Dynamic Banking in Continuous Time

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Abstract

We use a continuous time setting to show how banking capital evolves over time. The model frame work is related to so called "harvesting" models in mathematics. In this type of model there exist an optimal rule for harvesting, in the case of banking this is the dividends. This implies that there also exist some "optimal" level of primary capital as profits not distributed as dividends adds to primary capital. Given an optimal level of capital in relation to lending there are macro economic consequencies as banks' desire to maintain some debt to equity ratio through credit squeeze and credit crunch.

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1 Introduction

Since the eighties it has been shown that banks' primary capital are substantially larger than that required of regulating agencies, see e.g. Clark (1988) or Bank for International Settlments (2009). This means that capital adequacy rules rarely binds. Does this mean that regulating banks' capital is not necessary? Probably not. A lot of research around the issue of capital adequacy focus on liquidity and systemic risk, e.g. Allen and Gale (2005). Larsson (2010) address the solvency and optimal banking, but only in a two period setting.

Larsson (2010) address the solvency and optimal banking, but only in a two period setting. To focus on solvency in a dynamic setting means solving some intricate mathematical problems. In Borch (1968) he showed that if one assumes an optimal dividend distribution strategy that keeps capital below some optimal level and where the firm has decreasing returns to scale: the optimal level of capital is so low that firms default in finite time. This is due to discounting and the fact that if only capital above a certain level is distributed to shareholders waiting time to first dividend makes high levels of capital inefficient He also exploits an alternative dividend strategy which maximizes the longevity of the firm. Note that the optimal strategies are assumed and not derived.

In mathematics these problems of optimal management of "resources" in a stochastic frame-work is known as "harvesting" problems. Oksendal and Lungu (1997) derives the optimal harvesting function from a stock that maximizes the flow of discounted future harvest where the future harvest is dependent on current stock as well as some stochastic shock. There diffusion is multiplicative. Radner and Shepp (1996) derives a similar solution for corporate decision making. They solve both for multiplicative diffusion and additive diffusion. The multiplicative diffusion assumes constant returns to scale which is inappropriate for a single firm even if it works for a single investor as in Black and Scholes (1972). The additive diffusion means that firms can go bankrupt due to a shock being larger than the firms capital meaning that the firm has negative cash position and cannot run their business any longer. Radner and Shepp (1996) establishes what Borch (1968) assumed, optimal dividend policy will be to distribute all capital above some threshold as dividend. As a consequence the firm will default almost surely in finite time.

In the above papers the manager controls the dividend or harvesting process. To analyze banks we would like to add one dimension: scale. This since banks indeed can increase profits and/or risks through increased lending. Radner (1998) derives such a model under additive diffusion process. His solution is similar to the solutions without scale: there is some optimal scale given capital, firms go bankrupt almost surely in finite time, and the optimal dividend policy is to distribute everything above some threshold.

We use Radner (1998) to address the optimal banking problem in terms of solvency. We first derive the general problem in the same way as Radner (1998), then we make a simple parameterization showing the path of capital adequacy for the bank.

Section 2 describes the problem and solution method. Using Rothschild and Stiglitz (1970) section 3 gives some insights into how the model should be parameterized. Next we do a simple example of optimal path of capital adequacy of a bank in 4. Finally we conclude in section 5.

2 General Model

Consider an environment where there exist a Bank and potential costumers, costumers are either people with a project idea that can earn money in the future if launched or savers that have an endowment that needs to be spread out over several periods. In practice banks' lend to known costumers first, in our setting we let potential borrowers be of deteriorating quality, the more you lend to the more ´´unknown" credit quality do you obtain in your portfolio. Moreover, the loans driven by a stochastic process determining the outcome for their project that is financed with the loan if granted. In order to perform banking, i.e. act as intermediary between savers and borrowers services banks need to carry capital as reserves in case of credit losses when some borrowers default due to bad luck with project. For a full derivation of technical aspects of why and when the need for capital as reserves to cushion credit losses are established in Larsson (2010).

The formal model starts off with the process of costs in terms of the cost for financing for banks, $\rho(x, y)$ where x is total loan stock and y is capital used as reserves with. The central variable for the bank is its equity which is used as reserves, y, and consist of initial investment and retained earnings.

Banks "make" money on their stock of loans, i.e. their gross return stems from the interest their borrowers pay. This will be some interest rate r times the stock of loans x. As in standard finance the core value to owners of the bank is the flow of dividends, the withdrawal rate will be denoted w.

In addition to the cost for capital, $\rho(x, y)$, the bank suffers credit losses. Credit losses are stochastic and depends on amount of loans. The variance of credit losses will be denoted $\sigma(x)$ and the shock $\mathrm{d}z$. The process for the bank's equity will therefore be,

$$\mathrm{d}y = (r \cdot x(t) - w(t) - \rho\,(x(t), y(t)))\ \mathrm{d}t + \sqrt{\sigma(x)}\,\mathrm{d}z, \ \texttt{with}\ \sigma'(x) > 0. \tag{1}$$

The $\sigma(x)$ means that: the more loans, the more volatility in project outcomes in terms of shocks and hence more loans magnifies the shock $\mathrm{d}z$ in terms of realized credit losses. This since we assume some deteriorating quality of borrowers ability to pay back loans. For a more compact notation we will define banks' flow of gross profit as $m(x, y) \equiv r \cdot x(t) - \rho\left(x(t), y(t)\right)$.

Banks' expected value V is a function of its capital/equity y. Given some initial level of $y(0)$ and optimal controls w and x, which are dividends and volume of lending respectively the expected value is described as:

$$V\left(y\left(0\right)\right) = \max_{w,x} \int_{0}^{T} w\left(t\right) e^{-\mu t}\, \mathrm{d}t, \quad T = \inf\left\{t \mid y\left(t\right) \leq 0\right\}. \tag{2}$$

The existence of a final T stems from decreasing returns to scale which in turn will lead to default almost surely.

The dynamic problem to solve will be,

$$\mu V\left(y\right) = \max_{w,x} \left\{ w + V'\left(y\right)\left\{r \cdot x(t) - w(t) - \rho\left(x(t), y(t)\right)\right\} + \frac{1}{2}\sigma\left(x\right) V''\left(y\right) \right\}. \tag{3}$$

Radner (1998) shows that the optimal withdraw rate is a "bang-bang" solution where you either withdraw with the rate w or no dividends are withdrawn. The threshold depends on the level of capital y, if $y < b$, $w = 0$ and if $y > b$, $w = w^*$ where b is the threshold level of capital and $w*$ is some level. Further, when the capital is greater than b $V'(y) < 1$ and if capital is lower than b $V'(y) > 1$. This implies that $V'(b) = 1$.

The optimal condition of (3) with respect to x is then,

$$V'\left(y\right)\left(r - \rho_x\right) + \frac{1}{2}\sigma'\left(x\right) V''\left(y\right) = 0. \tag{4}$$

This implies that,

$$V''\left(y\right) = -\frac{2\left(r - \rho_x\right)}{\sigma'\left(x\right)} V'\left(y\right). \tag{5}$$

Together with 3 optimality solution for V" in 5 yields,

$$\mu V\left(y\right) = w + \left(r \cdot x - \rho(x, y) - w - \frac{\sigma\left(x\right)\left(r - \rho_x\right)}{\sigma'\left(x\right)}\right) V'\left(y\right). \tag{6}$$

The optimal condition is now a difference equation in *V(y)* and *V'(y)*. Also define the expression with *V'(y)* as:

$$h(x, y) \equiv r \cdot x - \rho(x, y) - \frac{\sigma\left(x\right)\left(r - \rho_x\right)}{\sigma'\left(x\right)} \tag{7}$$

to get,

$$\mu V\left(y\right) = w + \left(h\left(x, y\right) - w\right) V'\left(y\right). \tag{8}$$

To find out how the value responds to changes in level of capital reserves 8 is differentiated with respect to y. This yields,

$$\mu V'\left(y\right) = \left(h_x x'(y) + h_y\right) V'\left(y\right) + \left(h\left(x, y\right) - w\right) V''\left(y\right). \tag{9}$$

Using 5 we can eliminate $V''(y)$,

$$\mu V'(y) = (h_x x'(y) + h_y) V'(y) + (h(x,y) - w) - \frac{2(r-\rho_x)}{\sigma'(x)} V'(y). \tag{10}$$

Note that this means that we can eliminate the dependence of the value function. This allow us to solve for *x'(y)* as,

$$x'(y) = \frac{\mu - h(x,y)g(x,y) - h_y}{h_x}, \tag{11}$$

where we have defined $g(x,y)$ as:

$$g(x,y) = -2\frac{r-\rho_x}{\sigma'(x)}. \tag{12}$$

This is for capital below the cut-off point b, the equivalent for capital levels y above the cut-off point is,

$$x'(y) = \frac{\mu - (h(x,y) - w)g(x,y) - h_y}{h_x}. \tag{13}$$

The cut-off point b can be found by the two differental equations for x' where the first crossing point will define the cut-off point b. By integrating the first order condition for the bellman function the value function can be retrived

$$\begin{aligned} y > y_0 &\quad : \quad V(y) = y_0 + exp\left\{\int_{y_0}^{y} \frac{h(x(t),t)}{\mu}\,\mathrm{d}t\right\} \\ y \leq y_0 &\quad : \quad V(y) = y_0 \end{aligned}$$

Where we have used

$$\frac{\mathrm{d}}{\mathrm{d}y} \ln V' = \frac{h(x,y)}{\mu}. \tag{14}$$

3 Banks' Financing Cost

The problem of financial intermediation via credit, i.e. standard bank loans, is that there are three parties involved. Banks are in the middle as intermediary and they contract both with their borrowers and lenders. In the contracting with its borrowers banks run the risk that the borrowers project does not result in enough resources to repay the loan fully and the bank then suffer a credit loss. Agents lending to the bank suffer a similar problem, if a bank suffer a large amount of defaulted loans it may erode the bank's capital to the point where the bank no longer can fulfill its obligations, at least in the short run. It is clear that this is a problem in "two-steps" but contingent on the same underlaying risk: agents who borrowed from the bank cannot repay their loans fully.

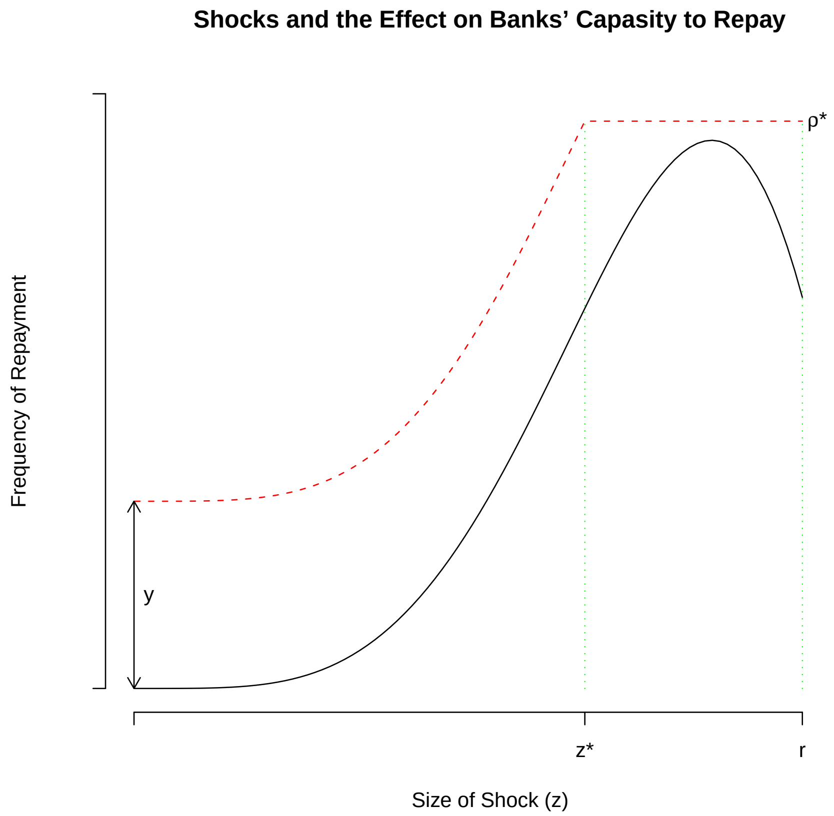

Figure 1: The relationship between the shock to borrowers and the the ability for the bank to pay the committed interest rate on its own debt, ρ^*. Note that the level rho is not in the same plane as frequency of repayment but drawn just for illustrations of the forces at work.

The bank's committed interest on its debt ρ^* in Figure 1 is in some sense exogenously given by the market. That is, it is the given market interest rate, actually expected return, given the specific loan stock x and amount of primary capital y and will be denoted with i.

The banks ability to pay the market interest rate is defined by the distribution of shocks and the level of primary capital given a level of the loan stock x.

$$\int_0^{z*} (y + z) \, g(z) \, \mathrm{d}z + \rho \int_{z*}^{\infty} g(z) \, \mathrm{d}z = i \tag{15}$$

Note that the level of shock $z*$ where the bank can fulfill its obligations is dependent on the level of primary capital y as they together constitute the banks ability to repay its debt.

$$z^* = z^*(y), \quad y + z^* = \rho \tag{16}$$

To characterize the function for banks' cost of capital ρ we differentiate 15,

$$(y+z^*)g(z^*)\frac{\mathrm{d}z^*}{\mathrm{d}y}+\int_0^{z^*} g(z)\,\mathrm{d}z-g(z^*)\frac{\mathrm{d}z^*}{\mathrm{d}y}\rho+\int_{z^*}^{\infty} g(z)\,\mathrm{d}z\rho_y=0 \tag{17}$$

Using the equality of ρ in 16 equation 17 collapse to:

$$\frac{\int_0^{z^*} g(z)\,\mathrm{d}z}{\int_{z^*}^{\infty} g(z)\,\mathrm{d}z}=-\rho_y \;\Rightarrow\; \frac{G(z^*)}{1-G(z^*)}=-\rho_y>0. \tag{18}$$

That is, the cost of capital, ρ is diminishing with respect to primary capital, y. Moreover, 18 shows that the riskier the bank, large z*, the more sensitive the bank is to changes in its primary capital. Note that this is regardless of assumed distribution for the shocks to their borrowing clients. This seems highly intuitive and mimics that of the behaviour on the market (needs reference). To find out if this effect is increasing or decreasing we can differentiate again to find,

$$\frac{g(z^*)\frac{\mathrm{d}z^*}{\mathrm{d}y}}{(1-G(z^*))^2}=-\rho_{yy}>0. \tag{19}$$

When investigating the relationship between cost of capital, $\rho(x,y)$, and lending, x, we use Rothschild and Stiglitz (1970) and their definition of increasing risk. Define the mean preserving spread in lending x such that x'>x with,

$$\int_{-\infty}^{\infty} zg(z,x')\,\mathrm{d}z=\int_{-\infty}^{\infty} zg(z,x)\,\mathrm{d}z, \tag{20}$$

$$\int_{-\infty}^{z'} [G(z,x')-G(z,x)]\;\mathrm{d}z\geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} [G(z,x')-G(z,x)]\;\mathrm{d}z=0 \tag{21}$$

The sensitivity of size of loan stock x on banks' borrowing cost $\rho(x,y)$ can be analyzed the same way as for y above starting with expression 15 but a pdf depending also on x.

$$G(z,x)(y+z)\Big|_{-\infty}^{z^*}-\int_{-\infty}^{z^*} G(z,x)\,\mathrm{d}z+\rho\int_{z^*}^{\infty} g(z,x)\,\mathrm{d}z=i \tag{22}$$

Using the identity in 16 for ρ 22 can be written as,

$$\rho-\int_{-\infty}^{z^*} G(z,x)\,\mathrm{d}z=i. \tag{23}$$

Differentiating 23 yields

$$\rho_x-G(z^*,x)\frac{\mathrm{d}z^*}{\mathrm{d}x}-\int_{-\infty}^{z^*}\frac{\partial G(z,x)}{\partial x}\,\mathrm{d}z=0. \tag{24}$$

From 16 we have that $\frac{\mathrm{d}z^*}{\mathrm{d}x}\equiv\rho_x$ allowing us to write 24 as

$$\rho_x=\frac{\int_{-\infty}^{z^*}\frac{\partial G(z,x)}{\partial x}\,\mathrm{d}z}{1-G(z^*,x)}>0. \tag{25}$$

The positivity stems from the conditions for a mean-preserving spread as

$$\int_{-\infty}^{z^*} \frac{\partial G(z,x)}{\partial x}\,\mathrm{d}z = \int_{-\infty}^{z^*} G(z,x')\,\mathrm{d}z, \tag{26}$$

which is defined to be positive yielding a positive numerator and the denominator is obviously positive. Differentiating again reveals whether the increase in banks' borrowing cost is at an increasing or decreasing speed.

$$\rho_{xx} = \frac{\left(\int_{-\infty}^{z^*} \frac{\partial^2 G(z,x)}{\partial x^2}\,\mathrm{d}z + G(z,x)\right)(1-G(z,x)) + g(z,x)\rho_x \int_{-\infty}^{z^*} \frac{\partial G(z,x)}{\partial x}\,\mathrm{d}z}{(1-G(z,x))^2} \tag{27}$$

To determine whether ρ_{xx} is positive or negative we need to know $\frac{\partial^2 G(z,x)}{\partial x^2}$ which is not obvious. When it comes to the cross derivatives addressing 18,

$$\frac{\partial \frac{G(z^*,x)}{1-G(z^*,x)}}{\partial y} = -\frac{g(z^*,x)\frac{\mathrm{d}z^*}{\mathrm{d}x} + \frac{\partial G(z^*,x)}{\partial x}}{(1-G(z^*,x))^2} = \rho_{yx} < 0 \tag{28}$$

This leaves us with,

$$\rho_x > 0, \quad \rho_y < 0 \text{ and } \rho_{yy} < 0. \tag{29}$$

4 An example

In order to illustrate the dynamics a simple example is shown where we have used the following ρ-function,

$$\rho(x,y) = \gamma \cdot exp\left\{\frac{x + \psi x^2}{y}\right\} + \mu \cdot y \tag{30}$$

The ratio of debt to capital for the bank under the optimal policy then have the following path More work is needed on the parameterization as the solution is sensitive to choices of ρ-function and its parameters.

It should be noted from the picture how the "optimal" capital adequacy is falling with capital so a credit squeeze is not one to one as banks lower lending less than capital falls with bad shocks.

5 Conclusion

This work is in its initial face but are to our knowledge the first to use harvesting models to analyze banking. It is not only possible to study solvency with this model frame work, but also problems of correlated borrowers can also be addressed.

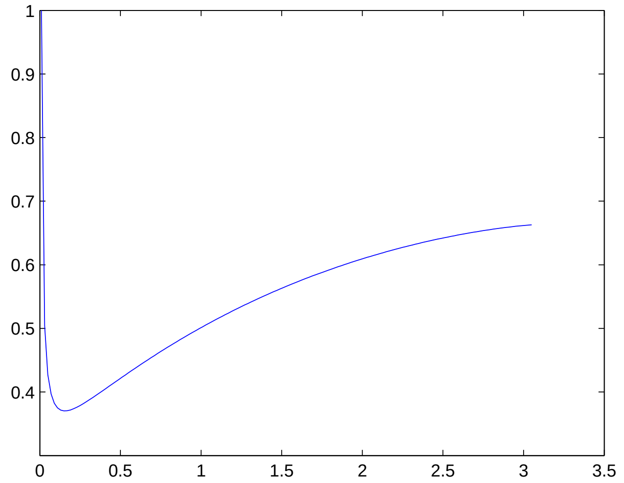

Figure 2: The relationship between the bank's lending and its primary capital. For capital levels close to zero the value of the bank is less than the capital.

References

Allen, Franklin and Douglas Gale, "Systemic Risk and Regulation," *WP*, 2005.

Bank for International Settlments, *79th Annual Report*, Vol. 79, Basle: Bank for International Settlments, 2009.

Black, F. and M. Scholes, "The valuation of option contracts and a test of market efficiency," *Journal of Finance*, 1972, *27*, 399–418.

Borch, Karl, *The economics of uncertainty*, Princeton,NJ: Princeton University Press, 1968.

Clark, J., "Economics of Scale and Scope at Depository Financial Institutions: A Review of the Literature," *Federal Reserve Bank of Kansas City Economic Review*, 1988.

Larsson, Bo, "Banking and Optimal Capital Ratio in an Equilibrium Model," *International Journal of Banking, Accounting and Finance*, 2010, *XX*, SSS.

Oksendal, Bernt and E. Lungu, "Optimal harvesting from a population in a stochastic, crowded environment," *Math. Biosciences*, 1997, *145*, 47–75.

Radner, Roy, "Profit maximization with bankruptcy and variable scale," *Journal of Economic Dynamics and Control*, 1998, *22*, 849–867.

_ and Larry Shepp, "Risk vs. profit potential: A model for corporate strategy," *Journal of Economic Dynamics and Control*, 1996, *20*, 1373–1393.

Rothschild, Michael and Joseph E. Stiglitz, "Increasing risk: I. A definition," *Journal of Economic Theory*, 1970, *2*, 225–243.