The optimum Quantity of Money in a Neoclassical Model with Idiosyncratic Risk

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Abstract

I tackle the problem of the optimum quantity of money in a model in the spirit of Friedman's (1969) seminal work. Provided the seignorage proceedings are distributed uniformly, I prove that the Friedman rule is (ex-ante) pareto suboptimal, is dynamically inconsistent,and strains capital formation.

1 Introduction

I will resent an economy that resembles that by Friedman (1969), where he argues heuristically that the government should issue money at a fixed rate which makes the opportunity cost of holding money as low as possible. The insight is that, as money provides transaction services, the optimal rate of inflation should have all consumers satiated with real balances, provided the social cost of producing money is negligible. That article prompted an immediate some economists' reaction, who sought to set out formally the axiomatic corpus proposed by Friedman. For instance, Brock (1974, 1975) and Bewley (1977, 1980), among others, confirmed the Friedman rule (FR) by means of formal general equilibrium (GE) models. Phelps (1973), on the contrary, contradicted the FR in the presence of excise taxes when money is a final good[1].

This article shows that the optimal inflation rate is strictly higher than the FR regardless the presence of distorted taxes. To this scope, I construct a Bewley-type economy where money is used as a store of value as well as a medium of exchange. I have also imposed the condition that the existing transaction technology induces consumers to hold a decreasing real balances-capital ratio with respect to wealth. This environment lies in harmony with Friedman's outlook on money demand. Besides, it is assumed that households face idiosyncratic risk materialized in their labor productivity; and that they smooth their consumption path by holding both capital and inconvertible money.

[1] See Krimbourgh (1986), Faig (1988), Guidotti and Veigh (1993).

incomplete market models with idiosyncratic risk was first introduced by Bewley (1977, 1980, 1983) to formalize classical issues such as the permanent income hypothesis, and the FR itself. In the same spirit, Aiyagari (1994) and Huggett (1995) addressed the capital accumulation patterns and the determination of the interest rate under the presence of idiosyncratic uncertainty, and demonstrated the presence of precautionary motive of savings. Krussell and Smith (1998) showed that, in a Bewley economy, the aggregate dynamics were mostly explained by the first moment of wealth distribution. By contrast, this article provides an example where idiosyncratic uncertainty matters substantially in terms of policy implications.

As well as the distributional channel, other reason why welfare might be increased by bidding up inflation is the interplay between aggregate capital and consumption. On the one hand, when inflation is low enough, it promotes capital accumulation because it makes more costly holding money related to capital. This phenomenon, known as the Tobin effect, reverses once inflation reaches a threshold level. On the other hand, as fluctuating-income consumers are given assets whose rate of return equals the opposite of the discount rate, they are willing to accumulate wealth boundlessly (Schechtman, 1976; Schechtman and Escudero, 1977; Sotomayor, 1984). Thus, as inflation decreases in a neighborhood of the FR, the portfolio substitution effect dominates the precautionary effect, and aggregate consumption and capital grow with the rate of injection of liquid balances.

Market incompleteness has been proven to be key in the literature on the OQM. In Levine (1991) and Kehoe et al. (1998), the distribution of real balances is exogenous. Other examples where an expansionary monetary policy dominates the FR are provided by Levine, Green and Zhou (2002), Mehrling (1995), Paal and Smith (2000), Deviatov and Wallace (2001), Smith (2002).

Along the next section I offer the description of the economy and some properties displayed by aggregate demand for assets, necessary to characterize equilibria. In section 3, the competitive equilibrium is defined. In section 4, the main proposition on optimality is presented. Section 5 concludes and offers future line of research. An appendix is devoted to proving the main result of the paper.

2 The Economy

There is a continuum of agents with identical preferences, who seek to maximize the same utility function defined over stochastic paths of consumption and cash. Time is discrete and denoted by $t \geq 0$, and the temporal horizon is infinite. Each period, consumers suffer form an idiosyncratic shock materialized in their units of efficiency labor. This is the only source of uncertainty, so that the aggregates shall evolve deterministically over time. The total amount of labor is normalized to one unit. Available stores of value are money and the single good which can be sold to the productive units, which are assumed to be identical and operating in a perfectly competitive market under a neoclassical production function.

Money is issued by the government at a fixed rate

$$M_{t+1}^S = \sigma M_t^S.$$

According to whether σ is greater or less than one, money is either injected or withdrawn. The proceedings of the inflation tax (subsidy) is assumed to be distributed uniformly among consumers. With regards to the rest of the financial market, it is assumed that no market for borrowing and lending exist, and thus the only possibility for smoothing the path of consumption is by holding the available assets (money and capital).

Although money is dominated by capital in terms of rate of return, real balances are held because they procure non pecuniary services, which are reflected in the appearance of cash in the households' utility function. This formulation captures some frictions in the financial market, the shoe-leather costs or simply represents the transactions technology.

At the aggregate level, the economy is described by the evolution of the distribution of assets. Uncertainty is restricted to individuals: by the large of law numbers (Judd, 1985) the shocks are purely idiosyncratic and not affect the evolution of the whole economy. I assume that these shocks are governed by a sequence of iid random variables which are cot correlated among individuals.

Firms produce out of labor and capital supplied by consumers. They have identical neoclassical production function and operate in a perfectly competitive regime. As the total labor is one, the production per capita can be expressed this way

$$f(K) = F(K,1) - \delta K.$$

f satisfies the usual Inada conditions. The firm buys capital and hire workers so as to maximize their profits. The necessary and sufficient conditions establish that the real wage and the real interest rate are paid their marginal productivity:

$$\begin{aligned} r &= f'(K) \\ w &= f(K) - f'(K)K. \end{aligned}$$

2.1 Consumers

2.1.1 Preferences and Transaction Technology

Each consumer maximizes a separable utility function defined over stochastic streams of consumption and real balances,

$$(1-\beta)\, E \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) + \varphi\left(m_t\right) \right], \tag{1}$$

for $0 < \beta < 1$.As usual, c denotes the non storable units of consumption and m stands for real balances available for transactions[2]. E is the expectation

[2] $\varphi(\frac{M_t}{P_t})$ would be a better representation of the implicit transaction technology that the utility function is intended to rather than of $\varphi(\frac{M_{t+1}}{P_t}) = \varphi(m_{t+1})$. However I use the alternative specification for convenience of notation without altering the implications of the model.

operator. Preferences (including β) are identical among individuals, and the current utility functions (u, φ) obey the usual Inada conditions. Moreover, it will be assumed that u has finite asymptotic exponent and that dominates φ in terms of marginal utility. Formally:

$$1 < \lim_{c\to\infty} -\frac{\ln u'(c)}{\ln c} < \infty \tag{2}$$

$$\lim_{x\to\infty} \frac{\varphi'(x)}{u'(x)} = 0. \tag{3}$$

The separability of the utility function simplifies the analysis and should be innocuous. The finite asymptotic exponent rules out the possibility that consumers behave as if there continue to be uncertainty as wealth diverges to infinity. A class of functions that does not exhibit asymptotic finite exponent is the exponential utility function, whose graphic displays huge variations in marginal utility of consumption as compared to values very close in proportion. This assumption works as a sufficient condition for which the demand for assets does not diverge to infinite. The fact that consumption dominates money in terms of utility reveals the very nature of consumption and rescues the its predominant weight on welfare in a context with money in the utility function.

When u and φ belong to the CRRA class of utility functions, (3) would simply imply that the elasticity of intertemporal substitution of φ would be less than that of u. The insight is that low values of m bring about a relatively high willingness of cash, even though a consumer has an acceptable level of consumption. This reflects an interaction between the productive system and financial markets, as if a given level of production could not be efficiently allocated without enough liquidity. The implications of this model must be interpreted on the basis that it represents an reduced version of a more complex and implicit transaction technology. The nature of this technology and its microeconomic foundations are a subject of further research.

2.1.2 The Budget Constraint

The intertemporal budget constraint being faced by consumers is given by

$$\begin{aligned} c_t + m_{t+1} + k_{t+1} &= (1+\pi_t)^{-1} m_t + (1+r_t) k_t + \theta_t w_t + \tau_t \equiv x_t \qquad (4) \\ x_0 &> 0, \text{ given.} \end{aligned}$$

I am assuming that the government distributes uniformly the inflation tax as lump-sum subsidies (if $\sigma > 1$) transfers (otherwise). Accordingly,

$$\tau_t = \frac{\sigma M_t^S}{P_t}.$$

The variable k represents the units of capital sold to the firms. In this context, this is foregone consumption rented at a rate r. Labor is supplied inelastically, since leisure does not enter the utility function. Yet, units of efficiency labor are idiosyncratic, and follow a iid stochastic process inside a compact real interval $\Theta = [\underline{\theta}, \overline{\theta}]$ with $\underline{\theta} > 0$, and according to a distribution (measure) ψ. It is possible to define the probability space $(\Omega, \mathcal{F}, P)$ which captures all the intrinsic uncertainty of the model: any element of Ω is a sequence $\{\theta_t\}$ of realizations of the stochastic process; $\mathcal{F}$ is the infinite product of Borel fields of Θ; and P is the probability measure naturally inferred from ψ[3]. Individuals are assumed to be rational: the expectation operator appearing in (1) and in the sequel is on the basis of the intrinsic uncertainty just defined.

An individual's position at period t is described by x_t, the total amount of resources available for consumption. I shall refer to it as wealth[4].

In Appendix B, it is proven that the economy is a globally stable system. Therefore, in the sequel I will mainly consider stationary states, which are characterized by displaying a constant series of asset returns $z = (\sigma^{-1}, 1 + r)$. As the wage rate can be deduced from the real interest rate, it can be dropped away from the state variable. Note that z defines the law of motion of the (individual) asset accumulation. Defining $a = (m, k)$, and denoting by ' the forward operator,

$$x' = a \cdot z + \theta w + \tau.$$

Note that θw represents the stochastic component of asset accumulation, and of income.

2.1.3 A Recursive Formulation

The maximization of the utility function (1) subject to the intertemporal budget constraint (4) may be written in a recursive formulation. Let us define $Z = \{(\sigma^{-1}, 1 + r) : \sigma^{-1} \leq \beta \leq 1 + r\}$ and assume that $z \in \text{int}Z$. First, it is convenient to have defined the current utility[5] $U = u + \varphi$:

$$v(x, z) = \max_{a \geq 0} \left\{ U(x, a) + \beta \int v(a \cdot z + \theta w + \tau, z) \psi(d\theta) \right\}. \tag{5}$$

As proven in Hernández-Lerma et *al* (1996), there is a unique *value function* v solving the functional equation (5). Moreover, the stationary policy rules are *unique* and *continuous*. It is well known that the value function is strictly increasing, strictly concave and continuously differentiable in x. From the maximum theorem, v varies continuously with the parameter z. I introduce the following notation. Functions $c(x, z), a(x, z)$, with $a = (m, k)$ are the optimal consumption and asset rules. This rule, together with the particular history $\{\theta_t\}$ define the law of motion of individual wealth

$$x'_z = a(x, z) \cdot z + \theta w + \tau. \tag{6}$$

[3] See Doob (1953) and Ionescu-Tulcea (1950) for the details.

[4] This is justified because in continuous time, indeed, the decision variable is wealth.

[5] $U(x, z, a) = u(x - a^+) + \varphi(m)$, where $a^+ = m + k$.

2.2 Lump-Sum Taxes

That the FR is suboptimal crucially hinges on the distributional impact of inflation. To understand this point, assume that during a period t, the distribution of money holdings is[6] λ. Were the government to allocate the injected money to subsidies proportional to the beginning-of-period individuals' money holdings, the income from money once subsidies proceeding are considered will be given by

$$\frac{m}{\sigma} + \frac{(\sigma - 1)\, m}{\sigma} = m.$$

In other words, the return of money is independent of secular inflation. The hypothesis made on the distributional effects of monetary policy emphasizes the interplay between fiscal and monetary policy

The condition of implementability applies for negative inflation rates. Under this condition, there might be a positive mass of consumers who are unable to pay the tax associated with such policy. The point has extensively been discussed by Bewley (1983). Given an initial distribution of wealth, a negative inflation rate is said to be implementable whenever all ($P-$a.s) agents can afford to pay the implicit tax that finances the withdrawal of money. This means that households have enough resources as to pay the present value of the flow of tax liabilities. Note that the Inada condition implies that $\lim_{c\to 0} u'(c) = \infty$. This condition implies that individuals will have an incentive to accumulate assets pay the tax, since otherwise they face a risk of not consuming in a finite period. It will be convenient to have the budget constraint written in terms of money balances relative to the average, say $b_t = m_t - M_t^S/P_t$.

$$\begin{aligned} c_t + b_{t+1} + k_{t+1} &= (1+\pi_t)^{-1} b_t + (1+r_t) k_t + \theta_t w_t \equiv \widehat{x}_t \\ \widehat{x}_t &= x_t - \sigma \frac{M_t^S}{P_t} \;\; \forall t \geq 0. \end{aligned}$$

Here I am assuming that the sequence of gross returns $z_t = \left((1+\pi_t)^{-1}, (1+r_t)\right)$ converges to a steady state[7] $z = \left(\sigma^{-1}, 1+r\right) \in \mathrm{int}Z$. Analytically, implementability means that the *ex-ante* probability of default is zero. This means the existence of a plan $\left\{b_{t+1}^0, k_{t+1}^0\right\}_{t=0}^{\infty}$ such that, for any $t > 0$ (a.s):

$$\widehat{x}_0 + \underline{\theta} \sum_{n=1}^{t} w_n R_n^{-1} \geq \sum_{n=1}^{t} \frac{i_n}{1+i_n} b_n R_n^{-1} + R_t^{-1} \left(b_{t+1}^0 + k_{t+1}^0\right), \tag{7}$$

where R_t is the discount factor:

$$R_t = \prod_{n=1}^{t} (1+r_n),$$

[6] The distribution of money holdings will be $\lambda = \mu_{t-1} \circ m_t^{-1}$.

[7] Details can be found in Appendix B.

and i is the nominal interest rate, namely $(1+i) = (1+r)(1+\pi)$. Therefore, the implementability condition incorporates the opportunity cost of holding money $(i/1+i)$. Equation (7) says that the present value of the flow of sure income up to period t plus the initial wealth must be at least greater than the present value of the flow of costs incurred by holding money. An inflation rate $\sigma < 1$ is *implementable* when (7) is satisfied by all consumers (a.s) for any $t \geq 0$. In steady state, the implementability condition is

$$b_t \geq \frac{\sigma}{1-\sigma}\theta w. \tag{8}$$

The right hand side of (8) is the present value of the labor income, or the *natural debt limit* because the ratio $\frac{\sigma}{1-\sigma}$ is the interest rate paid by money. Thus (8) means that money balance must not differ from average more than the natural debt limit. Maintaining this level of real balances guarantees that consumption will ever be positive. This property is non trivial, since it has been assumed that $u'(0) = \infty$.

2.3 Euler Equations and the Optimal Plan

It is convenient to set out the Euler Equations of (5) in steady state:

$$u'(c_t) \geq \beta\sigma^{-1}\left[\varphi'(m_t) + \int u'(c_{t+1})d\psi\right] \tag{9}$$

$$u'(c_t) \geq \beta(1+r)\left[\varphi'(m_t) + \int u'(c_{t+1})d\psi\right]. \tag{10}$$

The conditions (9,10) hold with equality when (8) hold strictly and $k_{t+1} > 0$, respectively. When $\sigma > 1$, there is no tax to pay and (8) holds trivially, and (9) holds with equality.

Next proposition says that the optimal plan is continuous with x. Both consumption and assets are *normal* – increasing with x. See Appendix A.

Proposition 1 *The optimal plan $c(x,z)$ and $g(x,z)$ are continuous in their domains. They are increasing in x and*

$$\lim_{x\to\infty} c(x,z) = \lim_{x\to\infty} a(x,z) = \infty$$

$$0 \leq \frac{c(x+h,z) - c(x,z)}{h} \leq 1.$$

From now on, I will assume that the factor prices, in the long run, belong to the set

$$Z = \left\{z = \left(\sigma^{-1}, 1+r\right) : \sigma^{-1} \leq 1+r \leq \beta^{-1}\right\}.$$

If r were less than the interest rate paid by money $(\sigma^{-1} - 1)$ the supply of capital would be zero. When any of the portfolio assets earns a rate of return greater than the rate of discount no equilibrium exists. In such a case, as show

Schechtman and Escudero (1977) and Sotomayor (1984), consumers' resources would diverge to infinite (a.s).

Next proposition says that the ratio money-to-capital tends to zero as wealth diverges to infinite. This implies that inflation is a regressive tax.

Proposition 2 *For* $z \in intZ$,

$$\begin{aligned}
\lim_{x\to\infty} c(x,z) &= \lim_{x\to\infty} m(x,z) = \lim_{x\to\infty} k(x,z) = \infty \\
\lim_{x\to\infty} \frac{m(x,z)}{c(x,z)} &= 0 \\
\lim_{x\to\infty} \frac{k(x,z)}{c(x,z)} &< \infty.
\end{aligned}$$

As a corollary of Proposition 2, the rich are eager to accumulate in terms of capital rather than money. This can be justified in terms of robbery costs (Friedman, 1969).

3 Competitive Equilibrium

A steady state *perfect foresight equilibrium* is defined as a set of optimal policies, value functions and a price system $(a_\sigma, v_\sigma, z_\sigma)$ together with a distribution of wealth μ_σ, such that

1. a_σ is optimal given v_σ, and the latter solves the Bellman equation (5).
2. $z_\sigma = \left(\sigma^{-1}, 1 + r_\sigma\right)$, with $r_\sigma = f'\left(\int k_\sigma d\mu_\sigma\right)$.
3. Markets clear:
$$\int c_\sigma d\mu_\sigma = f\left(\int c_\sigma d\mu_\sigma\right).$$
4. Government budget is balanced,
$$\tau_\sigma = \frac{\sigma}{1-\sigma}\int m_\sigma d\mu_\sigma.$$
5. The distribution is ergodic:
$$\mu_\sigma(A) = \int H(x,A)\mu_\sigma(dx),$$
where A is a Borel set and
$$\begin{aligned}
H(x,A) &= \psi(B) \\
B &= \left\{\theta : x'_\sigma = a_\sigma \cdot z_\sigma + \theta w_\sigma + \tau_\sigma \in A\right\}.
\end{aligned}$$

The rational expectations hypothesis is incorporated in the definition, since households' plans agree with the effective asset returns. The two first conditions establish that individuals are rational. The third one is the market clearing condition. The fourth one is the government budget constraint and the fifth one establishes that the wealth distribution is ergodic.

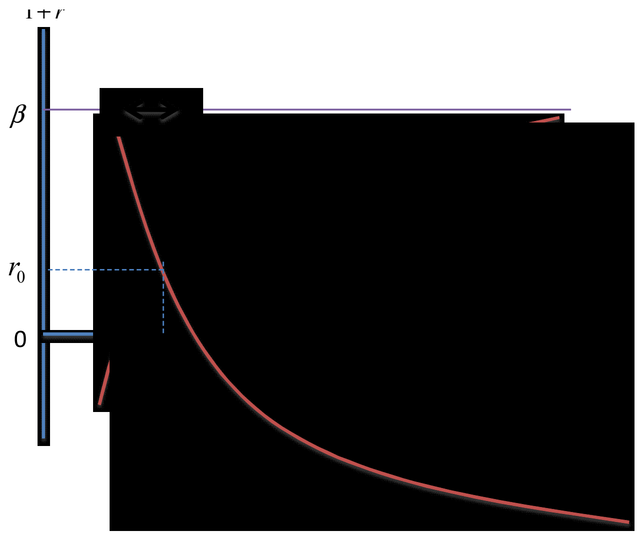
Existence is proven geometrically. Nothing can be said of uniqueness since the supply curve in figure 2 is not necessarily monotone, as argued in Aiyagari (1994). This is so because a raise in the interest rate has a twofold effect. On the one hand it increases capital income; on the other hand, it makes wages down. Since capital supply becomes perfectly elastic as the interest rate approaches the discount rate, the former effect will outweigh the latter for r high enough and thus the equilibrium is unique.

From now onwards, I will denote the monetary equilibrium allocation by a 3-tupla[8] $E_\sigma = (a_\sigma, v_\sigma, \mu_\sigma)$. Next proposition states the existence of equilibrium, which is an application of the Schauder fixed-point theorem (see Appendix B for the details). The uniqueness for σ near β allow us to write unambiguously $\lim_{\sigma \to \beta} E_\sigma$.

Proposition 3 *There exists at least one stationary monetary equilibrium. There exists an open neighborhood* (β, σ_0) *for which the equilibrium is unique.*

Since, as proven in proposition B.1, $\lim_{\sigma \to \beta} x_\sigma = \infty$ (a.s.) the capital supply would behave likewise, by virtue of proposition 1. This argument proves the following proposition.

[8] Note that z and any other equilibrium variable, like c_σ can be stated in terms of E_σ.


β
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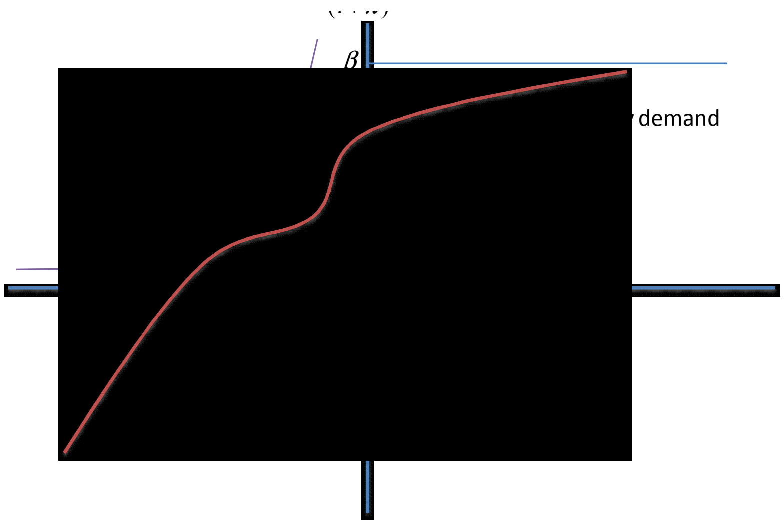

Figure 1:

Proposition 4 *Let the level of capital associated with the representative-agent version of the model be* k_β. *Then,* $\lim_{\sigma \to \beta} k_\sigma = k_\beta$.

In terms of welfare, the transition path matters, and so the initial condition, determined by historical and institutional variables, can influence the optimal policy. For this reason we may be interested in defining the non-stationary equilibrium as a 3-tupla $E_\sigma^\lambda = \left(a_\sigma^\lambda, v_\sigma^\lambda, \mu_\sigma^\lambda\right)$, where

$$\begin{aligned} (a_\sigma^\lambda, v_\sigma^\lambda) &= \{(a_{\sigma t}, v_{\sigma t})\}_{t=1}^\infty \\ \mu_\sigma^\lambda &= \{(\mu_{\sigma t})\}_{t=0}^\infty \to \mu_\sigma, \text{ with } \mu_{\sigma 0} = \lambda. \end{aligned}$$

Here $a_{\sigma t}$ are the optimal decision plans and the convergence of wealth the distribution is weak convergence. In Appendix B we will deal with the convergence of E_σ^λ to a steady state E_σ.

The sequence of optimal plans, resources and welfare can be written as a random variable defined in the product probability space[9]

$$(R_+ \times \Omega, \mathcal{B}(R) \times \mathcal{F}, \lambda \otimes P).$$

The sequence of σ-fields $\mathcal{B}(R_+) \times \mathcal{F}_t = \mathcal{B}(R_+) \times \mathcal{B}(\Theta^t)$ make up a filtration. Each $\mathcal{F}_t$ is the σ-field generated by the history of events θ_n for $1 \leq n \leq t$, and

[9] $\mathcal{B}(R_+)$ stands for the Borel sets of R_+.

the initial condition λ. In other words, $\mathcal{F}_t$ is the minimum σ-field generated by $\cup\mathcal{F}_t$ is by definition $\mathcal{F}$. Given the economy-wide initial state, the individual initial condition x plus the full realization of shocks will determine the path of future asset returns, which in turn determine uniquely the optimal decision plans $(a_{\sigma,t+1}, c_{\sigma t})$. According to this notation, when the initial condition coincides with the limiting distribution, $(a_\sigma^\lambda, v_\sigma^\lambda) = \left(a_\sigma^{\mu_\sigma}, v_\sigma^{\mu_\sigma}\right) = (a_\sigma, v_\sigma)$. This work deals with the ex-ante aggregate welfare assuming that the government commits itself to a fixed rule of monetary policy. The lifetime utility is a random variable in the probability space previously defined, which can be written in terms of of the optimal plans of consumption and money holdings:

$$W_\sigma^\lambda = (1-\beta)\int\int\sum_{t=0}^{\infty}\beta^t\left[u(c_{\sigma t}) + \varphi(m_{\sigma t})\right]d\lambda dP. \qquad (11)$$

In steady state, $W_\sigma^\lambda = W_\sigma^{\mu_\sigma} = W_\sigma = u(c_\sigma) + \varphi(m_\sigma)$. The main result of this work proves the existence of policy rules which improve the ex-ante welfare of the Friedman rule. First, and in order for this result make sense, I first prove that the Ramsey problem

$$\max_{\sigma\geq\beta} W_\sigma^\lambda$$

(subject to the optimal plans) has a non-empty solution for any initial distribution λ.

Proposition 5 *The Ramsey problem has a non-empty solution.*

Unfortunately, the Ramsey problem is too complex an object as to be resolved in closed form, as it embeds a full sequence of decision rules. However, it is not difficult to show that the Friedman rule is dominated in terms of welfare by increasing the money growth rate. Within the context of this Ramsey problem, Mulligan and Sala-i-Marti (1997) and Faig (1988) assume that current consumption can be expressed as a function of the future discounted stream of income. Consequently, the distributional issues I am dealing with are dropped from the analysis. The main difficulty of dealing with distribution is that the optimal inflation rate depends in general on the initial distribution, so that it is not possible to draw general conclusions about what the optimal inflation may be.

4 Optimality

In a pure monetary, representative-agent economy, the optimality of the Friedman rule is not much more than a truism as superneutrality holds and liquidity is a public service that facilitates transactions. Bewley (1980) proved the validity of the Friedman rule in a monetary economy where individual income fluctuates randomly. In Bewley (1983), it was proven that the implementability of a negative inflation rate could not be attainable. The following result constitutes the most important result of this work. optimality fails to hold because in the neighborhood of the Friedman rule, it is possible to increase aggregate consumption and capital by increasing σ. The proof is provided in Appendix C.

Proposition 6 *If $\underline{\theta}$ is low enough, the function W_σ^λ as defined in (11) are strictly increasing in σ in an open neighborhood of β.*

In other words, the proposition says that, no matter the initial condition, and provided the inflation rate is close enough to the FR, it is possible to increase welfare by increasing inflation. Two major difficulties have been overcome to prove the proposition: (1) there will be a temporary slowing down in aggregate consumption; and (2) the welfare loss due to a reduction in the transaction services is shown to be of second order compared to the welfare gains, because the interest rate rises very slowly with inflation when the money interest rate is close to zero. As a corollary of this proposition it is possible to show (see Appendix C) that in a neighborhood of the FR there is no comparable pair of inflation rates given a sufficiently high $\overline{\theta}$. On average, the transfer from the rich towards the poor raises aggregate welfare. Yet, for a sufficiently high $\overline{\theta}$, there is a positive mass rich consumers who unambiguously worse off under a expansionary policy. For low $\overline{\theta}$ it could be the case that there exists some interval (β, σ_0) such that, within it, an improvement in ex-ante welfare can be obtained by increasing inflation. Another straightforward corollary drawn from Proposition 6 is that the result on optimality is robust under alternative welfare criteria whose weights are non decreasing in (x, ω).

The crucial factor that determine the non-optimality of the FR is the distribution of the efficiency units of labor –the unique source of uncertainty. As in Levine and Zame (2000), when agents are very patient, the economy virtually collapses to its representative-agent version of the model as far as policy is concerned. Since the FR fails to smooth the individual consumption flows, the analysis suggests that the welfare benefits of an expansionary policy are important so long as heterogeneity is quantitatively relevant. Suitable fiscal arrangements which keep the low-type workers permanently away form zero labor income would make the welfare cost of a deflationary policy more innocuous. İmrohoroğlu (1992) computed the inflation cost in a pure monetary economy similar to ours. She found that the welfare costs of inflation are higher than those reported by Bailey (1956).

The analysis of this paper differs from İmrohoroğlu (1992) and Kehoe et al (1998) in that the minimal amount of income is here very small, stressing the distributive issue. Moreover, the comparison of alternative steady states as in İmrohoroğlu (1992) may potentially be lame, since the maximization of the steady state level of consumption does not necessarily lead to Pareto optimal allocations, as proven in Cass (1965). By contrast, in this article a time-inconsistency problem may arise for low inflation rates. when the policymaker applies a rate of money growth close enough to the FR, there is a certain threshold beyond which ex-ante welfare may be improved by increasing inflation.

5 Conclusion

The potential distributive role of monetary policy is an empirical issue which deserves serious study before we can adopt any definitive normative conclusion. It crucially depends on the criteria with which financial institutions conduct their monetary policy. Especially relevant issues are how the system allows for the possibility of issuing inside money not directly controlled by central authorities.; the channel through which the financial system distributes the injected outside money among the public; and of course, the quantitative importance of the inflation tax. Regarding the last point, the very nature of the secular inflation is related to alternative fiscal variables, in line with the contributions of Sims (1994) and Woodford (1995). Accordingly, the results presented here might well be robust even though the distributive role of money is apparently small in a broad range of countries.

The results presented have been obtained by taking up the axiomatic corpus of Friedman (1969), except for the fact that prices adjust instantaneously. Friedman argues that money demand should be a decreasing function of the rate of foreseen inflation, independently of the way inflation revenues are redistributes amongst consumers. This fact yields Friedman to abstract away from the distributive effects of money issuance. Yet this issue does not hold in a standard general equilibrium setup, for reasons pointed out above.

Moreover, it should be remarked that this article aimed at proving that aggregate welfare increases as policy departs from the FR, rather than computing the effective optimum quantity of money. Moreover, while the Tobin effect, by which inflation stimulates the accumulation of capital through a substitution effect between liquid and productive assets, prevails in steady state when inflation is around the rate of discount, no clear insight indicates that this relation should hold for higher inflation rates.

Finally, we are not sure that the result obtained in this work should be interpreted as a way to explain why the real rate of inflation is in the real world greater than that prescribed by Friedman, or rather to suggest possible normative implications which should guide monetary institutions. My view is that a deflation rate may well have adverse effects in welfare through its effects on distribution of wealth and income. The deflation episode of the Japan economy makes it plausible that low money interest rates can eventually be associated with poor economic performance. Whether this work should constitute a theoretical framework prescribing the inadequacy of an excessively tight monetary policy is yet to be discussed.

Appendix A

Let i be the expected nominal interest rate. Let ω be the forward operator applied to sequences. Then the following inequalities hold for any individual state

$$\varphi'(m_t) \geq i_{t+1} \int u'(c_{t+1}) d\psi \tag{12}$$

$$\varphi'(m_t) \geq \frac{i_{t+1}}{\beta(1+r_{t+1})} u'(c_{t+1}). \tag{13}$$

The inequalities follow straightforwardly by rearranging suitably the Euler equations (9) and (10) with the former holding with equality, namely, when consumers are not obliged to hold extra amount of real balances to pay the tax when $\sigma < 1$. When such is the case, take the marginal consumer whose wealth obliges her to maintain the minimum money holdings which allows her to pay the tax. Note that she must be borrowing constrained in the capital market, since otherwise, she could hold a marginal unit of capital rather than money, rendering a higher return. Therefore, in this case, both inequalities hold. For consumers who are less wealthy than the marginal one, the same must be true due to a parallel argument. They must be borrowing constrained in the capital market, so that both Euler equations hold with strict inequality. Yet, as far as the total resources decrease, the right-hand side of each Euler equation (9) and (10) diminishes respectively by $\beta(1+\pi_{t+1})^{-1} \int u'(c_{t+1}) d\psi$ and $\beta(1+r_{t+1})^{-1} \int u'(c_{t+1}) d\psi$: Since the latter amount is greater, the relations (13) and (13) follow.

Proposition 7 *The optimal plan $c(x,z)$ and $g(x,z)$ are continuous in their domains. They are increasing in x and*

$$\lim_{x\to\infty} c(x,z) = \lim_{x\to\infty} a(x,z) = \infty$$

$$0 \leq \frac{c(x+h,z) - c(x,z)}{h} \leq 1.$$

Proposition 8 *For $z \in intZ$,*

$$\lim_{x\to\infty} c(x,z) = \lim_{x\to\infty} m(x,z) = \lim_{x\to\infty} k(x,z) = \infty$$

$$\lim_{x\to\infty} \frac{m(x,z)}{c(x,z)} = 0$$

$$\lim_{x\to\infty} \frac{k(x,z)}{c(x,z)} < \infty.$$

The proof of both propositions are standard. the next appendix is devoted to the analysis of the asymptotic behavior of the economy on the side of the demand for consumption and the supply of production factors faced by consumers. It contains sufficient conditions by which there is an invariant distribution of the individual income, and, which is more important, there is a continuous dependence on a quite general parameter, which is to be interpreted as the expectation of future states of the economy.

Appendix B

This section is devoted to show that the model hereby described is globally stable, and displays uniqueness of equilibrium. While such result admittedly hinges on some critical hypothesis, it is needed to tackle an ulterior welfare analysis in line with Friedman (1969). We find it convenient recalling the reader that in any stationary equilibrium with a inflation rate strictly greater than the discount rate, there is a positive mass of households who happen to be borrowing constrained. This result is obtained straightforwardly by integrating (9) in steady state. The next proposition is aimed at justifying the property displayed by Figures 1 and 2, according to which the demand for assets diverge to infinity as the interest approaches the discount rate. It says that the sequence z_t converges weakly and has compact support if $\max z_t < 1$: let $\varsigma = \theta w + \tau$. It has a real compact interval with upper and lower limits provided that $\max z_t < 1$. The probability distribution governing the process depends fully on ψ and z; and will be denoted by the symbol (it conveys a linear transformation of θ). Subindices are omitted for the sake of convenience when there is no risk of confusion.

Proposition 9 *For each $z \in intZ$, the stochastic process $x_{z,t}$ converges to a random variable $x_z < \infty$ (a.s) which is independent of the initial state x_0. Such a limit is a measurable random variable. When $z \in \delta_\beta Z$, $\lim x_t = \infty$ a.s.*

Proof. Let $z \in int(Z)$. Using the Theorem 3.8 in Schechtman and Escudero (1977), the existence of a number b such that $x_{t+1} > x_t$ for any $x > b$ amounts to showing that

$$\lim_{x\to\infty} \frac{\int v'(x_{t+1})d\psi}{v'(\overline{x})} \leq 1,$$

where $\overline{x}$ is the upper bound of x_{t+1}, given x; namely $\overline{x} = a(x,z)\cdot z + \overline{\theta} w + \tau$. By assumption (2) there exists a bound in the exponent of u_0; say α (Schechtman and Escudero,1977) so that

$$\left(\frac{c_1}{c_0}\right)^\alpha < \frac{u'(c_0)}{u'(c_1)}$$

for $c_0 > c_1$ large enough. Thus, as proven in Proposition 2, $\lim_{x\to\infty} a(x,z) = \infty$; defining

$$\begin{aligned} \underline{x} &= a(x,z)\cdot z + \underline{\theta} w + \tau, \\ \underline{c} &= c(\underline{x}, z) \\ \overline{c} &= c(\overline{x}, z) \end{aligned}$$

and taking into account the above inequality,

$$\lim_{x\to\infty} \frac{\int v'(x_{t+1})d\psi}{v'(\overline{x})} \leq \lim_{x\to\infty} \frac{u'(\underline{c})}{u'(\overline{c}))} \leq \lim_{x\to\infty} \left(\frac{\overline{c}}{\underline{c}}\right)^\alpha.$$

By Proposition 1, for x large, there exists a function $0 < d(x,z) < 1$

$$\overline{c} = \underline{c} + w\left(\overline{\theta} - \underline{\theta}\right) d(x,z).$$

Then,

$$\lim_{x\to\infty} \frac{\int v'(x_{t+1})d\psi}{v'(\overline{x})} \leq 1 + w\left(\overline{\theta} - \underline{\theta}\right) \lim_{x\to\infty} \frac{d(x,z)}{\underline{c}} = 1.$$

Next, let us prove that the sequence x_z as defined in (6) converges weakly. Define $\xi = \sup\{x : k(x,z) = 0\}$. Define $X_t = -v'(x_t)$. The stochastic process so defined is $\mathcal{F}_t-$adapted. In order to prove that the stochastic process converges almost surely, we need to show that $I_{(X_t>\xi)}X_t$ converges. The stochastic process $I_{(X_t>\xi)}X_t$ is a submartingale. If the inflation rate is implementable, then X is bounded. Therefore, the martingale convergence theorem applies and the process $I_{(X_t>\xi)}$ converges almost surely to a bounded, $\mathcal{F}_t$ measurable random variable. Note that the random variable x_t can be expressed (see equation 6) as a function of the history of realizations of the θ's and of the initial wealth, that is $x_{t+1} = x_{(t)}(h_t, x_0)$. Let $\underline{h}_t$ be the worst history of events up to period t. The $\lim x_{(t)}(\underline{h}_t, x_0)$ is well defined and finite. Moreover, with probability one, individual wealth will reach a value at least greater than

$$\underline{x} = \sup_{x_0>0} \lim x_{(t)}(\underline{h}_t, x_0).$$

Let $x_\xi = (-v')^{-1}(\xi)$. It is clear that $x_\xi > \underline{x}$, since otherwise the mass of people who are borrowing constrained in stationary equilibrium would be zero. By Assumption (3), the function a must be concave, since the willingness to consume relative to money holdings increases as wealth increases. Accordingly, $\lim x_{(t)}(\underline{h}_t, x_0) \in [\underline{x}, x_\xi]$ for any $x_0 \in [\underline{x}, x_\xi]$, provided that $\theta \in [\underline{\theta}, \theta_\xi] \subset \Theta$. Besides, there is a one-to-one, increasing relationship between the sets $[\underline{\theta}, \theta_\xi]$ and $[\underline{x}, x_\xi]$. Such a set is invariant along $I_{(X_t>\xi)}X_t$. Let Ξ be such mapping. The distribution of wealth is then $\psi \circ \Xi^{-1}$ along the set for which $\theta_t < \theta_\xi$ except for a finite number of times. Accordingly, the set

$$\left\{\omega \in \Omega : \limsup I_{(X_t>\xi)}X_t > \liminf I_{(X_t>\xi)}X_t\right\}$$

is the set for which both $X_t > \xi$ and $\theta_t < \theta_\xi$ infinitely many times. Yet, this set is of measure zero, for otherwise, the stochastic process $I_{(X_t>\xi)}X_t$ would not converge. Therefore, if both $I_{(X_t>\xi)}X_t$ and $I_{(X_t\leq\xi)}X_t$ converge weakly, the stochastic process X_t converges and so does x_t as v' is monotonic. If $\beta\max(z) = 1$, the Euler equation for accumulation of capital amounts to

$$v'(x_t) \geq \int v'(x_{t+1})d\psi$$

with equality if $k(x;z) > 0$. Therefore, the induced process X_t is a non-positive submartingale. By Theorem 4.1s of Chapter 7 in Doob (1953), and by strict monotonicity of v', there exists $\lim x_t = x$; with probability one. (The proof is similar to that provided in this theorem for the case $\beta\max(z) < 1$, without conveying the transformation of the process X.) If the limit were finite, then there should be a maximum level of the support of wealth $x_{\max}$. By the Euler equation

$$v'(x_{\max}) \geq v'(a(x_{\max}) + \theta w + \tau) > v'(x_{\max})$$

a contradiction. ■

As proven in Clarida (1987)indeed, the support is a closed interval.

Next proposition enables us to aggregate whatever Borel-measurable real valued functions. Since policy function is continuous and thus measurable, it allows for integrating with respect to the probability P. It shows the far from obvious issue that such aggregates vary continuously with respect to the parameter z. To such scope, it will be useful to define μ_z to be the distribution of the asymptotic consumer's wealth by $\mu_z = P \circ x_z^{-1}$. This is well defined because x_z is measurable, as proven in Proposition 1. At this point, it is convenient to index the random variable with z. The weak-topology continuity ensures that, for any real valued, continuous function f, the expectation $\int f d\mu_z$ varies continuously with respect to the parameter z. In the sequel we will use the notation $\|\cdot\|$ to denote the norm of a probability measure induced from the weak topology. It is known that the space of probability measures over a metric space X can be metricized as a separable metric space, according to the weak topology, if and only if the space X is itself separable Parthasarathy (1967, Chapter 2). The key issue that guarantees the continuity consists of the fact that the speed of convergence is geometric, as proven in Kakutani and Yosida (1941).

Proposition 10 *The aggregate asset demand $\int a(x_z, z)dP$ varies continuously with respect to the parameter z in the weak topology:*

Proof. Since the function a is continuous, the above integral is well defined as long as total resources are bounded. Note that the expression (6) defines an operator T_z over the space of measures which determines the law of motion of μ. Assume that the economy starts, at time 0, from a state λ. We know, from Kakutani and Yosida (1941), that there are positive numbers a, ρ ($\rho < 1$) such that, for any t,

$$\left\|T_z^t \lambda - \mu_z\right\| \leq a\rho^t.$$

This means that

$$\left\|T_z^t \lambda - \mu_z\right\| \leq \rho^t o\left(T_z \lambda - \lambda\right).$$

Proving the proposition amounts then to proving that, for any given sequence $z_n \to z$, the sequence $\mu_{z_n} = T_{z_n}\lambda$ converges weakly to μ_z. The first step consists in noting that the operator T is "continuous"in the sense that, for any λ; the sequence $T_{z_n}\lambda$ converges weakly to $T_z\lambda$. Since by definition $T_z\mu_z = \mu_z$,making $\lambda = \mu_z$ in (**??**), yields

$$\|T_{z_n}\mu_z - \mu_z\| \leq \rho^t o\left(T_{z_n}\mu_z - \mu_z\right),$$

implying weak convergence. ■

To conclude this appendix, we state formally the global stability of the economy. The following definitions will be useful to characterize the space of beliefs.

Definition 11 *Let Z be the set of sequences of gross asset returns such that*

1. there are at most a finite number of periods in which (1 + r) is greater than β^{-1}; and
2. their first element is strictly smaller than the second, namely, the nominal interest rate is positive.

Definition 12 *Let $\partial_\beta Z$ be the sequences of asset returns for which the interest rate converges to the inverse of the discount rate, that is, such that* $\lim r_t = \beta^{-1} - 1$.

Z allows us to pay attention to a reduced space of beliefs inasmuch as to allow us to find self-fulfilling equilibrium. Its definition makes sense since the services rendered by money will imply that the nominal interest rate is positive. Confining attention to such a set can be interpreted as the agents' internalization of this statement, causing no loss of generality whatsoever. The set $\partial_\beta Z$ is the effective boundary beyond which wealth would diverge almost surely to infinite. Since this cannot be positive in equilibrium under standard assumptions made on preferences, and technology, we (and rational agents) know that equilibrium asset returns turn to be in $Z \setminus \partial_\beta Z$. Although the wage rate matters in order to infer future wealth, it is assumed that it can be taken from r. This point shall become clearer once the discussion of the productive sector is made

Proposition 13 *If beliefs are homogeneous among agents, for each implementable σ there exists a non-stationary equilibrium. Moreover, for each stationary equilibrium, there exists a non-stationary equilibrium which converges uniformly to the stationary equilibrium.*

Proof. Let $T : Z \to Z$ be the operator defined in the following way: given $z \in Z$, the transition function governing the individual states is given by $H_z : R_+ \times B(R+) \to [0;1]$ thus defined: ■

$$H_z(x; A) = \psi \left\{ \theta : L(x; z) + w' + \tau' \in A \right\}.$$

(The symbol $'$ stands for the forward operator). H_z determines the law of motion of states which in turn implies, by an aggregation-based method, a sequence Tz which will be so defined: given that the current state as of t is λ, the future state moves according to $\Psi_z \mu$ defined by

$$\mu_{t+n} = \Psi_z^{(n)} \lambda$$

with (ω being the forward operator)

$$\begin{aligned} \Psi_z^{(n+1)} \lambda(A) &= \int H_{\omega^n z}(s_n, A) \int H_{\omega^{n-1} z}(s_{n-1}, A) \cdots \int H_z(s, ds_1) \lambda(ds) \\ \Psi^{(0)} \lambda &\equiv \lambda \end{aligned} \quad (14)$$

The sequence $\Psi_z^{(n)}$in turn maps a sequence of factor prices in a natural way: let the first and second components of $(Tz)_{n+1}$ be respectively

$$\pi_{n+1} = \frac{\int m(\cdot, \omega^n z) d\Psi_z^{(n)}\lambda}{\int m(\cdot, \omega^{n+1} z) d\Psi_z^{(n+1)}\lambda} - 1,$$

and

$$r_{n+1} = \max\left\{ (f')^{-1} \left[\int k\left(\cdot, \omega^n z\right) d\Psi_z^{(n)}\lambda \right], -\pi_{n+1}(1+\pi_{n+1})^{-1} \right\}.$$

Note that we impose the condition that the nominal interest rate is positive. For the sequence z to belong to Z, its first component must converge to the rate of money growth , which turns out to be strictly greater than β. Therefore, there is at most a finite number of periods after which the inflation rate falls below the rate of discount. By virtue of Proposition in the next appendix, for any initial condition ; the number

$$\sup_n \{r_n : z \in Z\}$$

is well defined, which insures that the set Z is compact (for Z endowed with the supremum norm $\|\cdot\|$). Further, by construction, $T(Z) \subset Z$. In order for the expectations to be fulfilled, it must be the case that beliefs are a fixed point of T. The next step consists of proving that the mapping T is continuous. From the definition of T, proving continuity amounts to proving the continuity of Ψ_z for any $z \in Z$ in the weak topology. For if Ψ_z were continuous, then, by induction, $\Psi_z^{(n)}$ is easily proven to be continuous. Continuity is clear from the continuity of the policy function and the continuity of Q: for any sequence z_n convergent to z_0,

$$\Psi_{z_n}\lambda(A) = \int H_{z_n}\left(\cdot, A\right) d\lambda$$

converges to $\Psi_{z_0}\lambda(A)$, due to identity (14). The continuity of the policy function implies that the second term of the right hand side of (14) is measurable. By hypothesis, the first term of the right-hand side is continuous in its first argument. Proposition 1 and the monotonicity of a guarantees that H is monotonic. Equation (14) yields

$$\lim \Psi_{z_n}\lambda(A) = \int H_{z_0}\left(\cdot, A\right) d\lambda$$

The maximum theorem guarantees that Ψ_z varies continuously in the parameter z and thereby $\Psi_{z_n}\lambda$ converges in the strong topology (for a reference see Lucas and Stokey 1987, Chapter 11). Strong continuity implies weak continuity. By induction it is immediately proven that, by applying the Schauder Theorem, T has a fixed point. Let N be the set of asset returns which constitute a stationary equilibrium. Call Z_ν with $\nu \in N$, the subset of elements of Z converging to ν . The property that N is finite is generic, so that Z is compact. I will prove that the image of Z under T is a subset of Z, implying the existence of a sequence z converging to ν, which constitutes in turn an equilibrium. In Gil Martín (2001, Proposition 2.5), it is shown that $\Psi_{z_\nu}\lambda$ converges weakly to μ_ν, being μ_ν the equilibrium distribution of wealth corresponding to the equilibrium ν.

Appendix C: Proof of Proposition 5

Let $m_{\sigma'}, w'$ be the optimal money rule and wage associated with σ' and m_σ, w the same associated with σ. Then,

$$\lim_{\sigma\to\beta}\lim_{\sigma'\to\sigma}\frac{\sigma\left(w'-w\right)+\left(\sigma'-\sigma\right)w'}{\int\left(m_{\sigma'}-m_\sigma\right)dP}=0.$$

The proof is straightforward from the fact that

$$\lim_{\sigma\to\beta}\int m_\sigma dP=\infty.$$

As discussed in Section 3, this convergence is monotonic (the key assumption is A2). As a consequence,

$$\lim_{\sigma\to\beta}\lim_{\sigma'\to\sigma}\frac{\sigma'-\sigma}{\int\left(m_{\sigma'}-m_\sigma\right)dP}=0.$$

Proving the lemma, then, amounts to showing that

$$\lim_{\sigma'\to\sigma}\frac{\sigma'-\sigma}{w'-w}$$

is bounded away by zero. But this is clear from Proposition 4.

Lemma 14 *Assume u' is convex and that the equilibrium converges to a unique steady state. If at time t* $(1+f\prime(K_t))\geq 1$*; the transition path of accumulated capital is monotone from then onwards.*

Proof. The proof is similar to that Theorem 2 in Huggett (1995). ■

Proof of proposition 5

The proof proceeds in several steps. Firstly, we make use of the fact that, as the rate of injection of money equals the rate of discount, individual consumption differs from labor income by an arbitrarily small amount. Further, we already know that the golden rule level of consumption is strictly greater than the limit of the average consumption in steady state equilibrium as $\sigma\to\beta$. Moreover, as proven in Appendix B, the dependence on the parameter σ is continuous. Thus, from Proposition 4,

$$\lim\int k_\sigma d\mu_\sigma=k_\beta,$$

where the left hand side denotes the representative-agent steady state level of capita, namely the one which equals the real interest rate to the inverse of β. It follows that there exists a number $\varepsilon_1 > 0$ such that for $\sigma \in (\beta, \beta + \varepsilon_1)$,the average level of capital is increasing. Besides, since the golden rule level of consumption is strictly less than the one attained by[10] k_β, there should be another real number $\varepsilon_2 > 0$ for which consumption increases in capital. By Proposition B.1, there exists an invariant distribution of states which constitutes the equilibrium of the economy for every value of σ; and therefore an invariant distribution for consumption, money, capital and any other measurable function in suppμ_σ.

Let us define a sequence of mappings $\iota^x_{t+1} : R \to \Theta$ in the following way: for any initial condition λ households are assumed to perfectly foresee the path of inflation rates converging to σ. The optimal decision rule is determined by a sequence of functions a_{t+1} converging pointwise to a . Let the family of functions be defined recursively as follows:

$$\begin{aligned} N^\theta_1(x) &= a_1(x) \cdot z + \theta w_1 + \tau_1 \\ &\cdots \\ N^\theta_{t+1}(x) &= a_{t+1}(x) \cdot z + \theta w_{t+1} + \tau_{t+1} \end{aligned}$$

for any $t \geq 0$. It is taken for granted that such sequence depends on the initial condition. Now, each of the functions N^θ_{t+1} in turn defines a one-to-one correspondence between a level of wealth and a level of θ. In symbols, let define $\iota^x_{t+1}(x') = \theta$ if and only if $N^\theta_{t+1}(x) = x'$. The monotonicity of the policy function is passed on to ι^x_{t+1}. Further, as $a_t \to a_\sigma$; and sup(sup pμ_σ) is finite, then a fortiori $\lim N^\theta_{t+1}(x)$ is finite and independent of x. By construction, $\lim \iota^x_{t+1} = \iota$, a function independent of x; and by definition $\iota^x_{t+1}(x_\theta) = \theta$. The convergence is as well pointwise.

Moreover, by construction, for any sequence of $\mathcal{F}_t$-measurable, real valued functions f_t,

$$\int\int f_t d\lambda dP = \int\int f_t \circ \left(\iota^x_{t+1}\right)^{-1} d\lambda d\psi.$$

If f_t converges weakly to a function, say, f_0, independent of the initial state (as the functions relevant for our purpose), then

$$\begin{aligned} \int\int f_t d\lambda dP &\to \lim \int\int f_t \circ \left(\iota^x_{t+1}\right)^{-1} d\lambda d\psi = \\ \int\int f_0 \circ (\iota^x)^{-1} d\lambda d\psi &= \int d\lambda \int f_0 \circ (\iota^x)^{-1} d\psi = \int f_0 \circ (\iota^x)^{-1} d\psi \end{aligned}$$

[10] To see this point, note that the golden rule level of consumption maximizes, by definition, the average of consumption for a given level of capital (always in steady state). By concavity of production function, the level of capital satisfies the first order condition

$$f'(k^*) = 1;$$

whereas $f'(k_\beta) = \beta^{-1} > 1$: rearranging, the level of consumption is strictly increasing in k for a small enough interval containing k_β.

The last but one equality stems from the fact neither f_0 nor ι^x, as argued above, depend on the initial state. Consequently, for any sequence of $\mathcal{F}_t$-measurable real random variables f, there exists a sequence of transformations ι^x depending on the initial condition x, such that

$$\int\int fd\lambda dP = \left\{\int\int f_t \circ \left(\iota^x_{t+1}\right)^{-1} d\lambda d\psi\right\}.$$

Let first do the welfare analysis in steady state. The transform so obtained, induces an *equivalent class* of the form $\iota^{-1}(\theta)$. When $f_t = f$ for any t in steady state, the following equality holds:

$$\int\int fd\lambda dP = \int fd\lambda,$$

where λ denotes the invariant distribution of wealth. Alternatively, in steady state, any level of wealth belonging to the support of the steady state can be associated with a θ by means of ι. By construction,

$$\int fd\lambda = \int f \circ \iota^{-1} d\psi. \tag{15}$$

Let A be defined as $\{x \in R_+ : c_{\sigma+\varepsilon} - c_\sigma \geq 0\}$. Let f_θ denote $f \circ \iota^{-1}(\theta)$ for a generic $\lambda \otimes P$ measurable function f. Thus constructed, the equilibrium invariant distribution μ_σ defines a random variable $x_\sigma : \Omega \to \sup \mathrm{p}\mu_\sigma$ by

$$x^t_\sigma \rightharpoonup x_\sigma,$$

where $\rightharpoonup$ means weak convergence or limit in probability, and x^t_σ is obtained by recursion of (6). The limiting random variable is well-defined as established in Appendix B. For any measurable real valued function $f : R \to R$, the change of variable formula gives rise to the equality

$$\int fd\mu_\sigma = \int f_\theta d\psi.$$

Our claim is to show the existence of a sufficiently small real number $\varepsilon > 0$ such that for levels of inflation close enough to the inflation rate σ, $\sigma + \varepsilon$ leads to a higher expected value of welfare. Let us call ξ_0 the limiting distribution of a generic equilibrium variable for an arbitrary $\sigma \in (\beta, \beta + \min(\varepsilon_1, \varepsilon_2))$, and ξ_ε the same for a strictly greater level of inflation $\sigma + \varepsilon$. Likewise, throughout the proof, ρ_0, i_0 and $\rho_\varepsilon, i_\varepsilon$ shall denote, respectively, the gross real interest and nominal interest rate under regimes σ and $\sigma + \varepsilon$. By the reasoning undergone earlier, we know that

$$\int c_0 d\psi < \int c_\varepsilon d\psi.$$

The levels of inflation are bounded away from $\min(\varepsilon_1, \varepsilon_2) \equiv \varepsilon$. By (strict) concavity of the utility function,

$$u(c_\varepsilon) - u(c_0) > u'(c_\varepsilon)(c_\varepsilon - c_0). \tag{16}$$

By Euler equation (the one that corresponds to capital), the last equation, together with Theorem 8.3 in Clarida(1987) yields

$$u(c_\varepsilon) - u(c_0) > \beta\rho_\varepsilon \left(\int u'(c_\varepsilon) d\psi \right) (c_\varepsilon - c_0). \tag{17}$$

The proof of the first part of the proposition then amounts to showing that there exists a level of inflation (close enough to the rate of discount), and a number $\varepsilon > 0$ such that

$$\int [u(c_\varepsilon) - u(c_0)]\, d\psi > \int [\varphi(c_\varepsilon) - \varphi(c_0)]\, d\psi.$$

By monotonicity of the consumption plan, the random variable $(\mu_\sigma \otimes P) \circ c_\sigma^{-1}$ has range $C_\sigma = [c_\sigma(a); c_\sigma(b)]$; for some elements of Ω, a and b having an infinite number and converging to $\underline{\theta}$, and $\overline{\theta}$, respectively. The mapping $c_\sigma \circ \iota_\sigma^{-1}$ is monotonic in θ due to the monotonicity of ι_σ^{-1} in θ. Besides, the following identity holds:

$$c_\sigma \circ \iota_\sigma^{-1}(\theta) = \theta w_\sigma + r_\sigma k_\sigma \circ \iota_\sigma^{-1}(\theta) + \frac{1-\sigma}{\sigma}\tau_\sigma.$$

Since consumers are borrowing constrained so long as is close enough to ; it follows that

$$\begin{aligned} c_\sigma \circ \iota_\sigma^{-1}(\theta) &= (\theta - \underline{\theta})\, w_\sigma, \\ c_\sigma \circ \iota_\sigma^{-1}(\overline{\theta}) &> \overline{\theta} \end{aligned}$$

when the interest rate paid on money is positive ($\sigma < 1$). The tangent of the mapping $c_\sigma \circ \iota_\sigma^{-1}$ decreases globally in σ when the inflation rate is chosen to be close enough to the discount factor. Since, as discussed above, aggregate capital is locally increasing in inflation and the wage rate is in turn an increasing function of aggregate capital, there will exist a unique θ_0 such that $c_0 \circ \iota_0^{-1}(\theta_0) = c_\varepsilon \circ \iota_\varepsilon^{-1}(\theta_0)$. Hence, the set A above defined as

$$A = \bigcup_{\theta_0 \leq \theta} \iota_0^{-1}(\theta) = \bigcup_{\theta_0 \leq \theta} \iota_\varepsilon^{-1}(\theta),$$

implying

$$\int u'(c_\sigma) dP \leq \int u'(c_{\sigma+\varepsilon}(x,s)P(ds), \ \forall x \in A.$$

By integration of (16) and equation (17) we get to

$$\begin{aligned} & \int \left[\int u(c_{\sigma+\varepsilon}) d\mu_{\sigma+\varepsilon} - \int u(c_\sigma) d\mu_\sigma \right] dP \\ = & \int [u(c_\varepsilon) - u(c_0)]\, d\psi \\ > & \int u'(c_\varepsilon)(c_\varepsilon - c_0) d\psi \\ \geq & \int u'(c_\varepsilon) d\psi \int (c_\varepsilon - c_0) d\psi. \end{aligned} \tag{18}$$

The first equality comes out at once of (15). The first inequality is a consequence of the fact that the welfare gain first order dominates the welfare lose. Anew, proceeding similarly with the Euler equation corresponding to money, exploiting the fact that φ is strictly concave, we obtain

$$\varphi(m_0) - \varphi(m_\varepsilon) > \varphi'(m_\varepsilon)(m_0 - m_\varepsilon).$$

Let us define J_ε to be the excess of contemporaneous marginal transaction services of money over the marginal utility of consumption multiplied by the relative cost of holding money under the policy regime $\sigma + \varepsilon$. By Euler equation, J_ε differs from zero if and only if the household happens to be borrowing constrained. That is,

$$J_\varepsilon = \varphi'(m_\varepsilon) - \frac{i_\varepsilon}{\beta\rho_\varepsilon} u'(c_\varepsilon)$$

which is finite as far as $\underline{\theta} > 0$. By the Euler equations,

$$\varphi(m_0) - \varphi(m_\varepsilon) < \left[\frac{i_\varepsilon}{\beta\rho_\varepsilon} u'(c_\varepsilon) + J_\varepsilon\right](m_0 - m_\varepsilon). \tag{19}$$

On the other hand, note that, due to the fact that $u'(0) = \infty$, consumers need to hold, in steady state, an amount of real balances that does not differ from the average by a quantity

$$\frac{\sigma}{1-\sigma}\underline{\theta} w_\sigma$$

(see 8). Note that for any random variables X, Y defined over the same probability space, and provided their two first moments exist,

$$EXY = EXEY + \operatorname{cov}(X, Y)$$

where cov indicates the covariance operator. If $|Y - EY| \le \gamma$, then we can write the inequality

$$EXY \le EXEY + \gamma \sup |X - EX|$$

Taking expectations on (19), applying the last formula, and considering the upper bound (8) on money holdings, we get to

$$\begin{aligned}
& \int\int \left[\varphi(m_\sigma) d\mu_\sigma - \varphi(m_{\sigma+\varepsilon}) d\mu_{\sigma+\varepsilon}\right] dP \\
= \; & \int \left[\varphi(m_0) - \varphi(m_\varepsilon)\right] d\psi \qquad (20) \\
< \; & \frac{i_\varepsilon}{\beta\rho_\varepsilon} \int (m_0 - m_\varepsilon) d\psi \int u'(c_\varepsilon) d\psi + \\
& \frac{i_\varepsilon}{\beta\rho_\varepsilon} \frac{\underline{\theta}\,|w_0 - w_\varepsilon|}{1-\sigma} \sup_{\omega\in\Omega} \left| u'(c_\varepsilon) - \int u'(c_\varepsilon) d\psi \right| + \\
& \int J_\varepsilon (m_0 - m_\varepsilon) d\psi \qquad (21)
\end{aligned}$$

for σ close enough to β and ε close enough to zero. Next, I shall prove that there exists a number small enough for which the last two summands of the right-hand side of the last equation are negligible. Let

$$\begin{aligned} \gamma_f &= \sup_{\omega\in\Omega} \left| f'(m_\varepsilon) - \int f'(m_\varepsilon) d\psi \right| \\ f &= u, \varphi \end{aligned}$$

For the third summand of the right-hand side we have

$$\begin{aligned} & \frac{\int J_\varepsilon (m_0 - m_\varepsilon) d\psi}{\int (m_0 - m_\varepsilon) d\psi} \\ = & \frac{\int \varphi(m_\varepsilon)(m_\varepsilon - m_0) d\psi}{\int (m_0 - m_\varepsilon) d\psi} - \frac{i_\varepsilon}{\beta\rho_\varepsilon} \frac{\int u'(c_\varepsilon)(m_0 - m_\varepsilon) d\psi}{\int (m_0 - m_\varepsilon) d\psi} \\ < & \frac{\int \varphi(m_\varepsilon) d\psi \int (m_\varepsilon - m_0) d\psi}{\int (m_0 - m_\varepsilon) d\psi} + \\ & \frac{i_\varepsilon}{\beta\rho_\varepsilon} \frac{\int u'(c_\varepsilon) d\psi \int (m_0 - m_\varepsilon) d\psi}{\int (m_0 - m_\varepsilon) d\psi} + \\ & \frac{\underline{\theta} i_\varepsilon \left| w_0 - w_\varepsilon \right|}{\beta (1-\beta) \rho_\varepsilon} \frac{\gamma_u + \gamma_\varphi}{\int (m_0 - m_\varepsilon) d\psi}. \end{aligned}$$

Since the nominal interest rate and $\int \varphi(m_\varepsilon) d\psi$ tend to zero as ε tends to zero, the last expression tends to zero so long as

$$\lim_{\sigma\to\beta} \lim_{\varepsilon\to 0} \frac{\sup_{\omega\in\Omega} \left| u'(c_\varepsilon) - \int u'(c_\varepsilon) d\psi \right|}{\int (m_0 - m_\varepsilon) d\psi} = 0. \tag{22}$$

But this equality holds trivially once we make use of Lemma 1. To demonstrate it, it suffices to divide the numerator and denominator of the right- hand side of the last expression by and make use of Lemma 1. Likewise, the second term in the right-hand side of (21) is dominated asymptotically by the first one. Proving this fact indeed amounts to (22). Rearranging (18) and (21), a sufficient condition in order for the existence of an sufficiently small ε such that $W_\varepsilon > W_0$ is

$$\beta\rho_\varepsilon \int (c_\varepsilon - c_0) d\psi > i_\varepsilon \int (m_0 - m_\varepsilon) d\psi.$$

It amounts to showing that

$$\lim_{\sigma\to\beta} \lim_{\varepsilon\to 0} \frac{i_\varepsilon}{\varepsilon} = 0, \tag{23}$$

for we know, from a previous reasoning made earlier, that

$$\lim_{\sigma\to\beta} \lim_{\varepsilon\to 0} \frac{\int (c_\varepsilon - c_0) d\psi}{\varepsilon} > 0.$$

A sufficient condition for this (23) to hold is that

$$\lim_{\sigma \to \beta} \lim_{\varepsilon \to 0} i_\varepsilon \int m_\varepsilon d\psi = 0. \tag{24}$$

The last equality and the fact that

$$\lim_{\sigma \to \beta} \lim_{\varepsilon \to 0} \varepsilon^{-1} \int m_\varepsilon d\psi = \infty,$$

(proven on the basis that $\int \int m_\sigma d\lambda dP$ diverges to infinite as σ approaches the rate of discount in the way shown diagrammatically in Figure 1). Then (24) in turn reduces to (23), as the speed of convergence to zero of money interest dominates over that of money. In order to prove the statement (23), I make use of the assumption (2). It implies

$$\lim_{m \to \infty} \varphi'(m) m = 0.$$

Thus, making anew use of the Euler equation, and the fact that the set of borrowing-constrained individuals becomes of P-zero measure as the inflation rate collapses to the discount factor,

$$\lim_{\sigma \to \beta} \lim_{\varepsilon \to 0} \left[\varphi'(m_\varepsilon) m_\varepsilon - \frac{i_\varepsilon}{\beta \rho_\varepsilon} u'(c_\varepsilon) m_\varepsilon \right] = 0.$$

Of course, ρ_ε and $u'(c_\varepsilon)$ are finite magnitudes: accordingly, (24) holds, as we wanted to show.

A parallel reasoning can be made to demonstrate the second part of this proposition, namely that $W_\sigma^{\mu_{\beta+\varepsilon}}$ is increasing in σ for a given interval. The second part of this proposition establishes that, starting from a steady state characterized by an inflation rate σ, welfare would be improved by increasing expected inflation. We start by showing that consumption is increasing, from the first period after the implementation of the new regime onwards. Recall the notation for a non stationary equilibrium as $E_\sigma^\lambda = \left(a_\sigma^\lambda, v_\sigma^\lambda, \mu_\sigma^\lambda\right)$. As usual, we shall omit the dependence on the parameter σ when it is no risk of confusion. Jensen inequality and the assumption convexity of marginal utility of consumption (2) guarantee that

$$\int u'(c_t) d\psi > u' \left(\int c_{t+1} d\psi \right).$$

If ε is small enough, $\beta \rho_\varepsilon$ is close enough to one, implying

$$u'(c_t) \geq u' \left(\int c_{t+1} d\psi \right).$$

Note that both sides of the inequality depend on the level of wealth x. Integrating both sides of the inequality,

$$c_t \leq \int c_{t+1} d\psi,$$

and applying Theorem 8.3 in Stokey and Lucas (1989), a version of the iterated rule of conditional expectations,

$$\int c_t d\mu_t \leq \int c_{t+1} dd\mu_{t+1}.$$

Namely, the sequence of consumption increases *monotonically* to the steady state. This assertion indeed implies that aggregate consumption in the first period is strictly less than that of period zero. However this decrease in consumption, ex-ante (at time zero, immediately after implementing the new rule) one-period ahead instantaneous utility increases, due to a redistribution of wealth which provokes a gain in terms of utility that overwhelms the losses. To show this point, I need to introduce some notation. Let b_t be the excess of money holdings with respect to the average and b_0 the same variable for the steady state characterized by . As pointed out above, under the FR, there is barely any discrepancy between consumption and labor income. From the definition of the function ι_0, it is clear that consumption in period t may be rewritten as a function $c_t^{\varepsilon} : \Theta \times \Theta^t \to R_+$. For period 1, $c_1^{\varepsilon}(\underline{\theta}, \underline{\theta})$ turns out to be strictly greater than $\underline{\theta}$ because the previous plan characterized by consuming $\underline{\theta}$ is attainable and there exists a sequence of sufficiently small numbers $\{\eta_t\}_{t=1}^{\infty}$ (function of $\underline{\theta}$ and ε among other variables) such that $\underline{b}_t + \eta_t = \underline{b}_0$, $\underline{b}_t$ being the infimum of money holdings among every possible realizations. Let us define $\underline{c}_t^{\varepsilon}$ and c_0 likewise. Let $\psi^{(t)}$ be defined as the product measure of ψ on itself t times. Let h be a operator defined on Θ defined in this way: $h(\theta) = \theta'$ if and only if the following equation holds:

$$\int u\left(c_1^{\varepsilon}(\theta, s)\right) \psi(ds) = u(c_0 \circ \iota_0^{-1}(\theta')).$$

Since the divergence of $c_0 \circ \iota_0^{-1}$ from identity diverges (in the supremum norm) by an arbitrary small number, the mapping h (which is parametrized by ε) has a unique fixed point lying in the interior of Θ for the same reasons pointed out in the first part of the proof. By continuity and monotonicity, there is a unique fixed point, say $\theta_{\varepsilon} > \underline{\theta}$. Any θ in the interval $\Gamma_1 = [\underline{\theta}, \theta_{\varepsilon}]$ satisfies

$$\int u\left(c_1^{\varepsilon}(\theta, s)\right) \psi(ds) > u(c_0 \circ \iota_0^{-1}(\theta)).$$

A similar argument leads to the existence of a family of non empty measurable subsets Γ_t such that, for any $\theta \in \Gamma_t$,

$$\int_{\Theta^t} u\left(c_t^{\varepsilon}(\theta, s)\right) \psi^{(t)}(ds) > u(c_0 \circ \iota_0^{-1}(\theta)).$$

Let $\Gamma = \cap \Gamma_t$. It is non empty because, as the capital is an increasing sequence, so it is the sequence of the wage rate. Hence, there exists an attainable plan of consumption consisting of holding an amount of money holdings equaling the average $b = 0$, and consuming the labor income. This plan gives a period by period utility strictly greater than the one achieved by the optimal plan under the FR. The proposition is concluded once we show that

$$\lim_{t\to\infty}\int\int u(c_t^{\varepsilon})d\psi\otimes d\psi^{(t)} > u(c_0\circ\iota_0^{-1}(\theta)).$$

Note that, from the fundamental theorem of calculus,

$$\int_{\Theta^t} u\left(c_t^{\varepsilon}\left(\theta,s\right)\right)\psi^{(t)}(ds) - u(c_0\circ\iota_0^{-1}(\theta)) = \int\left[\int_{c_0\circ\iota_0^{-1}(\theta)}^{c_t^{\varepsilon}(\theta,s)} u'(q)dq\right]\psi^{(t)}(ds)$$

for any t. Define a measurable mapping $\ell:\Gamma\to\Gamma^c$where Γ^c is the complement of Γ such that the mapping ℓ is mean-preserving as of period 1; namely

$$\int\int_{\Gamma\times\Theta} c_1^{\varepsilon}d\psi^{(2)} = \int\int_{\ell(\Gamma)\times\Theta} c_1^{\varepsilon}d\psi^{(2)}.$$

The function ℓ has the additional properties (not essential yet facilitating the remaining analysis):

1. ℓ is decreasing and continuous
2. If $\theta\in\ell(\Gamma)$, $\theta'\in\ell(\Gamma)$ for any $\theta'<\theta$.

Clearly, since marginal utility is strictly decreasing,

$$\int\left[\int_{c_0(\theta)}^{c_1^{\varepsilon}(\theta,\theta')} u'(q)dq\right]\psi(d\theta') > \int_{\ell(\Gamma)}\left[\int_{c_0(\theta)}^{c_1^{\varepsilon}(\theta,\theta')} u'(q)dq\right]\psi(d\theta'),$$

or

$$\begin{aligned}&\int_{\Gamma\times\Theta}\left[u\left(c_1^{\varepsilon}\left(\theta,\theta'\right)\right)-u(c_0\circ\iota_0^{-1}(\theta)\right]d\psi^{(2)}\\ > &\int_{\ell(\Gamma)\times\Theta}\left[u\left(c_1^{\varepsilon}\left(\theta,\theta'\right)\right)-u(c_0\circ\iota_0^{-1}(\theta)\right]d\psi^{(2)}.\end{aligned}$$

Next, we shall argue that long-run utility of agents belonging to $\Gamma^c\cup\ell(\Gamma)^c$ increases, despite reducing their consumption at time 0. The reason is that the previous plan continues to be optimal, since we chose to be close enough to β as to make the discrepancy of money holdings with respect to the average negligible. From Lemma the capital stock is increasing and converging to $\int k_\varepsilon d\psi > \int k_0 d\psi$. As the initial condition allows households to maintain b up to average, and capital (and wage) grows with σ,

$$c_t^{\varepsilon}\geq\underline{\theta}w_t^{\varepsilon}>\underline{\theta}w_0.$$

Since w_0 differs from c_0 by an arbitrary small amount independent of ε, it must be the case that

$$\lim_{t\to\infty}\int\int u(c_t^{\varepsilon})d\psi\otimes d\psi^{(t)} > u(c_0\circ\iota_0^{-1}(\theta)).$$

The proposition is then concluded, because then consumption is reduced at the expense of increasing b with the prospective of pooling risk, but their long-run utility cannot be reduced.

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