Why Politicians Don't Care - Results from Intertemporal Optimization and Viscosity Solutions

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Abstract

Abrupt policy changes are often associated with multiple equilibria. However, we show that changing policy instantly can be an optimal reaction - even in a continuous environment. We employ a stylized model of intertemporal optimization with endogenous exit where the optimization horizon exceeds a possible regime change, and show that for this class of problems viscosity solutions are typical, and indicate instant changes in policy.

The most astonishing result is that the optimal policy is to ignore the danger of a crisis until it is immediate and to then sharply change the policy by fending off the crisis.

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1 Introduction

Previous literature modelling financial crises and speculative attacks highlighted particularly the aspects of speculators attacking a currency, but not incorporated the main role of the central bank adequately. In fact, setting the interest rate influences the fundamentals and obviously the costs of speculators. Thus the behavior of the central bank is not just a passive reaction due to speculative pressure and is also more than sole signalling, it changes the state of the system.

If the central bank chooses to defend the regime by raising the interest rate, it accepts that fundamentals degrade and furthermore accepts that the degrading fundamentals aggravate the future attack and thus worsens its future position. Hence the behavior of the central bank now is crucial for both the time path of the economy and for the own future position. On the other side, the speculators know, that attacking weakens the position of the central bank but also have to consider the costs if the central bank decides to defend as a reaction on the attack.

The trade-off for the central bank is that one control influences the possibilities to benefit from the regime and the duration of the regime as well as the probability to bear the costs of a regime change which occurs if the attack strength exceeds the reserves of the central bank.

We apply an infinite horizon intertemporal optimization framework where the time horizon exceeds the duration of the regime. The time when the central bank chooses to abandon the peg is determined endogenously. First, we describe the general framework, where we introduce the objective function and the two state processes, fundamentals and attack. Second, we offer a solution for a simple case of the model where states are just linearly dependent on the interest rate. Third, we describe an extended linear model with feedback, mean reversion and herding effects.

2 Model

There are two actors the central bank and speculators. The central bank maximizes utility:

$$U_0(\theta, A) = \int_0^T e^{-\rho t} u(\theta(t))\, dt + e^{-\rho T} \upsilon(\theta(T) - c) \tag{1}$$

where instantaneous utility u is derived from the state of the fundamentals $\theta(t)$ and discounted by factor ρ. The overall utility is the sum of the discounted instantaneous utility up to the terminal time T, which denotes the time when the central bank is forced to devalue, and the discounted terminal value of the regime.[1] The terminal time is endogenously defined by the state process. The terminal value υ is a function of the fundamentals at the terminal time less an amount c representing the costs of the regime change. For the remainder of the paper, we assume that the value υ of the system after the regime change is the value of a control and pressure free system converging to the steady state,

[1] For the given setup we find that $\lim_{T\to\infty} e^{-\rho T} \upsilon(\theta(T) - c) = 0$, i.e. if the regime persists forever, the second term vanishes.

i.e. as if fundamentals would start to develop at $\theta(T) - c$ and the central bank would keep the interest rate on its steady state value $\bar{r}$.

The central bank tries to maximize the objective equation (1) through its control variable the interest rate $r(t)$ which is always non-negative $r(t) \geq 0$. The optimization problem is subject to the state of the system which is summarized by the state vector x:

$$\dot{x} = \begin{pmatrix} \dot{\theta}(t) \\ \dot{A}(t) \end{pmatrix} = \begin{pmatrix} f(r(t), \theta(t)) \\ g(r(t), \theta(t), A(t)) \end{pmatrix} \tag{2}$$

There are two state variables, the fundamentals $\theta(t)$ and the strength of the attack $A(t)$. The first state variable $\theta(t)$ enters the utility directly while the second $A(t)$ determines the terminal time $T = \inf\{t : A(t) > D\}$, i.e. the first time when the strength of the attack exceeds a threshold D.[2] The central bank's control thus yields two effects: firstly, it influences the fundamentals and thus its potential to gain utility and secondly, it influences the terminal time at which it is forced to switch the regime.

Let $V(\theta, A)$ be the value function of this optimization problem, i.e. the total utility of the central bank given it chooses an optimal control r^*

$$\begin{aligned} V(\theta, A) &= \sup_{r:[0;\infty)\to[0;\infty)} \{U_0(\theta, A)\} \\ &= U_0(\theta, A) \text{ with } \begin{pmatrix} \dot{\theta}(t) \\ \dot{A}(t) \end{pmatrix} = \begin{pmatrix} f(r^*(t), \theta(t)) \\ g(r^*(t), \theta(t), A(t)) \end{pmatrix} \\ &\qquad \text{and } \begin{pmatrix} \theta(0) \\ A(0) \end{pmatrix} = \begin{pmatrix} \theta_S \\ A_S \end{pmatrix}. \end{aligned}$$

Intertemporal optimization theorems then imply that V yields the following Bellman equation (Waelde 2008):

$$\rho V(\theta, A) = \sup_r \left\{ u(\theta) + \frac{dV(\theta, A)}{dt} \right\} \tag{3}$$

Since V is not continuously differentiable at any feasible point, a more general interpretation of this partial differential equation is necessary. As we will show below, the concept of viscosity solutions applies.

The motion of the fundamentals is influenced by the interest rate and by its own level. In fact the motion of fundamentals is often represented by a mean-reverting process (cf. Cox et al. (1985)), where a higher level of fundamentals than the natural level $\bar{\theta}$ leads to a move back to its natural level. The same holds for the interest rate, where a rate lower than the natural level $\bar{r}$ leads to an increase in the fundamentals. If the central bank chooses an interest rate that is higher than its natural level, fundamentals worsen.

$$\dot{\theta} = f(r(t), \theta(t)) = -f_1(r(t)) - f_2(\theta(t)) \tag{4}$$

$\frac{\partial f_1(.)}{\partial r(t)} > 0$ is the interest rate elasticity of the fundamentals and $\frac{\partial f_2(.)}{\partial \theta(t)} > 0$ is the elasticity of the mean reversion, showing how strong the fundamentals

[2] Naturally, we restrict the initial state vector to be feasible, i.e. $A(0) \leq D$.

are forced back to their natural level. This means that there is a stabilizing mechanism which turns bad fundamentals (below the natural level) to the better and degrades good fundamentals (higher than the natural level). Obviously, such a fundamentals process possesses a steady state $(\theta, r) = \left(\bar{\theta}, \bar{r}\right)$ if $f_1(\bar{r}) = f_2\left(\bar{\theta}\right) = 0$.

The change in the attack strength depends on the costs $r(t)$, the fundamentals $\theta(t)$ and on strategic complementarities, i.e. a herding effect $A(t)$. We treat the attack strength as a reduced form equation of the global games mechanism (Morris and Shin 1998).

The speculators take position against the currency if fundamentals are worse than their natural level. They refrain from attacking when the interest rate is higher than its natural level and greater strength in the attack induces more speculators to participate in the attack, since the probability to succeed rises.

$$\dot{A} = g(r(t), \theta(t), A(t)) = -g_1(r(t)) - g_2(\theta(t)) + g_3(A(t)) \tag{5}$$

Where $\frac{\partial g_1(.)}{\partial r(t)} > 0$ is the interest rate elasticity of the attack, $\frac{\partial g_2(.)}{\partial \theta(t)} > 0$ is the fundamentals elasticity of the attack and $\frac{\partial g_3(.)}{\partial A(t)} > 0$ is the elasticity of herding.

If we assume as above that $g_1(\bar{r}) = g_2\left(\bar{\theta}\right) = g_3(0) = 0$ then the system (θ, r, A) possesses a steady state at $\left(\bar{\theta}, \bar{r}, 0\right)$, i.e. the fundamentals are in the steady state and there is no speculative pressure.

3 Linear Version

As a first illustration, we set the fundamentals elasticities $\frac{\partial f_2(.)}{\partial \theta(t)}, \frac{\partial g_2(.)}{\partial \theta(t)}$ and the elasticity of herding $\frac{\partial g_3(.)}{\partial A(t)}$ equal to zero. The interest rate elasticities should be linear coefficients. Thus we set $\frac{\partial f_1(.)}{\partial r(t)} = \alpha$ and $\frac{\partial g_1(.)}{\partial r(t)} = \gamma$. With this modification we get as state vector:

$$\begin{pmatrix} \dot{\theta} \\ \dot{A} \end{pmatrix} = \begin{pmatrix} -\alpha(r(t) - \bar{r}) \\ -\gamma(r(t) - \bar{r}) \end{pmatrix}$$

With $\alpha, \gamma > 0$; and α being the interest rate elasticity of the fundamentals and γ being the interest rate elasticity of the attack. In this simple model, the control of the central bank directly implies a perfect correlation of fundamentals and attack, i.e. a low (high) interest rate increases (decreases) fundamentals as well as strengthens (weakens) speculative pressure.

As a first step, we "guess" the optimal control r^*, then show that the corresponding value function satisfies the Bellman equation, and finally take a closer look at the Bellman equation at the border.

The optimal control r^* depends on the state, and two cases have to be analyzed separately: the interior $A < D$, where the attack is yet emergent, and the border case $A = D$, where any further pressure would lead to a breakdown of the regime.

1. The interior case $A < D$

   The Bellman equation reads (cf. Waelde (2008), ch. 6; Fleming and Soner (2006), ch. 1.7):

$$\begin{aligned}\rho V(\theta, A) &= \sup_r \left\{ u(\theta) + \frac{dV(\theta, A)}{dt} \right\} \\ &= \sup_r \left\{ u(\theta) + DV \cdot \begin{pmatrix} \dot{\theta} \\ \dot{A} \end{pmatrix} \right\} \\ &= \sup_r \left\{ u(\theta) - (V_\theta \alpha + V_A \gamma)(r - \bar{r}) \right\}.\end{aligned} \tag{6}$$

   with ρ as the discount factor. Thus the argument in the supremum is linear in r and the optimization problem (6) has a border solution $r = 0$, if and only if:

$$V_\theta \alpha + V_A \gamma > 0 \tag{7}$$

   As we show in appendix 6.1.1, this condition holds true.

2. The border case $A = D$

   The value of leaving the regime $\upsilon(\theta - c)$ is strictly lower than staying within the system $V(\theta, A = D)$ for all possible values of θ (see appendix 6.1.2). Any further increase in A thus would lead to an infinitely negative slope of V and therefore must be avoided. The optimization problem thus is to maximize θ subject to $\frac{dA}{dt} \leq 0$. Since $\frac{dA}{dr} > 0$ and $\frac{d\theta}{dr} > 0$, i.e. any control increasing θ also increases A, the optimal solution is to not let A decrease and therefore

$$r^* = \bar{r}; \ \frac{dA}{dt} = 0; \ \frac{d\theta}{dt} = 0 \tag{8}$$

   Summarizing, the optimal control is

$$r^*(\theta, A) = \begin{cases} 0 & \text{if } A < D \\ \bar{r} & \text{else} \end{cases}.$$

We derive the following intuition, starting at an arbitrary point, where the strength of the attack is less than the reserves $A < D$. To improve utility (1) the central bank maximizes the fundamentals. Therefore the central bank sets the interest rate to zero,[3] which implies that the fundamentals increase depending on their initial value θ_S, the interest rate elasticity α, the natural interest rate $\bar{r}$ and obviously the elapsed time t. Thus we get as time path of the fundamentals:

$$\theta(t) = \theta_S + \int_0^t \alpha \bar{r} d\tau = \theta_S + \alpha \bar{r} t \tag{9}$$

Setting the interest rate to zero also implies that attacking is very cheap, so as the fundamentals improve also the attack strength grows. Depending also on the initial value of the attack A_S, the respective interest rate elasticity γ, the

[3] As noted earlier we, require the interest rate to be non-negative. Obviously without this conditon the optimal interest rate would be minus infinity.

natural interest rate $\bar{r}$ and the elapsed time t. Which implies as time path for the attack:

$$A(t) = A_S + \int_0^t \gamma \bar{r} d\tau = A_S + \gamma \bar{r} t \tag{10}$$

The optimal policy of the central bank to set the interest rate to zero is accompanied by a growing strength of the attack, which means that to keep the exchange rate peg the central bank has to intervene in the currency market, i.e. to sell foreign currency, thus depleting reserves. Since a devaluation would mean a decrease in the argument of the utility function by c, the central bank starts to defend the peg additionally through raising the interest rate in the instant before the reserves are exhausted. The time when the central bank raises interest rates to stop speculation but does not yet devalue is thus denoted by $\bar{T}$ and is called defense time. $\bar{T}$ is reached when strength of the attack equals reserves $A\left(\bar{T}\right) = D$. Inserting (10) gives:

$$\bar{T} = \frac{D - A_S}{\gamma \bar{r}} \tag{11}$$

The defense time is reached earlier the lower reserves D, the higher the initial attack level A_S, the interest rate elasticity of the attack γ and the natural interest rate $\bar{r}$ are.

When the central bank applies a restrictive monetary policy, both the speculative pressure and the fundamentals stop growing. Therefore we get the following time paths given the optimal control r^*

$$A(t) = \begin{cases} A_S + \gamma \bar{r} t & \text{if } t < \bar{T} \\ D & \text{else} \end{cases}$$

$$\theta(t) = \begin{cases} \theta_S + \alpha \bar{r} t & \text{if } t < \bar{T} \\ \theta_S + \alpha \bar{r} \bar{T} & \text{else} \end{cases}$$

For simplicity we assume exponential utility $u(\theta) = -\exp(-\chi\theta)$ and calculate the value function:[4]

$$\begin{aligned} V &= U_0(r^*) = \\ &= -\int_0^{\bar{T}} \exp(-\rho t) \exp(-\chi(\theta_S + \alpha \bar{r} t))\, dt \\ &\quad - \int_{\bar{T}}^{\infty} \exp(-\rho t) \exp\left(-\chi\left(\theta_S + \alpha \bar{r} \bar{T}\right)\right) dt \\ &= -\frac{\exp(-\chi\theta)}{\rho + \chi \alpha \bar{r}} \left(\frac{\chi \alpha \bar{r}}{\rho} \exp\left(-(\rho + \chi \alpha \bar{r}) \bar{T}\right) + 1 \right) \end{aligned}$$

We can now show that this value function indeed solves the Bellman equation (6) of this problem.[5] Rearranging and deriving with respect to the state variables θ and A delivers the costate variables V_θ and V_A.

$$V_\theta = -\chi V, \ V_A = \frac{1}{\gamma \bar{r}} \left((\rho + \chi \alpha \bar{r}) V + \exp(-\chi\theta) \right)$$

[4] Detailed calculations are shown in the appendix 6.1.1.

[5] Inserting in the Bellman equation shows easily that the solution is feasible.

Inserting into (7) and using the value function gives

$$V_\theta \alpha + V_A \gamma = \frac{\chi \alpha \bar{r}}{\rho + \chi \alpha \bar{r}} \exp\left(-\chi \theta\right) \left(1 + \exp\left(-\bar{T}\left(\rho + \alpha \gamma \bar{r}\right)\right)\right) > 0$$

which is obviously positive. We conclude that for the interior case $A < D$, the Bellman equation

$$\rho V\left(\theta, A\right) = \sup_r \left\{u\left(\theta\right) - \left(V_\theta \alpha + V_A \gamma\right)\left(r - \bar{r}\right)\right\} \tag{12}$$

has an argument which is linear in $r\left(t\right)$ with negative slope and thus the solution to the optimization problem (6) is the minimal r, i.e. $r\left(t\right) = 0$.

To look at the border case $A = D$ we utilize the Hamiltonian notation of the problem as used in (Fleming and Soner 2006, cf. ch. 2, lemma 8.1) and define the subsolutions $D^- V$ and supersolutions $D^+ V$ accordingly. A value function belonging to both $D^- V$ and $D^+ V$ is called a viscosity solution. The following Corollary helps to keep the notation simple. For infinite horizon time-homogeneous optimization problems with discounted utility the value function takes the form $V\left(t, x\right) = \exp\left(-\rho t\right) V\left(x\right)$ where ρ is the discount factor and x the state variable (Fleming and Soner 2006, ch. I.7).

Corollary 1 *: For infinite horizon time-homogeneous optimization problems with discounted utility each feasible value function is continuously differentiable with respect to the time variable t. Thus $\frac{\partial}{\partial t} V\left(t, x\right)$ enters each element in $D^- V$ and $D^+ V$ and it is sufficient to define $D^- V$ and $D^+ V$ without the time differential.*

We now define the subsolutions $D^- V$ and supersolutions $D^+ V$.

$$D^+ V\left(\theta, A\right) = \left\{\left(p, q\right) \in \mathbb{R}^2 : \limsup_{\substack{(y,a) \to (\theta, A) \\ a \leq D}} \frac{V\left(y, a\right) - V\left(\theta, A\right) - p\left(y - \theta\right) - q\left(a - A\right)}{\left\|\left(y, a\right) - \left(\theta, A\right)\right\|} \leq 0\right\} \tag{13}$$

$$D^- V\left(\theta, A\right) = \left\{\left(p, q\right) \in \mathbb{R}^2 : \liminf_{\substack{(y,a) \to (\theta, A) \\ a \leq D}} \frac{V\left(y, a\right) - V\left(\theta, A\right) - p\left(y - \theta\right) - q\left(a - A\right)}{\left\|\left(y, a\right) - \left(\theta, A\right)\right\|} \geq 0\right\} \tag{14}$$

Since $V\left(\theta, A\right)$ is continuously differentiable in all feasible states, we have $D^+ V\left(\theta, A\right) = D^- V\left(\theta, A\right) = \left(V_\theta\left(\theta, A\right), V_A\left(\theta, A\right)\right)$ which solve the Bellman equation. In addition to this standard definition we also define the sub- and supersolutions from the outside of the feasible states, i.e. the region of states in which the regime ends. We will apply this to the Bellman equation to include controls which might end the regime.

$$D^{+}_{out}V(\theta, D) = \left\{ (p,q) \in \mathbb{R}^2 : \limsup_{\substack{(y,a)\to(\theta,D) \\ a>D}} \frac{V(y,a) - V(\theta, D) - p(y-\theta) - q(a-D)}{\|(y,a) - (\theta, A)\|} \leq 0 \right\} \tag{15}$$

$$D^{-}_{out}V(\theta, D) = \left\{ (p,q) \in \mathbb{R}^2 : \liminf_{\substack{(y,a)\to(\theta,D) \\ a>D}} \frac{V(y,a) - V(\theta, D) - p(y-\theta) - q(a-D)}{\|(y,a) - (\theta, A)\|} \geq 0 \right\} \tag{16}$$

Since the value after the regime change $\nu(\theta - c) = V(\theta, a)$ is strictly smaller than the value of remaining in the regime $V(\theta, D)$, we have $D^{+}_{out}V(\theta, D) = \left\{(p,q) \in \mathbb{R}^2 : \limsup \mathbb{R}^2\right\} = (\infty, \infty)$ and $D^{-}_{out}V(\theta, D) = \left\{(p,q) \in \mathbb{R}^2 : \liminf \emptyset\right\} = (-\infty, -\infty)$. We know that for all $(p,q) \in D^{+}_{out}V(\theta, D)$ we have $\rho V(\theta, A) \geq \sup_{r<\bar{r}} \{u(\theta) - (p\alpha + q\gamma)(r - \bar{r})\}$ and for all $(p,q) \in D^{-}_{out}V(\theta, D)$ we have $\rho V(\theta, A) \leq \sup_{r<\bar{r}} \{u(\theta) - (p\alpha + q\gamma)(r - \bar{r})\}$. The optimal control and the border, i.e. $A = D$, thus must satisfy the following viscosity formalization of the Bellman equation[6]

$$\begin{aligned} \rho V(\theta, A) \geq u(\theta) &- \sup_{r<\bar{r}} \{(p_{out}\alpha + q_{out}\gamma)(r - \bar{r})\} I(r < \bar{r}) \\ &- \sup_{r\geq\bar{r}} \{(p\alpha + q\gamma)(r - \bar{r})\} I(r \geq \bar{r}) \\ \text{for } & (p_{out}, q_{out}) \in D^{+}_{out}V(\theta, D) \text{ and } (p,q) \in D^{+}V(\theta, D) \end{aligned}$$

$$\begin{aligned} \rho V(\theta, A) \leq u(\theta) &- \sup_{r<\bar{r}} \{(p_{out}\alpha + q_{out}\gamma)(r - \bar{r})\} I(r < \bar{r}) \\ &- \sup_{r\geq\bar{r}} \{(p\alpha + q\gamma)(r - \bar{r})\} I(r \geq \bar{r}) \\ \text{for } & (p_{out}, q_{out}) \in D^{+}_{out}V(\theta, D) \text{ and } (p,q) \in D^{+}V(\theta, D) \end{aligned}$$

The only control r that fulfills both conditions is $r(t) \equiv \bar{r}$. Any $r(t) < \bar{r}$ would violate both conditions.

Intuitively, any further increase in A leads to an infinitely negative slope of V and therefore must be avoided. The optimization problem thus is to maximize θ subject to $\frac{dA}{dt} \leq 0$. Since $\frac{dA}{dr} > 0$ and $\frac{d\theta}{dr} > 0$, i.e. any control increasing θ also increases A, the optimal solution is to not let A decrease and therefore

$$r = \bar{r};\ \frac{dA}{dt} = 0;\ \frac{d\theta}{dt} = 0. \tag{17}$$

This solution is a viscosity solution, i.e. the natural extension of the solution concept for the Bellman equation (6). It is well known that value functions in general are not continuously differentiable in any feasible state[7] and thus for

[6] Here $I(x)$ is the indicator function with value 1 if x is true and 0 else. By abuse of notation we assume $\infty \cdot 0 = 0$.

[7] In fact, the value function is differentiable only at regular points.

these points the classical solutions do not apply. Viscosity solutions do apply also in many cases, where the value function is not continuously differentiable but necessarily coincides with the standard solution otherwise. This means, that we could have restricted our analysis to the approach used for the border case $A = D$. However, for reasons of clarity and intuition we first, showed the classical approach and then the viscosity approach.

The viscosity solution implies that optimal policy is to maximize the instantaneous utility and to not care about its fragility, i.e. the rising speculative pressure. The fragility is recognized but not accounted for in the decision about optimal the interest rate until the immediate danger of a crash emerges. In that opting out is associated with costs, the decision maker raises the interest rate to fend off the attack. Thereby it is necessary that costs exist no matter how big they are. Thus it could also be private costs, which would arise with a breakdown of the regime, that prompt the decision maker to raise interest rates. In a more elaborate model, there also exists the possibility that the policy maker abandons the option to defend or that he decides to exit the regime after some time.

4 Extended Linear Version

Now we consider the case when $f_2(.)$, $g_2(.)$ and $g_3(.)$ are also linear functions. We will further use the coefficients α and γ and define $\frac{\partial f_2(.)}{\partial \theta(t)} = \beta, \frac{\partial g_2(.)}{\partial \theta(t)} = \delta$ and $\frac{\partial g_3(.)}{\partial A(t)} = \varepsilon$.[8] So the state vector now is:

$$\begin{pmatrix} \dot{\theta} \\ \dot{A} \end{pmatrix} = \begin{pmatrix} -\alpha\left(r\left(t\right)-\bar{r}\right)-\beta\left(\theta\left(t\right)-\bar{\theta}\right) \\ -\gamma\left(r\left(t\right)-\bar{r}\right)-\delta\left(\theta\left(t\right)-\bar{\theta}\right)+\varepsilon\left(A\left(t\right)\right) \end{pmatrix}$$

The Bellman equation has the same form as in (6), but with a different derivation of the value function with respect to time

$$\rho V(\theta, A) = \sup_r \left\{ u(\theta) + \frac{dV(\theta, A)}{dt} \right\} \tag{18}$$

$$= \sup_r \{ u(\theta) - (V_\theta \alpha + V_A \gamma)(r - \bar{r}) - V_\theta \beta(\theta(t)) - V_A(\delta(\theta(t)) - \varepsilon(A(t))) \}. \tag{19}$$

Since both interest rate elasticities α and γ are still linear, the optimal strategy for the central bank remains a border solution.

The difference to the simple model is that the attack not only increases due to a low interest rate but also due to herding (ε) and bad fundamentals (δ). This creates a far richer set of policy options, trade-offs and realistic settings. E.g. a zero interest rate policy not necessarily leads to an attack. It might be possible that the herding effect and the interest rate effect are outweighed through the effect of good fundamentals, implying speculators refrain from attacking, $\dot{A} \leq 0$.

In addition we restrict the set of feasible controls to controls leading only to non-negative attack strengths, i.e. $0 \leq A \leq D$.

[8] Where β and δ are the mean-reversion elasticities of the fundamentals and of the attack and ε is the elasticity of herding.

4.1 Model Dynamics and Optimal Behavior

For a better understanding of the policy options of the central bank we draw the dynamics of the model in the state space $(A;\theta)$ of the system for the zero-interest-rate policy (cf. figure 1).

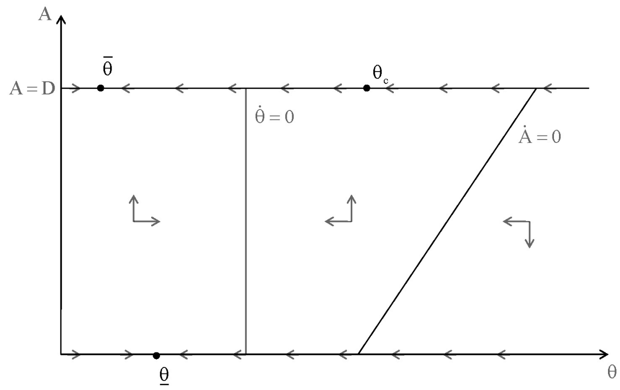

Figure 1: Dynamics of the Model

The zero-motion-lines $\dot{\theta} = 0$ and $\dot{A} = 0$ as well as the directional arrows are drawn for a zero interest rate policy. In case of $A = 0$ and $A = D$ the interest rate, as in the simple model, is set to offset a change in the attack, i.e. $r = \inf\left\{r : \dot{A} = 0\right\}$, leading to convergence to the equilibria $\overline{\theta}$ and $\underline{\theta}$. Left to the zero-motion-line of θ the mean-reversion leads to increasing fundamentals where right to the zero-motion-line we get decreasing fundamentals. Left to the zero-motion-line of A, fundamentals are relatively bad and thus lead to higher speculative pressure. The herding effect implies that higher attack levels ceteris paribus allow higher fundamentals to still increase the attack. Right to the zero-motion-line of A, fundamentals are high enough to outweigh the herding effect leading to decreasing speculative pressure.

Three regions with different policy options emerge. First, with $A = 0$ the central bank either sets $r = 0$, which boosts A and θ or sets $r = \inf\left\{r : \dot{A} = 0\right\}$, preserving the attack free state, with the system converging to $\underline{\theta}$.

Second, with $0 < A < D$ the central bank could also apply a zero interest rate policy with fundamentals and speculative pressure growing. In contrast to the simple case, the alternative is to massively raise the interest rate to offset the attack completely and then subsequently stabilize the system in an attack free state (i.e. $A = 0$). We model this policy by setting the interest rate $r \to \infty$ for $t \to 0$. If we let $r \to \infty$ all other terms cancel out and θ decreases by $\frac{\alpha}{\gamma}D$, which is the relative slope of θ and A.[9] Thus we get a jump in the state

[9] Putting aside mathematical rigor for a moment, we find that:

$$\dot{\left(\frac{\theta}{A}\right)} = \frac{\dot{\theta}}{\dot{A}} = \frac{-\alpha\left(r\left(t\right) - \bar{r}\right) - \beta\left(\theta\left(t\right) - \bar{\theta}\right)}{-\gamma\left(r\left(t\right) - \bar{r}\right) - \delta\left(\theta\left(t\right) - \bar{\theta}\right) + \varepsilon\left(A\left(t\right)\right)} \underset{r\to\infty}{\rightarrow} \frac{\alpha}{\gamma}$$

The intuition behind this result is that if the interest rate grows beyond bound all other effects including the feedback effect of A and θ become irrelevant and only the ratio of the interest elasticities remains relevant.

variables from $(\theta_S; A_S)$ to $\left(\theta_S - \frac{\alpha}{\gamma}D; 0\right)$, i.e. defending the attack entails a loss in the fundamentals. After the jump the central bank sets the interest rate so that another attack won't start, causing convergence to $\underline{\theta}$.

Third, at the upper boundary where $A = D$, the central bank has the three following options. It can opt-out, stay at the boundary with speculative pressure $A = D$, and also go back as in the previous case with $r \to \infty$ till $A = 0$. (1) If the central bank opts out, it has to bear costs c of the regime change (e.g. damaged reputation). The opt-out is optimal if fundamentals are sufficiently strong, i.e. $\theta > \theta_c$, since convergence to the bad equilibrium $\bar{\theta}$ would cause higher welfare losses than to immediately opt-out (see 6.2.2).[10] (2) The option to preserve the speculative pressure $A = D$ in order to neither harm fundamentals more than necessary nor carry the opt-out costs, is comparable but different from the calm option in the first case. Here, again the interest rate r is set to match $\dot{A} = 0$, but the high value of A implies strong herding effects which require higher interest rates to offset this additional pressure. Therefore $\bar{\theta} < \underline{\theta}$, i.e. the zero attack equilibrium $\underline{\theta}$ is better than the attack equilibrium $\bar{\theta}$. (3) Whether the central bank sets the interest rate to jump to the pressure free state ($A = 0$) depends on the relationship of time preference rate ρ and herd effect ε. If $\rho > \varepsilon$, high current values of θ are preferred over lower values of θ in the future, implying that a jump is not optimal, since it would cause an immediate loss in fundamentals (for a derivation see 6.2.3).

The derivation of the Bellman equation as direct derivation of the partial differential equation has to be done to show in which states (θ_S, A_S) with $0 < A_S < D$ the zero interest rate policy and accordingly the "jump"-policy would be optimal.

5 Conclusion

We applied an infinite horizon intertemporal optimization model with endogenous exit on a simple speculative attack framework. The central bank sets the interest rate which influences both fundamentals and attack strength. Hence with one variable the central bank enhances fundamentals but also boosts speculative pressure. The central bank's role is beyond solely responding to speculative pressure or signalling, thereby incorporating possible changes through a depreciation.

The model's degree of abstraction is high to support a broad range of applications, amongst others debt crises (e.g. Greece), bank runs, investment projects, renewable resources or dictatorships.

We show that optimal policy is to set controls in a way that allows to extract maximum benefits from the system by ignoring the danger of a crisis and thriving up crisis pressure until the breakdown is immediate and then to sharply change the policy by fending it off. In doing so a central bank avoids costs associated with a regime change. Notably regardless, how small these costs are, defending is optimal. In the more general approach, after a certain time under pressure - which depends inter alii on the opt-out costs - it becomes optimal

[10] The same holds if the decision is between "jump and converge to good equilibrium" and "opt out". Where the relative location depends on the values of ε and ρ.

to leave the regime. Hence this easy and stylized model delivers a very good theoretical basis that sudden policy changes are indeed optimal.

6 Appendix

6.1 Linear Version

6.1.1 Value Function

Through integrating and rearranging we show that the value function is always negative:

$$
\begin{aligned}
V &= \sup_r (U_0) \\
&= -\int_0^{\bar{T}} \exp(-\rho t) \exp(-\chi(\theta + \alpha\bar{r}t))\, dt - \int_{\bar{T}}^{\infty} \exp(-\rho t) \exp\left(-\chi\left(\theta + \alpha\bar{r}\bar{T}\right)\right) dt \\
&= -\int_0^{\bar{T}} \exp(-\rho t - \chi(\theta + a\bar{r}t))\, dt - \exp\left(-\chi\left(\theta + \alpha\bar{r}\bar{T}\right)\right) \int_{\bar{T}}^{\infty} \exp(-\rho t)\, dt \\
&= \left[\frac{\exp(-\rho t - \chi(\theta + \alpha\bar{r}t))}{\rho + \chi\alpha\bar{r}}\right]_0^{\bar{T}} - \exp\left(-\chi\left(\theta + \alpha\bar{r}\bar{T}\right)\right) \left[-\frac{\exp(-\rho t)}{\rho}\right]_{\bar{T}}^{\infty} \\
&= \frac{\exp\left(-\rho\bar{T} - \chi\left(\theta + \alpha\bar{r}\bar{T}\right)\right)}{\rho + \chi\alpha\bar{r}} - \frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} - \frac{1}{\rho} \exp\left(-\rho\bar{T} - \chi\left(\theta + \alpha\bar{r}\bar{T}\right)\right) \\
&= \frac{\exp\left(-\rho\bar{T} - \chi\left(\theta + \alpha\bar{r}\bar{T}\right)\right)}{\rho + \chi\alpha\bar{r}} - \frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} - \frac{\exp\left(-\rho\bar{T} - \chi\left(\theta + \alpha\bar{r}\bar{T}\right)\right)\left(1 + \frac{1}{\rho}\chi\alpha\bar{r}\right)}{\rho\left(1 + \frac{1}{\rho}\chi\alpha\bar{r}\right)} \\
&= \frac{-\exp\left(-\rho\bar{T} - \chi\left(\theta + \alpha\bar{r}\bar{T}\right)\right)\frac{1}{\rho}\chi\alpha\bar{r} - \exp(-\chi\theta)}{\rho + \chi a\bar{r}} \\
&= -\frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} \left(\frac{\chi\alpha\bar{r}}{\rho} \exp\left(-(\rho + \chi\alpha\bar{r})\bar{T}\right) + 1\right) < 0.
\end{aligned}
$$

The partial derivative of V with respect to θ is

$$
\begin{aligned}
&\frac{d}{d\theta}\left(-\frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} \left(\frac{\chi\alpha\bar{r}}{\rho} \exp\left(-(\rho + \chi\alpha\bar{r})\bar{T}\right) + 1\right)\right) \\
&= -\chi V,
\end{aligned}
$$

which is always positive.

The partial derivative of V with respect to A is

$$
\begin{aligned}
&\frac{d}{dA}\left(-\frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} \left(\frac{\chi\alpha\bar{r}}{\rho} \exp\left(-(\rho + \chi\alpha\bar{r})\bar{T}\right) + 1\right)\right) \\
&= -\frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} \frac{\chi\alpha\bar{r}}{\rho} \left(-\frac{-(\rho + \chi\alpha\bar{r})}{\beta\bar{r}} \exp\left(-\bar{T}(\rho + \chi\alpha\bar{r})\right)\right) \\
&= -\frac{\chi\alpha}{\beta\rho} \exp\left(-\bar{T}(\rho + \chi\alpha\bar{r}) - \chi\theta\right)
\end{aligned}
$$

which is always negative.[11] The partial derivative can also be written as

$$
\begin{aligned}
&= \frac{\rho + \chi\alpha\bar{r}}{\beta\bar{r}} \left(V + \frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} \right) \\
&= \frac{1}{\beta\bar{r}} \left((\rho + \chi\alpha\bar{r}) V + \exp(-\chi\theta) \right).
\end{aligned}
$$

Using (7) we proof that an exterior solution exists through:

$$
\begin{aligned}
V_\theta \alpha + V_A \beta &> 0 \\
V_\theta \alpha + V_A \beta &= -\chi V \alpha + \frac{1}{\beta\bar{r}} \left((\rho + \chi\alpha\bar{r}) V + \exp(-\chi\theta) \right) \beta \\
&= -\chi V \alpha + \frac{1}{\bar{r}} \rho V + \chi\alpha V + \frac{1}{\bar{r}} \exp(-\chi\theta) \\
&= \frac{1}{\bar{r}} \left(\rho V + \exp(-\chi\theta) \right),
\end{aligned}
$$

thus

$$
\rho V + \exp(-\chi\theta) > 0.
$$

Inserting for V gives

$$
\begin{aligned}
\rho \left(-\frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} \left(\frac{\chi\alpha\bar{r}}{\rho} \exp\left(-(\rho + \chi\alpha\bar{r}) \bar{T} \right) + 1 \right) \right) + \exp(-\chi\theta) &> 0 \\
-\frac{1}{\rho + \chi\alpha\bar{r}} \left(\chi\alpha\bar{r} \exp\left(-(\rho + \chi\alpha\bar{r}) \bar{T} \right) + \rho \right) + 1 &> 0 \\
\frac{1}{\rho + \chi\alpha\bar{r}} \left(\chi\alpha\bar{r} \exp\left(-(\rho + \chi\alpha\bar{r}) \bar{T} \right) + \rho \right) &< 1 \\
\chi\alpha\bar{r} \exp\left(-(\rho + \chi\alpha\bar{r}) \bar{T} \right) + \rho &< \rho + \chi\alpha\bar{r} \\
\exp\left(-(\rho + \chi\alpha\bar{r}) \bar{T} \right) &< 1 \\
-(\rho + \chi\alpha\bar{r}) \bar{T} &< 0,
\end{aligned}
$$

which is true, since $\bar{T} \geq 0$.

6.1.2 Comparison of Values

For $t > \bar{T}$ the central bank sets $r = \bar{r}$, which implies that $\theta(t)$ is constant, i.e. $\dot{\theta} = 0$ and thus $\theta(t) = \theta_S$. For $\upsilon(\theta - c)$ we write:

$$
\begin{aligned}
\upsilon(\theta - c) &= \int_0^\infty \exp(-\rho\tau) u(\theta - c) \, dt \\
&= u(\theta - c) \frac{1}{\rho} \\
&= -\frac{1}{\rho} \exp(-\chi(\theta - c))
\end{aligned}
\qquad (20)
$$

[11] $\frac{d\bar{T}}{dA} = -\frac{1}{\beta\bar{r}}$.

In comparison V at point $A = D$ equals:

$$\begin{aligned} V(\theta, A=D) &= -\frac{\exp(-\chi\theta)}{\rho+\chi\alpha\bar{r}}\left(\frac{\chi\alpha\bar{r}}{\rho}\exp(0)+1\right) \\ &= -\frac{\exp(-\chi\theta)}{\rho+\chi\alpha\bar{r}}\left(\frac{\chi\alpha\bar{r}+\rho}{\rho}\right) \\ &= -\frac{\exp(-\chi\theta)}{\rho} \end{aligned} \tag{21}$$

If we compare equations (20) and (21), we see that indeed $V(\theta, A=D)$ is higher and thus, defending the regime at the corner is optimal.

6.2 Extended Linear Version

6.2.1 Solutions of the Differential Equations

For the time path of θ ,we get if $r = 0$

$$\theta_t = \left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}\bar{r}\right)\exp(-\beta t) + \bar{\theta} + \frac{\alpha}{\beta}\bar{r}$$

if $A = D$

$$\theta_t = \left(\theta_S - \bar{\theta} - \frac{\alpha\varepsilon D}{\alpha\delta - \beta\gamma}\right)\exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \bar{\theta} + \frac{\alpha\varepsilon D}{\alpha\delta - \beta\gamma}$$

and if $A = 0$

$$\theta_t = \left(\theta_S - \bar{\theta}\right)\exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \bar{\theta}$$

with θ_S being the initial point of θ.

6.2.2 Comparison of Values: Opt out?

Annotation: For simplicity, we assume that utility in the extended version is described by the identity function, i.e. $u(\theta) = \theta$.

We opt out, if the value of convergence to the bad equilibrium is smaller than the value of immediately opting out:

$$V(\text{"opt out"}) > V(\text{"converge to bad equilibrium"})$$

$$\begin{aligned} \textstyle\int_0^\infty \exp(-\rho t) \quad & (\theta_S - c)\,dt > \\ & \int_0^\infty \exp(-\rho t)\left(\left(\theta_S - \bar{\theta} - \frac{\alpha\varepsilon D}{\alpha\delta - \beta\gamma}\right)\exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \bar{\theta} + \frac{\alpha\varepsilon D}{\alpha\delta - \beta\gamma}\right)dt \end{aligned}$$

$$\begin{aligned}
\frac{1}{\rho}\left(\theta_S - c\right) &> -\frac{1}{\frac{\alpha\delta}{\gamma}-\beta-\rho}\left(\theta_S - \bar{\theta} - \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\right) + \frac{1}{\rho}\left(\bar{\theta} + \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\right) \\
\theta_S - c &> \bar{\theta} + \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma} - \frac{\rho}{\frac{\alpha\delta}{\gamma}-\beta-\rho}\left(\theta_S - \bar{\theta} - \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\right) \\
\theta_S\left(1 + \frac{\rho}{\frac{\alpha\delta}{\gamma}-\beta-\rho}\right) &> \bar{\theta}\left(1 + \frac{\rho}{\frac{\alpha\delta}{\gamma}-\beta-\rho}\right) + \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\left(1 + \frac{\rho}{\frac{\alpha\delta}{\gamma}-\beta-\rho}\right) + c \\
\theta_S\left(\frac{\alpha\delta-\beta\gamma}{\alpha\delta-\beta\gamma-\gamma\rho}\right) &> \bar{\theta}\left(\frac{\alpha\delta-\beta\gamma}{\alpha\delta-\beta\gamma-\gamma\rho}\right) + \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\left(\frac{\alpha\delta-\beta\gamma}{\alpha\delta-\beta\gamma-\gamma\rho}\right) + c \\
\text{with } \frac{\alpha\delta-\beta\gamma}{\alpha\delta-\beta\gamma-\gamma\rho} &> 0 \\
\theta_S &> \bar{\theta} + \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma} + c\frac{\alpha\delta-\beta\gamma-\gamma\rho}{\alpha\delta-\beta\gamma}
\end{aligned}$$

The comparison with convergence to the good equilibrium gives:

$$\begin{aligned}
V\left(\text{"opt out" }\right) &> V\left(\text{"jump and converge to good equilibrium"}\right) \\
\theta_S &> \bar{\theta} + \frac{\alpha\rho D}{\alpha\delta-\beta\gamma} + c\frac{\alpha\delta-\beta\gamma-\gamma\rho}{\alpha\delta-\beta\gamma}
\end{aligned}$$

6.2.3 Comparison of Values: Fend of attack?

The attack is fend off completely, if the value of convergence to the bad equilibrium is smaller than the value of convergence to the good equilibrium after a jump:

$$V\left(\text{"converge to bad equilibrium"}\right) < V\left(\text{"jump and converge to good equilibrium"}\right)$$

$$\begin{aligned}
\int_0^\infty \exp\left(-\rho t\right) & \left(\left(\theta_S - \bar{\theta} - \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\right)\exp\left(\left(\frac{\alpha\delta}{\gamma}-\beta\right)t\right) + \bar{\theta} + \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\right)dt \\
&< \int_0^\infty \exp\left(-\rho t\right)\left(\left(\theta_S - \frac{\alpha}{\gamma}D - \bar{\theta}\right)\exp\left(\left(\frac{\alpha\delta}{\gamma}-\beta\right)t\right) + \bar{\theta}\right)dt
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{\frac{\alpha\delta}{\gamma}-\beta-\rho}\left(\theta_S - \bar{\theta} - \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\right) + \frac{1}{\rho}\left(\bar{\theta} + \frac{\alpha\varepsilon D}{\alpha\delta-\beta\gamma}\right) &< -\frac{1}{\frac{\alpha\delta}{\gamma}-\beta-\rho}\left(\theta_S - \frac{\alpha}{\gamma}D - \bar{\theta}\right) + \frac{1}{\rho}\bar{\theta} \\
-\frac{\bar{\theta}\left(\alpha\delta-\beta\gamma\right) - \gamma\rho\theta_S + \alpha\varepsilon D}{\rho\left(\beta\gamma-\alpha\delta+\gamma\rho\right)} &< -\frac{\bar{\theta}\left(\alpha\delta-\beta\gamma\right) - \gamma\rho\theta_S + \alpha\rho D}{\rho\left(\beta\gamma-\alpha\delta+\gamma\rho\right)} \\
\varepsilon &> \rho
\end{aligned}$$

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