OLG}	42.92%	31.47%	22.02%
δ^{pa}	84.71%	77.76%	67.63%

We see that the values, in comparison with those of *Table Two*, are of a higher and more plausible order of magnitude. The similarity of relative modest responses to changes in the interest rate value, however, remains. The associated *average* tax rates appear to be quite low, but these could be increased significantly by raising the assigned value for the debt/tax-free growth rate. If they were to become very high, however, the question then arises in this case whether the maximum debt would be determined by the technical features of this model, as it now stands, or by further ones underlying a Laffer curve.