The Macrodynamics of Growing Open Economies under Financial Convex Adjustment Costs: Critical Dynamical Thresholds and Endogenous Cycles

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Abstract: *Departing from the basic optimal control setup for growing open economies in continuous time, we put forward one main hypothesis: Is the macrodynamics structure of open economies governed by endogenous growth and cycles? To tackle this hypothesis, we strip off the classical open economy problem from all institutional frameworks and evaluate our hypothesis assuming non linear convex financial adjustment in the form of risk premia on foreign holdings and investment adjustment costs on domestic capital. By imposing always dynamical rules for adjustment, we are able to obtain meaningful dynamical systems that confirm our hypothesis in a conceptual level and pave the route for the further introduction of endogenous cycles and growth dynamics in mainstream economic growth theory.*

Keywords: Macrodynamics of open economies, economic growth, risk premium on foreign assets/debt, investment adjustment costs, endogenous cycles, Hopf bifurcations
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1. Introduction

In this paper, we depart from the basic Romer (1986) hypothesis for long run endogenous growth in a standard optimal control framework, following the tradition of Ramsey (1928), Cass (1965) and Koopmans (1965) neo-classical long run growth model of optimal savings and capital accumulation, and put forward the hypothesis that long run structural growth in open economies will coexist with endogenous structural cycles, when assuming convex financial adjustment costs for domestic capital investment and holdings of foreign debt/assets. Our research proposal tackles this hypothesis by following a three step procedure. First, we put forward the fundamental optimal control problem for an open growing economy *a la Romer*, assuming there are risk premia on holdings of foreign assets/debt and investment adjustment costs on domestic capital, following the literature strains started by Bardhan (1967) and Hayashi (1982), respectively, and discuss the relevant functional forms and parameter restrictions for our specific applied proposals. Two applied examples of this style of formalization, very similar to our proposal in recent nonlinear macrodynamics policy oriented literature are the papers by Turnovsky (2002) and Eicher, Schubert and Turnovsky (2008), which carefully describe the literature background on nonlinear dynamic modelling of open economies in continuous time. Then, we show how it is possible to define our problem as a dynamical system in the control and state variables, assuming dynamic adjustment rules for the *Keynes-Ramsey* intertemporal consumption equations and a simple scaling procedure, for each specific convex adjustment hypothesis individually first and finally the complete adjustment case. Finally, we discuss analytically and numerically each of our three proposals with the purpose of determining the main qualitative dynamics on the phase space and identify possible bifurcation regions with economic meaning that confirm our hypothesis of growth and cycles coexistence. While we are able to confirm the existence of such regions analytically, when assuming risk premia convex adjustment in holdings of foreign assets/debt for a state dependent switching control dynamical system, we discard such a hypothesis for the convex investment adjustment costs case, where both long run growth and transitions are only possible when assuming specific endogenous parameter combinations. To conclude, we show how the introduction of both convex adjustment hypotheses, following the complete optimal control problem, is entirely represented by a three dimensional dynamical system, when assuming only our set of *a priori* modelling rules, and, with the aid of numerical methods introduce the hypothesis of further complex dynamics arising from our nonlinear modelling proposal.

To conclude this short introduction to our proposal, we will take this paragraph to stress the importance of our idea in the vast field of macroeconomic dynamics. Contemporary research on macroeconomic policy has undergone an important shift in its methodology, in order to deal with issues related to economic openness. This additional economic dimension imposes diverse challenges to modelling. One such challenge is the impact of broad access to international capital markets through the domestic financial sector, which has become a crucial theme of mainstream macroeconomic policy discussions since industrialized nations suffered the dire effects of excessive private debt to finance over evaluated domestic asset investment and the consequent macroeconomic imbalances, and boom/bust dynamics it implied. This issue, however, did not end with bad investment decisions by households; in a few months the problems spread to the payment system and the vast majority of financial sector firms suffered the consequences of their wrong decisions, as bad debt contracts circulated between financial institutions in the complex and interconnected world of international financial markets, leading eventually to a series of bankruptcies in the banking system and the fears of a systemic crisis leading to a collapse of the global payment system. Much speculation on the reasons that led to these outcomes arose and the usual suspects of bad regulation, corporate and political malpractice and irresponsibility, rapidly become the number one targets of mainstream commentators. However, the issue of global imbalances leading to boom/bust dynamics at a national and regional level had already become a hot topic in macroeconomic dynamics research, since the consecutive financial crises in Mexico, 1994, East Asia, 1997 and Russia, 1998. Research led on this topic suggested that in the root of the balance of payments problems faced by these economies was the financial openness of domestic capital markets. The conclusion was that openness contributes to growth by increasing the investment opportunities of a developing economy, but increased the volatility of growth dynamics. Some of the theoretical literature in the field, such as Aghion, Bacchetta and Banerjee (2004), even suggests higher amplitude endogenous cycles during the catching up transition period. The expansion of globalization and consequent growth of global commercial and financial links between nations, in a growing world during the last decade, just led to the inevitable consequence that all nations can now suffer the worst consequences of globalization and not only the ones still in its development stage. Theory reflected this outcome by ending the distinction between small open economies and relatively closed big economies. No economy is nowadays big enough to minimize the effects of global financial imbalances. We suspect that the distinction between developed and developing economies will soon be substituted by terms that reflect an economy's international financial position instead. Following this introduction on the implications of openness leading to global financial imbalances, it becomes clear that modelling economies, where access to international financial markets is total and specific structural nonlinear implications, resulting in more complex dynamics than the usual steady-state economics, such as endogenous cycles and growth, lies on the heart of the problem of defining the correct conceptual dynamical causes and implications of this phenomenon. Following this path has a greater probability of establishing the correct policy tools and targets for economic policymaking to tackle this problem in the future, rather than imposing solutions based on the contemporary charges that the usual suspects are the sole scapegoats responsible for this global economic problem.

2. *Non-linear dynamics and endogenous cycles in macrodynamics literature*

We start this brief literature review on the topic of non-linear growth dynamics and endogenous cycles by putting forward two book references, Barro and Sala-i-Martin (2004) and Gandolfo (1996)[2], which will allow readers not familiar with this field to get started in some of the economic growth topics and modelling issues discussed throughout this paper. From here on, we shall distinguish between papers that directly tackle the theoretical modelling framework for endogenous growth and cycles and the papers that propose specific applications in this specific field but assuming particular extensions of endogenous growth modelling methodology. Since we are of the opinion that our proposal is closer to the first group, we shall give particular attention to this literature strain and some few examples of specific applications, in order to complete our short review. The papers by Baumol and Benhabib (1989) deal mainly with both the economic interpretation and mathematical implications of nonlinear dynamics for economic modelling, but their scope is not limited to endogenous growth theory. On the specific field of macrodynamics, in particular the research dealing with the main modelling outcomes arising from the base endogenous growth setup, such as cycles, a variety of papers defined the field during the past decade. This literature strain is usually defined in macroeconomics, as indeterminacy and sunspot equilibria outcomes, following the intuition that an economic optimum should be reached in a formal dynamic economic program and further dynamical issues are considered as mathematical outcomes with limited economic interpretation[3], therefore secondary to analysis. The literature we present in this section argues that such outcomes should not be interpreted as marginal to macroeconomics and develops the endogenous cycle theory hypothesis, departing from classical endogenous economic growth hypothesis. Examples of this early literature are the papers by Lordon (1995), Greiner and Semmler (1996), Greiner (1996), Drugeon (1998), Benhabib and Nishimura (1998) and Asada, Semmler and Novak (1998). For more recent proposals on the subject, we highlight the papers by Wirl (2002), Nishimura and Shigoka (2006) and Slobodyan (2007) following the same base assumptions on endogenous growth theory and cycles.

As referred to in the previous paragraph, a wide range of applied extensions has gained a relevant place in modern macrodynamics literature, following the recent advances discussed for the base endogenous growth modelling setup and its nonlinear dynamic implications. These extensions include, among others, the early work by Matsuyama (1991) linking the Romer (1986) increasing returns hypothesis with indeterminacy of equilibrium in a model of industrialization. More recent applications include the proposal by Flashel (2000), which deals with insider-outsider effects in the labour market, Boucekkine et al (2005), dealing with vintage capital endogenous growth models, Slobodyan (2005), proposal on development and poverty traps, and finally, Greiner (2008) model with human capital financed by public expenditures.

3. *General optimal control problem for a centralized open economy with convex adjustment costs*[4]

The main objective of this paper is to assess analytically and numerically the dynamics of a standard infinite horizon optimization problem for an open growing economy. Departing from the simple endogenous growth assumption defined

[2]Lorenz (1993) provides a good introduction to modelling of complex nonlinear dynamic economics but its scope is wider and not restricted to the macrodynamics field.
[3]Or as it is usually described in economics as *Self Fulfilling Prophecies* or existence of *Sunspots* that enable an economy to be better off than its current state.
[4]For reasons of simplification, we discard the use of the time subscript in the time varying variables of our model. The meaningful variables are consumption, $C(t)$, investment on domestic capital, $I(t)$, domestic capital accumulation, $K(t)$, and foreign debt/assets accumulation, $B(t)$.

by Romer (1986), the extended benchmark optimization problem for an open economy can be defined by (1), in the centralized case, where we assume the existence of convex adjustment costs for foreign debt/assets, $rB\cdot\Psi\left(B,K\right)$, and investment, $I\cdot\Phi\left(I,K\right)$, following the usual definition for domestic capital, K, foreign debt, B, and investment in domestic capital, I. In this framework no policy functions and/or parameters are considered, except for those directly related to convex adjustment dynamics. This optimization problem can be described as a central planner consumption and investment optimal control problem, restricted by the physical constraint on domestic capital accumulation, which should be interpreted as a broad measure of capital, and the open economy intertemporal financial constraint, where access to foreign financial resources for domestic consumption and investment activities is perfect. We follow this formalization because we are only interested on characterizing the dynamics of convex adjustment in a conceptual model for an open economy. Further assumptions on the functional forms describing utility, output and convex adjustment follow in section *3.1.*. The remaining parameters are ρ, r and δ, which refer to the intertemporal discount factor, international interest rate and capital depreciation rate, respectively.

$$\begin{aligned} &\underset{C,I}{MAX}\ \ U=\int_0^{\infty}U\left(C\right)e^{-\rho t}dt \\ &\textit{subject to the solution of}\ : \\ &\begin{cases}\dot{B}=C+I\Phi\left(I,K\right)+rB\Psi\left(B,K\right)-Y\left(K\right)\\ \dot{K}=I-\delta K\end{cases}\end{aligned} \tag{1}$$

[5]

The present value Hamiltonian for this optimization problem is:

$$H^{*}=U\left(C\right)+q_1\left(I-\delta K\ \right)+q_2\left(C+I\Phi\left(I,K\right)+rB\Psi\left(B,K\right)-Y\left(K\right)\right)$$

The *Pontryagin* maximum necessary conditions for this optimal control problem are[6]:

Optimality Conditions

$$U_c^{'}\left(C\right)+q_2=0 \tag{2}$$

$$q_2\left[\Phi\left(I,K\right)+I\Phi_I^{'}\left(I,K\right)\right]+q_1=0 \tag{3}$$

Admissibility Conditions

$$B_0=B_{(0)}\ \ ,\ K_0=K_{(0)}$$

Multipliers Conditions

$$\dot{q}_2=q_2\left[\rho-r\Psi\left(B,K\right)-rB\Psi_B^{'}\left(B,K\right)\right] \tag{4}$$

$$\dot{q}_1=q_1\left(\rho+\delta\right)-q_2\left[rB\Psi_K^{'}\left(B,K\right)+I\Phi_K^{'}\left(I,K\right)-Y_K^{'}\left(K\right)\right] \tag{5}$$

State Conditions

$$\dot{B}=C+I\Phi\left(I,K\right)+rB\Psi\left(B,K\right)-Y\left(K\right) \tag{6}$$

$$\dot{K}=I-\delta K \tag{7}$$

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[5]In economic optimal control problems it is usual to include the necessary condition of strict concavity of the present value *Hamiltonian* in relation to the controls, known by the *Arrow necessary second order conditions for the existence of an optimum,* which is always guaranteed to exist for solutions defined by the transversality conditions (8) and (9). We discard from this presentation this discussion, because it is not clear the definition of similar conditions for endogenous growth problems with more than one state variable, as the solution of such optimal control problems is defined by the existence of steady-state solutions under specific scaling rules that guarantee the transversality conditions for optimum are fulfilled. For example, in endogenous growth models with two state variables and endogenous technical change, in the Uzawa (1965) and Lucas (1988) fashion, it is usual to impose distinct scaling rules and allow for different growth rates for the variables, in order to obtain solutions that fulfil the *Pontryagin* maximum conditions.

[6]As usual in optimal control economic modelling proposals, we shall consider transversality conditions as relevant parameter restrictions that allow for economic feasible outcomes arising from analytical and numerical analysis of the proposed dynamical systems. Loosely, we can define both transversality conditions in endogenous growth theory as follows: (a) condition (8) states that Ponzi games are not feasible in this economy, in other words, growth cannot be based on debt accumulation; and (b) condition (9) states that we shall only accept growth dynamics with at most balanced growth paths and increasing growth dynamics are not possible.

Transversality Conditions

$$\lim_{t\to\infty} q_2 B e^{-\rho t} = 0 \quad (8)$$

$$\lim_{t\to\infty} q_1 K e^{-\rho t} = 0 \quad (9)$$

3.1. Relevant functional forms

Utility Function

We follow the usual concavity assumption for the utility function, $U(C)$, where $U_c^{'}(C) \succ 0$ and $U_c^{''}(C) \prec 0$. A typical formulation is given by the family of strictly concave isoelastic utility functions with constant intertemporal elasticity of substitution, γ, such as the functional form for aggregate consumption given by equation (10):

$$U(C) = C^y \Rightarrow -\frac{U_c^{'}(C)}{U_c^{''}(C)} = -\frac{C}{y-1} \succ 0,\ where\ 0 \prec y \prec 1 \quad (10)$$

Output function

The output function, $Y(K)$, follows the usual increasing returns hypothesis from endogenous growth theory, originally proposed in Romer (1986), where $Y_K^{'}(K) \succ 0$ and $Y_K^{''}(K) = 0$. We shall assume in this paper the classic Romer (1986) proposal for increasing returns and long run growth and compare it with the neoclassical hypothesis, where A, the meaningful parameter, refers to the exogenous technology parameter.

$$Y(K) = AK,\ \begin{cases} A \succ 1\ ,\ Increasing\ Returns\ (Romer) \\ 0 \prec A \prec 1\ ,\ Constant\ Returns \end{cases} \quad (11)$$

Foreign debt/assets convex adjustment assuming exogenous and endogenous risk premium on holdings

The extended expression $rB \cdot \Psi(B,K)$ follows the original proposal by Bardhan (1967), but is here extended to accommodate the possibility of convex adjustment of both debt and foreign assets accumulation. Our formulation follows the hypothesis that there exists a risk premium associated with the borrowing of foreign debt or the holding of foreign assets. The risk premium depends on an institutional parameter, d, which represents the exogenous level of premium that investors place in the dynamic measure of a country foreign balance, given by the ratio of holdings of foreign debt/assets to domestic capital. We consider that the exogenous risk premium is also dependent on the level of the international interest rate, which holds a level of exogenous spread on foreign holdings defined by rd.

$$\Psi(B,K) = 1 + \frac{d}{2}\frac{B}{K},\ d \succ 0 \quad (12)$$

Investment convex adjustment costs

The extended expression $I \cdot \Phi(I,K)$ is an application of the familiar Hayashi (1982) cost of adjustment framework, where we assume that the adjustment costs, quantified by the institutional parameter h, are proportional to the rate of investment per unit of installed capital. The introduction of investment adjustment costs on the open economy budget constraint is a classical formalization in endogenous growth economics to impose regions of convergent transitions to long run steady states defined by balanced growth paths of the type, $X_t = \tilde{x}_t e^{\Omega_x t} \Rightarrow \dot{X}_t = \dot{\tilde{x}}_t e^{\Omega_x t} + \Omega_x \tilde{x}_t e^{\Omega_x t}$, where Ω is usually assumed to be common to all variables and define an endogenous growth rate dependent on model parameters. The introduction of investment adjustment costs in a open economy endogenous growth model context has the following interpretation: (i) if $h \prec 0$ then conditions impose a home bias on investment in domestic capital, (ii) if $h \succ 0$ then conditions impose a bias on investment in foreign assets.

$$\Phi(I,K) = 1 + \frac{h}{2}\frac{I}{K} \quad (13)$$

4. ***Dynamic analysis of an open economy facing risk premia on foreign debt/assets: A credit/debit card economy***

In this section, we shall assume that there are no convex adjustment costs in investment, $\Phi(I,K)=1$. This hypothesis yields changes to optimality condition (3), and in the co-state and state conditions (5) and (6), respectively, for the general optimal control problem defined in section *3.*. We state the new optimal control conditions:

$$q_2 + q_1 = 0 \tag{14}$$

$$\dot{q}_1 = q_1(\rho+\delta) - q_2\left[rB\Psi'_K(B,K) - Y'_K(K)\right] \tag{15}$$

$$\dot{B} = C + I + rB\Psi(B,K) - Y(K) \tag{16}$$

The strategy to solve this set of open economy growth systems follows the path of defining meaningful *Keynes-Ramsey* rules for the dynamics of consumption[7]. These rules are obtained, as usual in optimal control problems, by assuming optimality condition (2) and the respective time derivative and substituting in the co-state condition relating to the open economy budget constraint, (4). In this set of problems, the first assumption to be taken is that consumption is always available through foreign debt accumulation. This assumption yields the open economy *Keynes-Ramsey* consumption rule described in (16) for the consumption dynamics through foreign debt accumulation:

$$\dot{C}_B = \frac{U'_c(C)}{U''_c(C)}\left(\rho - r\Psi(B,K) - rB\Psi'_B(B,K)\right) \tag{17}$$

As we imposed an optimal control condition on domestic assets investment, in order to guarantee that there exists dynamic conditions that relate both asset accumulation decisions in an open economy setting. This condition, described by equation (14), can be used to define an alternative *Keynes-Ramsey* rule. By substituting (14) in (2) and taking the time derivative, we can use the co-state condition (15), associated with domestic assets accumulation, in order to define the second *Keynes-Ramsey* rule for consumption through domestic asset accumulation, assuming the optimality condition (14) holds always:

$$\dot{C}_K = \frac{U'_c(C)}{U''_c(C)}\left(\rho + \delta + rB\Psi'_K(B,K) - Y'_K(K)\right) \tag{18}$$

The path to follow from here is a tricky one, since we have two consumption rules and we are interested in defining a set of dynamical rules that define consumption decisions in the phase space. In order to do that, we start by defining the rule that guarantees indifference in asset accumulation from consumption decisions, $\dot{C}_B = \dot{C}_K$. This rule is:

$$\rho - r\Psi(B,K) - rB\Psi'_B(B,K) = \rho + \delta + rB\Psi'_K(B,K) - Y'_K(K) \tag{19}$$

By considering the functional forms defined in section *3.1.* we obtain the following rule for indifference in accumulation:

$$-r\left(1 + d\frac{B}{K}\right) = \delta - \frac{rd}{2}\left(\frac{B}{K}\right)^2 - A \tag{20}$$

In order to define a meaningful dynamical system that takes into account the indifference condition expressed in (20), we shall redefine our dynamical system assuming the following scaling rule, $Z_i = \frac{X_i}{K}$, where subscript *i* refers to the relevant scaled variables for this system, which are given by:

[7]By *Keynes-Ramsey* consumption rules, we mean the intertemporal dynamic consumption decisions that are obtained for the control variable in an optimal control problem with a constant intertemporal discount rate. In macroeconomics literature these dynamic equations are known by *Keynes-Ramsey* consumption rules, following the work by the two famous Cambridge scholars, which related intertemporal consumption decisions to the discounted value of expected future incomes and optimal savings for capital accumulation. However, it is our opinion that in open economy optimization problems with two state variables, this rule is not unique, since state defined income accumulation can vary in its source. Therefore, it is reasonable to impose two possible consumption paths that satisfy the optimal investment condition, (14), which for this model just states that the shadow price of domestic capital is equal to the marginal value of foreign assets, or in other words, optimal investment decisions must always fulfil the rule of equal intertemporal marginal adjustment for different assets, in order to allow for different accumulation decisions. The straightforward interpretation of this definition is that investors will always choose to accumulate assets that adjust faster to optimum outcomes rather than others that yield longer adjustment rates. In a market economy setup, this rule relates to investment decisions that are based on the discounted rate of capital return on investment, which are linked to the expected intertemporal discounted financial costs of investing between different assets.

$$Z_1=\frac{C}{K}\Rightarrow\dot{Z}_1=\frac{\dot{C}}{K}-\frac{C}{K}\frac{\dot{K}}{K},\ Z_2=\frac{B}{K}\Rightarrow\dot{Z}_2=\frac{\dot{B}}{K}-\frac{B}{K}\frac{\dot{K}}{K},\ \dot{Z}_3=\frac{\dot{K}}{K}=\frac{I}{K}-\delta \tag{21}$$

Assuming this scaling rule, we can define the indifference condition in the phase space for $\dot{Z}_1$ and $\dot{Z}_2$ as a threshold rule obtained from solving the quadratic equation (20), which sets boundaries for switching dynamics between the two relevant *Keynes-Ramsey* conditions (17) and (18). In order to define the meaningful dynamical system, we need to impose a final rule on the growth rate of capital accumulation. Since we have no meaningful information about investment dynamics, we shall assume that capital grows at a constant endogenous rate and the ratio of investment to capital is just a parameter, $\bar{Z}_4$. These set of assumptions are portrayed in (22):

$$\frac{I}{K}=\bar{Z}_4\succ\delta\Rightarrow K(t)=K(0)e^{(\bar{Z}_4-\delta)t} \tag{22}$$

The dynamics of this economy are given by a switching threshold system, where the relevant switching regions between the scaled differential equations obtained from (17) and (18) are defined by the solutions of (20), which we shall briefly describe as $Z_{2,-}^t$ and $Z_{2,+}^t$. The solution to this quadratic equation defines the two influence regions for each equation and is given below:

$$Z_2^t=1\pm\frac{\sqrt{(rd)^2+2rd(r+\delta-A)}}{rd} \tag{23}$$

On the other hand, it is also reasonable to consider that expression (23) does not define threshold influence regions, but the two possible equilibrium solutions for scaled debt/asset dynamics. Therefore, we can impose $\bar{Z}_2=Z_2^t$ and solve this optimal control problem of long run growth by imposing parameter restrictions in a classical endogenous growth fashion. This will lead to a solution where long run growth solutions are possible under specific parameter restrictions and transitions only exist if we consider discontinuous, instantaneous adjustment to exogenous shocks. We derive the existence of BGP solutions for this specific hypothesis of our optimal control problem in the first section of the appendix and put forward the meaningful parameter restrictions for the existence of economically feasible steady-states, along with some numerical results that demonstrate some possible BGP outcomes under reasonable parameter values. Still, as this hypothesis does not entail any interest to the dynamic perspective that we wish to pursue in this paper, we leave further discussion on this issue to another opportunity. Nevertheless, it is important to consider the solution presented in the appendix as a possible solution arising from a full information criteria on state adjustment, which may be compared in welfare terms with the switching threshold dynamic proposal we discuss thoroughly in the following paragraphs and subsections.

Recall from the threshold quadratic condition (20) that both the left and right hand side of the quadratic equation have to be negative due to the parameter restriction $\gamma\prec 1$. Therefore, we can use this result to define in the phase space the influence region described for this threshold system:

$$\begin{aligned}&\frac{rd}{2}(Z_2)^2-r(1+dZ_2)+A-\delta\prec 0\Rightarrow Z_{2,-}^t\prec Z_2^B\prec Z_{2,+}^t\\&\frac{rd}{2}(Z_2)^2-r(1+dZ_2)+A-\delta\succ 0\Rightarrow \dot{Z}_2^K\prec Z_{2,-}^t\wedge Z_2^K\succ Z_{2,+}^t\end{aligned} \tag{24}$$

We obtain the relevant expressions from (23), which define the switching phase space regions of influence, (24), for this system. We shall call these regions as the credit and debit threshold influence regions, which are defined by the following two dynamical systems with a common state condition and control switching state dependent dynamics obtained after the application of scaling rules (21) and (22). We shall further consider that dynamics on the threshold limits will be governed by either one of the systems, which will not imply further analytical assumptions since the threshold limits are not fixed points of this system. Nevertheless, for simulation purposes we shall assume an equal probability of decision, in order to portray the indifference in asset accumulation dynamics discussed previously, such rule is sufficient in our opinion to allow for all possible outcomes on the threshold limits without imposing further restrictions on the model.

$$\text{Debit Economy } \left(Z_i^K\right): \begin{cases} \dot{Z}_1^K = Z_1^K \left(\dfrac{\rho + \delta - \frac{rd}{2}\left(Z_2^K\right)^2 - A + \left(\delta - \bar{Z}_4\right)(\gamma - 1)}{\gamma - 1} \right) \\ \dot{Z}_2^K = Z_1^K + \bar{Z}_4 + \left(r + \frac{rd}{2} Z_2^K - \bar{Z}_4 + \delta \right) Z_2^K - A \end{cases} \tag{25}$$

$$\text{Credit Economy } \left(Z_i^B\right): \begin{cases} \dot{Z}_1^B = Z_1^B \left(\dfrac{\rho - r\left(1 + dZ_2^B\right) + \left(\delta - \bar{Z}_4\right)(\gamma - 1)}{\gamma - 1} \right) \\ \dot{Z}_2^B = Z_1^B + \bar{Z}_4 + \left(r + \frac{rd}{2} Z_2^B - \bar{Z}_4 + \delta \right) Z_2^B - A \end{cases} \tag{26}$$

4.1. Phase space dynamics and equilibrium

It is not straightforward to study the complete phase space of the threshold switching model defined in (25) and (26), as local bifurcation interactions may arise due to the threshold nature of our model. Taking into account this issue we shall proceed in a step by step process to able us to define the main equilibrium expressions and restrictions analytically and then undertake some numerical exploration of the underlying dynamics. There are three sets of distinct fixed points that are meaningful to explore: The static fixed points arising from threshold switching endogenous expressions given in (24); the universal fixed point given when we set scaled consumption dynamism to be equal to zero, $Z_1^* = 0$; and finally the specific to regime set of fixed points, which entail expected outcomes for economic dynamics and are obtained by setting $Z_1^{**} \succ 0$ in the switching threshold system given by (25) and (26).

We shall start by defining the system steady-states independently from threshold dynamics, in order to produce some economic and mathematical meaningful restrictions that allow us to reproduce correctly the phase portrait of this system:

Universal fixed points

$$Z_1^* = 0,\ Z_2^* = \frac{-\left(r - \bar{Z}_4 + \delta\right) \pm \sqrt{\left(r - \bar{Z}_4 + \delta\right)^2 - 2rd\left(\bar{Z}_4 - A\right)}}{rd} \tag{27}$$

Debit economy fixed points

$$Z_2^{K,**} = \pm\sqrt{2\frac{\rho + \delta - A + \left(\delta - \bar{Z}_4\right)(\gamma - 1)}{rd}},\ Z_1^{K,**} = A - \bar{Z}_4 - \left(r + rdZ_2^{K,**} + \delta - \bar{Z}_4\right) Z_2^{K,**} \tag{28}$$

Credit economy fixed point

$$Z_2^{B,**} = \frac{\rho - r + \left(\delta - \bar{Z}_4\right)(\gamma - 1)}{rd},\ Z_1^{B,**} = A - \bar{Z}_4 - \left(r + rdZ_2^{B,**} + \delta - \bar{Z}_4\right) Z_2^{B,**} \tag{29}$$

One main rule to guarantee the existence of economically feasible regions on this system, $\left(Z_2^* \in \mathbb{R}\right) \wedge \left\{ \left(Z_{2,-}^* \prec Z_2^{B,**} \in \mathbb{R} \prec Z_{2,+}^*\right) \vee \left(Z_{2,-}^* \prec Z_2^{K,**} \in \mathbb{R} \prec Z_{2,+}^*\right) \right\} \Rightarrow Z_1^{**} \in \mathbb{R}^+$, arises from the analysis of the steady states described above. This rule is obtained by assuming that the specific fixed points for scaled consumption, Z_1^{**}, are always positive. This restriction, described below in expression (30), imposes meaningful economic steady states for scaled debt/assets, Z_2^{**}, to be always contained in the region defined by the two roots obtained for the universal fixed point, Z_2^*.

$$\begin{gathered} Z_1^{**} \succ 0 \Rightarrow A - \bar{Z}_4 - \left(r + rdZ_2^{**} + \delta - \bar{Z}_4\right) Z_2^{**} \succ 0 \Leftrightarrow Z_2^* \prec 0 \\ Z_1^{**} \succ 0 \Rightarrow Z_{2,-}^* \prec Z_2^{**} \prec Z_{2,+}^* \end{gathered} \tag{30}$$

We continue this analytic analysis of the credit/debit economy by putting forward the generalized J*acobian* matrices, characteristic equations and respective eigenvalue expressions for the universal and specific fixed points qualitative analysis of the linearized system. These conditions are described below in expressions (31) and (32) for the universal and specific steady state, respectively:

$$J^* = \begin{bmatrix} \frac{d\dot{Z}_1}{dZ_1} & 0 \\ 1 & r + rdZ_2^* - \bar{Z}_4 + \delta \end{bmatrix}_{Z_i = Z_i^*}, \left(\left. \frac{d\dot{Z}_1}{dZ_1} \right|_{Z_2 = Z_2^*} - \lambda \right) \left(r + rdZ_2^* - \bar{Z}_4 + \delta - \lambda \right) = 0 \Rightarrow \begin{cases} \lambda_1^* = \left. \frac{d\dot{Z}_1}{dZ_1} \right|_{Z_2 = Z_2^*} \\ \lambda_2^* = r + rdZ_2^* - \bar{Z}_4 + \delta \end{cases} \tag{31}$$

$$J^{**} = \begin{bmatrix} 0 & \frac{d\dot{Z}_1}{dZ_2} \\ 1 & r + rdZ_2^{**} - \bar{Z}_4 + \delta \end{bmatrix}_{Z_i = Z_i^{**}}, \lambda^2 - \left(r + rdZ_2^{**} - \bar{Z}_4 + \delta \right) \lambda - 4 \left. \frac{d\dot{Z}_1}{dZ_2} \right|_{Z_i = Z_i^{**}} = 0 \Rightarrow$$

$$\Rightarrow \lambda^{**} = \frac{r + rdZ_2^{**} - \bar{Z}_4 + \delta \pm \sqrt{\left(r + rdZ_2^{**} - \bar{Z}_4 + \delta \right)^2 + 4 \left. \frac{d\dot{Z}_1}{dZ_2} \right|_{Z_i = Z_i^{**}}}}{2} \tag{32}$$

Although, we have taken the option of following a complete description of the linearized dynamics obtained from (31) and (32), it is not clear that local qualitative analysis of fixed points is sufficient to describe this system dynamics fully. The reason for such claim is obvious, when we take into account the interdependence between the switching threshold regions and local dynamics arising from different parameter combinations. In this scenario, parameter variation may lead to several local bifurcations and impose critical changes to the shape of the phase portrait of this system. Nevertheless, we present in section *2.1.* of the appendix a summary of the main restrictions and conditions for local qualitative dynamic analysis under the *Grobman-Hartman* theorem. This section is useful to describe all relevant mathematical conditions that are significant for a broader exploration of this system; still, taking into account the remaining economic and mathematical conditions describe in this section, it becomes clear that further analysis of this system must rely on thorough numerical analysis, since restrictions imposed are still too broad to allow for simple classification of the dynamics arising from this proposal.

Following our brief discussion of the problems arising from parameter combination and its impact on the phase portrait of this system, we finish this section by exploring feasible economic phase space regions with *Hopf* bifurcations. We follow this path for two reasons. First, because we are interested in introducing possibility of cycles in models of long run endogenous growth, in this case through local bifurcations arising from specific steady state for the debit economy, $Z_{2,-}^{K,**}$, as described in section 2*.1.* of the appendix. Second, because by imposing the existence of *Hopf* bifurcations in economic feasible regions of the plane, we are able to input enough restrictions, in order to give a partial perspective of the dynamics arising for the debit/credit economy in a economically feasible region with endogenous cycles[8].

We start this analysis by stating the conditions for *Hopf* bifurcation existence in economically feasible regions, dominated by debit economy dynamics:

> Theorem 1*: For regions with economic meaning dominated by debit economy dynamics, there will exist feasible Hopf bifurcations leading to local cycle dynamics, in the vicinity of* $Z_{2,-}^{K,**}$*, if the following necessary and sufficient conditions are fulfilled:*
> *Condition 1 –* $Z_{2,-}^{K,**} \prec Z_{2,-}^{t} \Rightarrow$ *Debit economy region*
> *Condition 2 –* $Z_2^{K,**}, Z_2^*, Z_2^t \in \mathbb{R} \Rightarrow Z_{2,-}^{K,**} \prec 0$
> *Condition 3 –* $Z_{2,-}^* \prec Z_{2,-}^{K,**} \prec Z_{2,+}^* \Rightarrow Z_{1,-}^{K,**} \succ 0 \Rightarrow \left. \frac{d\dot{Z}_1}{dZ_2} \right|_{Z_i = Z_i^{K,**}} \prec 0 \Rightarrow$ *Economic feasible region*

[8]When assuming the possibility of dynamical adjustment arising from high frequency threshold switching control variable dynamics based on state variable dynamical outcomes, we paved the way for introducing sliding mode control dynamic regions for this system. In section *1.2.* of the appendix, we extend the issues in order to allow for a broader comprehension of the global dynamics of this system and the possibility of local bifurcations interactions leading to the existence of regions dominated by sliding mode control dynamics or degenerate cycles.

When these conditions are fulfilled, Hopf bifurcations, defined by positive exogenous risk premia on foreign holdings, will exist in economic meaningful regions and these local bifurcations will be defined by the following expression, which renders the real parts of the eigenvalues defined in (32) equal to zero:

$$Z_{2,-}^{K,**} = \frac{\bar{Z}_4 - r - \delta}{rd} \Rightarrow d = \frac{\left(\bar{Z}_4 - r - \delta\right)^2}{2r\left[\rho + \delta - A + \left(\delta - \bar{Z}_4\right)(\gamma - 1)\right]} \succ 0$$

Taking into account the restrictions imposed by theorem 1 and the information for local linearized dynamics described in section *2.1.* of the appendix, we are able to define the phase portrait for this system for parameter combinations that correspond to possible cycle regions with economic interpretation. To construct this phase portrait we depart from one main assumption, regarding the sign of $\left.\frac{d\dot{Z}_1}{dZ_1}\right|_{Z_2=Z_2^*}$, and develop the two possible hypotheses that are feasible in this case, regarding the specific position of the positive root for the universal fixed point, $Z_{2,+}^*$. This main assumption is that this steady state will be a repellor, if it lies on the debit economy region, $Z_{2,-}^t \succ Z_{2,+}^*$ and $\left.\frac{d\dot{Z}_1^K}{dZ_1^K}\right|_{Z_2=Z_2^*} \succ 0$, or a saddle point if it lies on the credit economy region, $\left.\frac{d\dot{Z}_1^B}{dZ_1^B}\right|_{Z_2=Z_2^*} \prec 0$ and $Z_{2,-}^t \prec Z_{2,+}^*$. By assuming this hypothesis, we restrict the dynamics of the negative universal root, $Z_{2,-}^*$, to be a saddle path, following the analytical description provided in section *2.1.* of the appendix. Based on numerical simulations, we know that the second case is more likely to occur, when we take into account the feasible economic parameter space. Further, we can also define a generalized expression for the slope of the eigenvectors in the vicinity of the universal fixed points. For $Z_{2,+}^*$, the trajectories associated with λ_2^* will have a slope equal to zero and the trajectories associated with λ_1^*,will have a slope equal to $-\lambda_2^* + \lambda_1^*$. As $\lambda_1^* = \left.\frac{d\dot{Z}_1}{dZ_1}\right|_{Z_2=Z_2^*}$. As the sign of λ_2^* is equal to the sign the specific root, $Z_{2,+}^*$, we can define both trajectories governing the saddle points described. For the case where $Z_{2,-}^t \succ Z_{2,+}^*$, we will have only a saddle path associated with $Z_{2,-}^*$, in the debit economy region. The stable manifold for this saddle path will have a positive slope, while the unstable manifold lies on the Z_2 axis. For the case where $Z_{2,-}^t \prec Z_{2,+}^*$, we will have another saddle path associated with $Z_{2,-}^*$, in the credit economy region. The unstable manifold for this saddle path will have a negative slope, while the stable manifold lies on the Z_2 axis.

We conclude this phase portrait description by defining the critical dynamical transitions occurring for varying parameter d, through numerical simulation and the analytical results presented in the appendix, relating to the qualitative dynamics for $Z_{2,-}^{K,**}$. For regions defined by $d \succ d^{Hopf}$, the dynamics in the vicinity of this steady state will be governed by a node, as described by the local linearized dynamic conditions in table 9, and therefore impose a supercritical *Hopf* bifurcation transition to the system. In this case the steady state, $Z_{i,-}^{K,**}$, is increasing in $Z_{1,-}^{K,**}$ and decreasing in $Z_{2,-}^{K,**}$. For the opposite case, the dynamics are of a repellor and we have a subcritical *Hopf* transition with the steady state of the system following the opposite direction. Having put this, we are now able to draw the phase portrait of this system, which can be found in figure 1, and discuss further implications of this specific region of the phase space defined by theorem 1[9]:

[9]We discard from this analysis the remaining fixed points, as numerical simulations suggest that under theorem 1 conditions, both, $Z_{i,+}^{K,**}$ and $Z_i^{B,**}$ do not belong to the economic feasible region, defined on the first and fourth quadrants of the plane.

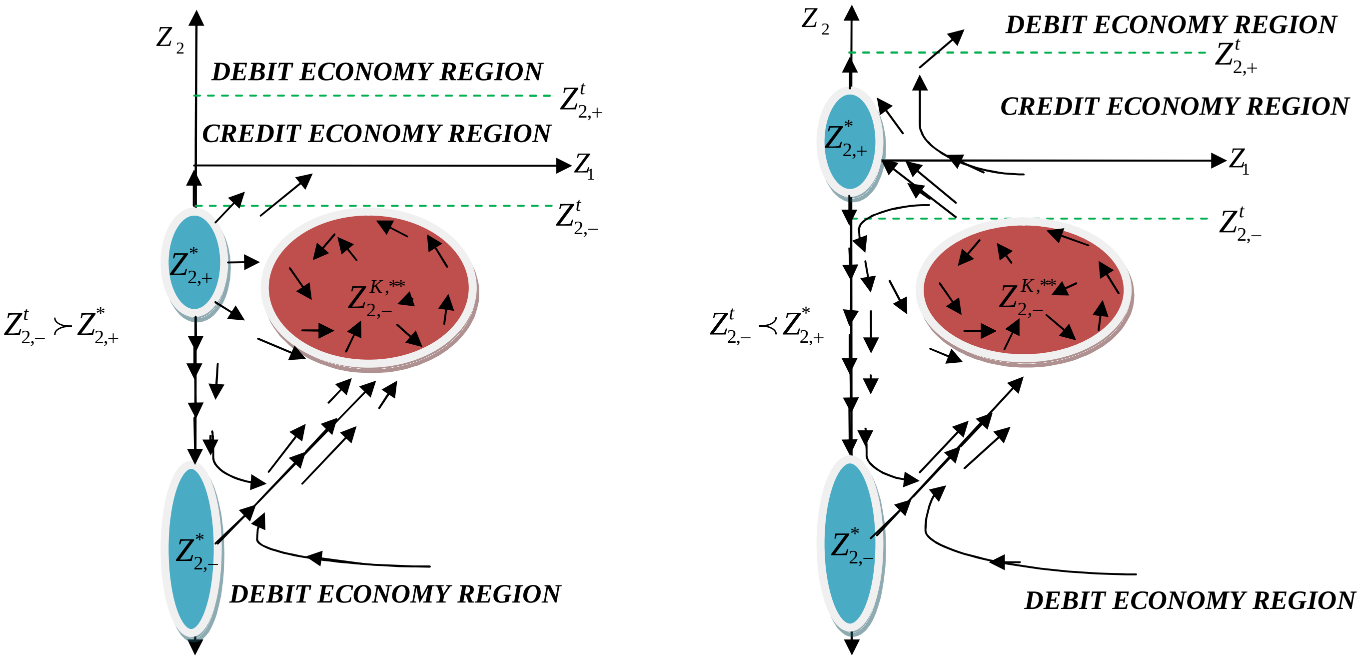

Fig. 1- Phase Space portrait for economic meaningful regions of the Debit/Credit Economy

A quick inspection to the vector fields depicted in figure 1 renders the first conclusion regarding this system dynamics. The eventual behaviour depends on initial conditions. Further, in the most likely case with two saddle points and switching threshold dynamics, the stable manifold of $Z^*_{2,+}$ and both the stable and unstable manifold of $Z^*_{2,-}$, may play a role in the creation of degenerate cycles, associated with the subcritical transitions near the *Hopf* bifurcation region and the interactions with the two saddle points vector fields, which may be able to lock in divergent subcritical trajectories within the inner region dominated by the stable and unstable saddle point manifolds. Such degenerate cycles would be stable if both the stable manifold of $Z^*_{2,+}$ and the unstable manifold of $Z^*_{2,-}$ were able to lock in the trajectories that pass through the subcritical region in the vicinity of $Z^{K,**}_{i,-}$. However, we were not able to determine a feasible set of parameters that is able to reproduce such dynamics. In these simulations, we are able to reproduce these wide amplitude degenerate cycles, but as the amplitude of the cycle increases in the subcritical region, there is always a trajectory that eventually crosses the stable manifold of $Z^*_{2,+}$ without locking in and therefore diverges alongside the Z_2 axis to infinity, until it passes the upper threshold and explodes to infinity. We shall discuss this case and other interesting features of this system, both in the next sections and in the appendix, by resorting to numerical methods and macroeconomic theory. This option is related to the existence of feasible regions defined by theorem 1, which we describe briefly in this section but that may have different economic implications, when we take into account the resulting parameter combinations that fulfil theorem 1.

4.2. Numerical Analysis: Local Hopf Bifurcations and Cycles in the Debit Economy Region

We conclude this presentation of the debit/credit economy by demonstrating the phase space dynamics discussed in the previous section. For this purpose, we evaluated numerically the parameter space for exogenous investment and technology, $(\bar{Z}_4, A) \in [0.051, 1.5]$ that fitted the conditions of theorem 1 for reasonable values of exogenous risk premium, $d \in]0,10[$,assuming as fixed the remaining parameters described in table 1. We assumed a high value of intertemporal substitution of consumption, $\gamma = 0.9$, because it allows to demonstrate the phase dynamics discussed in the previous section for the most likely case, $Z^t_{2,-} \prec Z^*_{2,+}$, and the interaction between the two universal saddle points and the cycle region in the neighbourhood of $Z^{K,**}_{i,-}$[10]. Figure 2 shows the feasible parameter space for values consistent with existence of long run growth, $\bar{Z}_4 \succ \delta$, and feasible outcomes for the *Hopf* bifurcation, $d = d^{Hopf}$. Following this numerical exploration, we are able to confirm that feasible values for exogenous technology and investment are mainly

[10]In section *1.2.* of the appendix, we explore the feasible parameter spaces for different values of γ , following the same criteria but with less numerical detail.

in a region where both the net marginal productivity of domestic capital and net investment in domestic capital are lower than the net marginal revenue on foreign assets, $A, \bar{Z}_4 \prec r+\delta$. Cycle regions are more common in economies with negative net inflows of capital when higher values for the intertemporal elasticity substitution of consumption, γ, are chosen. We decided to choose the remaining values assuming a reasonable long run growth rate for the variables of 2%. Figure 2 demonstrates the phase space dynamics of the limit cycle with initial values in the vicinity of $Z_{i,-}^{K,**}$ steady state.

ρ	r	δ	$\bar{Z}_4$	γ	A	d^{Hopf}
0.05	0.05	0.05	0.07	0.9	0.07	0.2813

Table 1- Parameter values for specific *Hopf* bifurcation

$Z_{2,-}^{K,**}$	$Z_{1,-}^{K,**}$	$Z_{2,+}^{K,**}$	$Z_{1,+}^{K,**}$	$Z_{2}^{B,**}$	$Z_{1}^{B,**}$	$Z_{2,-}^{*}$	$Z_{2,+}^{*}$	$Z_{2,-}^{t}$	$Z_{2,+}^{t}$
-2.1333	0.0320	2.1333	-0.0960	0.1422	-0.0044	-4.2667	0	-1.2949	3.2949

Table 2- Computed steady states and dynamical threshold boundaries

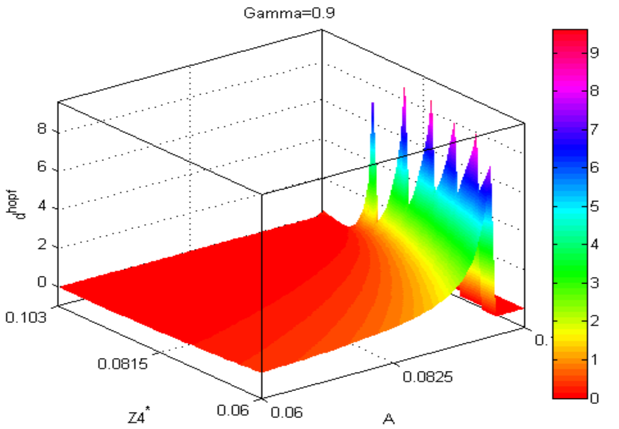

Fig. 2- Feasible $\left(\bar{Z}_4, A\right)$ parameter space for $d = d^{hopf}$

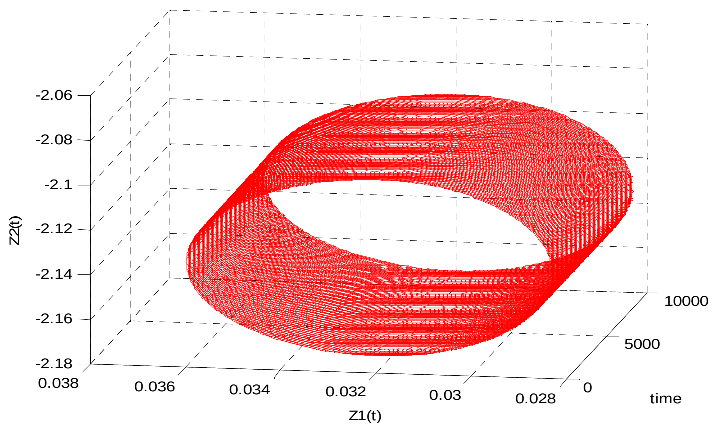

Fig. 3- Phase space dynamics for $d = d^{hopf}$

To demonstrate the interaction between the two saddle points and the cycle region, we may follow two different paths, which will render the same outcome for this set of parameters. The first option is to choose a set of initial values for the variables that lie on the region where the stable and unstable manifolds interact with the cycle region, as defined in figure 1. The other option is to vary slightly the *Hopf* bifurcation parameter and let the sub-critical *Hopf* region lead the trajectories until the moment they are unable to lock in with the stable manifold of $Z_{2,+}^{*}$ and end up exploding after crossing the upper switching threshold region. We follow the second path, and set it up to correspond to a permanent exogenous shock on exogenous risk premium parameter, $\Delta d = -0.05$, at $t = 1000$. Figures 4 to 7 demonstrate the dynamics in the phase space and the time paths of each variable:

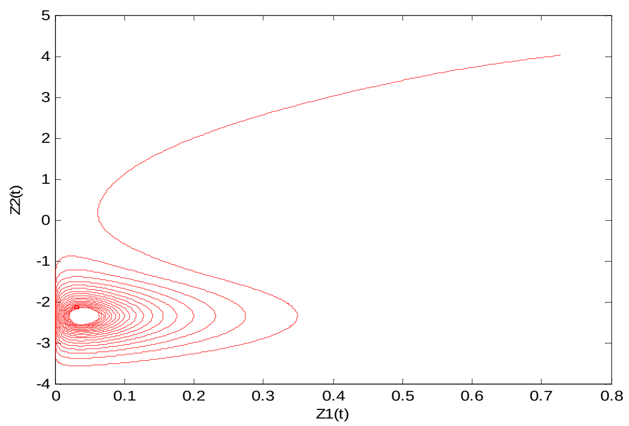

Fig. 4- Two dimensional phase space for unstable cycle dynamics

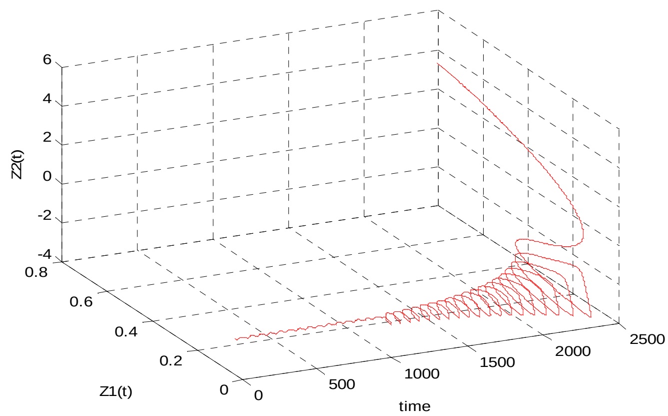

Fig. 5- Three dimensional phase space for unstable cycle dynamics

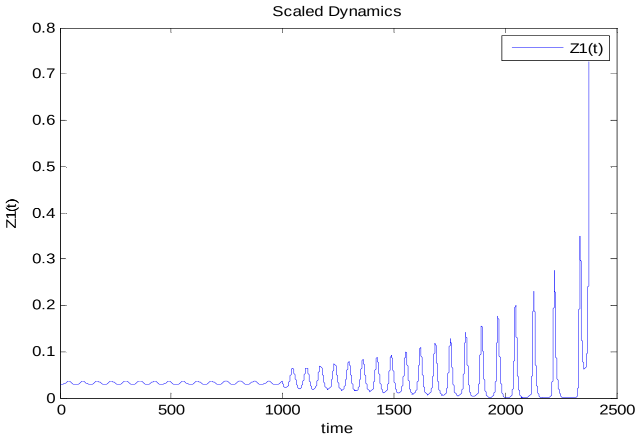

Fig. 6- $Z_1(t)$ time path for unstable cycle

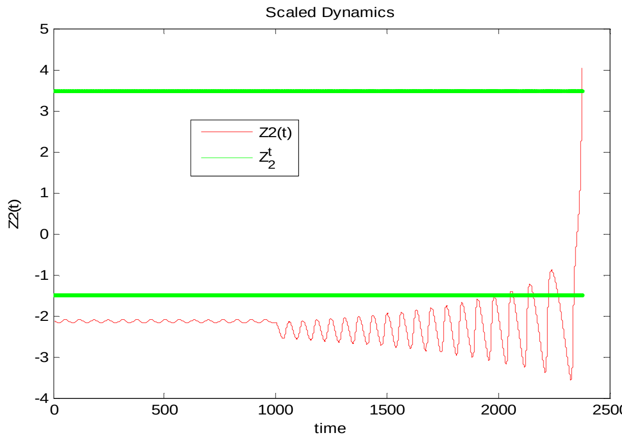

Fig. 7- $Z_2(t)$ time path for unstable cycle

This simple numerical demonstration of the dynamics described in the previous section represents only a taster of the potential dynamics that this system may contain. In section *2.3.* of the appendix, we demonstrate some further applications of the credit/debit economy by following two main directions that we believe still require further numerical exploration. These directions are data reconstruction for testing purposes and further exploration of the phase space through local dynamics and switching threshold interactions, which may lead to more complex dynamics than the ones here demonstrated.

4.3. Discussion

In this section, we showed how the introduction of financial convexities in the form of risk premia on holdings of foreign assets/debt may lead to cycle dynamics along a long run growth path. This is an important argument towards open capital markets as risk premium adjustment dynamics might lead to stable situations and allow for long run growth. This conclusion is reinforced by the possibility of infinite cycle dynamics arising from the interaction of saddle points and state dependent control switching dynamics. Our numerical analysis suggested that for economic feasible regions this extreme amplitude boom and bust dynamics are not stable in the long run and system dynamics eventually leave the degenerate cycle region and explode. This result could be considered as reassuring, if we consider that an economy which allows for such dynamics will eventually collapse in the long run, leaving no room for such macroeconomic outcomes[11]. Still, as we discuss in section *2.3.* of the appendix, such dynamics might be possible for economic feasible regions under different parameter combinations and uncertainty on risk premium might lead to stable cyclical dynamics, but with alternating moments of different boom and bust amplitudes. Nevertheless, the main conclusion of this proposal holds: risk premium regulates financial openness and access to foreign capital markets through the existence of cycle dynamics, while still allowing for long run growth.

Finally, we would like to finish this discussion on the debit/credit economy by putting forward a conceptual issue that might be subject to discussion on the aims of our conceptual proposal. Is the credit/debit economy model more appropriate for micro agents based modelling purposes or is it still an appropriate macro-model for policy decision? Since switching control state dependent financial position dynamics are not expected to occur at a macro level, it is possible that this proposal is more appropriate to consider in an agent-based level, where the individual decisions could be portrayed in a differential fashion and would be connected to a macro outcome to reproduce some sort of evolutionary dynamics. On the other hand, if we put aside the specific model mechanics and take into account the dynamic outcomes, the credit/debit card model is able to reproduce interesting dynamics, which can be easily tied to macroeconomic outcomes, therefore it is not clear what should be the more convenient path for future extensions relating to this specific proposal.

[11]By economic collapse, we assume dynamic outcomes that lead to time paths diverging to infinity in both directions, taking into account that such trajectories should be bounded by transversality conditions, (8) and (9), which our simulation routine do not account for. Since we assume that dynamical stability to long run growth must be given by economically feasible parameter restrictions in substitution of trajectories bounded by transversality conditions, such outcomes in numerical experiments should be considered as economic collapses, for the specific parameter space considered.

5. *Dynamic analysis of an open economy with investment adjustment costs: Transitions and long run endogenous growth*

In this section, we introduce the hypothesis of convex adjustment costs for investment in units of domestic capital. This type of configuration is commonly used to impose stable dynamic transitions in macroeconomic policy models of endogenous growth for economies with open capital markets. We shall define it as being dynamic economies with transitions *a la Turnovsky*, following the early and extensive application of this methodology by the famous contemporary mathematical economist. Examples of such applications in early works on the mathematics of endogenous growth can be found in Turnovsky (1996a,199b, 1999, 2002), among others. Further applications following this methodology can be widely found in endogenous growth theory literature, yet we restrict this introduction to the papers referred in order to keep this presentation as concise as possible and leave to the reader the opportunity to explore this literature trend in continuous time economic growth theory based on these main references.

Following the intuition described in the previous section, we set the conditions for defining the meaningful *Keynes-Ramsey* conditions for consumption dynamics in a setup where there is no risk premium associated with the holdings of foreign assets/debt, $\Psi(B,K)=1$. First, we redefine the optimal control conditions (4), (5) and (6) for the respective co-state and state conditions of this problem:

$$\dot{q}_2 = q_2(\rho - r) \quad (33)$$

$$\dot{q}_1 = q_1(\rho+\delta) - q_2\left[I\Phi'_K(I,K) - Y'_K(K)\right] \quad (34)$$

$$\dot{B} = C + I\Phi(I,K) + rB - Y(K) \quad (35)$$

Repeating the steps taken in section *4.*, in order to define the *Keynes-Ramsey* consumption rule arising from the optimality condition in consumption and taking into account the co-state condition (23), for foreign holdings accumulation, we obtain:

$$\dot{C}_B = \frac{U'_c(C)}{U''_c(C)}(\rho - r) \quad (36)$$

Recall now that the optimality condition for investment activities comes as

$$q_1 = -q_2\left(1 + hIK^{-1}\right) \quad (37)$$

$$q_1 = U'_c(C)\left(1 + hIK^{-1}\right) \quad (38)$$

Taking the time derivative we obtain:

$$\dot{q}_1 = -\dot{q}_2\left(1+hIK^{-1}\right) - q_2 hK^{-1}\dot{I} + q_2 hIK^{-2}\dot{K} \quad (39)$$

$$\dot{q}_1 = U''_c(C)\dot{C}\left(1+hIK^{-1}\right) - U'_c(C)hK^{-1}\dot{I} + U'_c(C)hIK^{-2}\dot{K} \quad (40)$$

Recall that after substituting by the relevant functional forms and the optimality condition (2), the co-state condition (34) is now given by:

$$\dot{q}_1 = q_1(\rho+\delta) + U'_c(C)\left[-\frac{h}{2}\left(\frac{I}{K}\right)^2 - A\right] \quad (41)$$

Substituting conditions (38) and (40) in the co-state condition (41) we obtain the second *Keynes-Ramsey* consumption rule:

$$\dot{C}_K = \frac{U'_c(C)}{U''_c(C)}\left(1+hIK^{-1}\right)^{-1}\left[-hK^{-1}\dot{I} + hIK^{-2}\dot{K} + \left(1+hIK^{-1}\right)(\rho+\delta) - \frac{h}{2}\left(\frac{I}{K}\right)^2 - A\right] \quad (42)$$

Again, we can use the indifference condition described in the previous sections, by equalizing (36) and (42), in order to define the relevant dynamical expression for indifference in assets accumulation for this economy. After some algebra, we obtain a differential equation for investment decisions:

$$\dot{I} = -\frac{1}{2}\frac{I^2}{K} + \left(r + \delta + \frac{\dot{K}}{K}\right)I + (r+\delta-A)\frac{K}{h} \quad (43)$$

Recall that from state condition for capital accumulation, (7), we can obtain the relation defined for $\dot{Z}_3$ in (21), for the domestic capital growth rate. Substituting in (43), we obtain the quadratic differential equation for investment that guarantees indifference in accumulation in this economy:

$$\dot{I} = \frac{1}{2}\frac{I^2}{K} + rI + \left(r + \delta - A\right)\frac{K}{h} \tag{44}$$

This dynamical relation for investment guarantees that both *Keynes-Ramsey* consumption rules for this economy are identical and there is a dynamical determined path of indifference in assets accumulation, satisfying our modelling condition for state dynamic adjustment through investment decisions. Taking that in consideration, we can define the meaningful dynamical system, which will be composed by equations (7), (35), (36) and (44). We shall not consider the possibility of switching threshold dynamics as in the previous economy. The reason for this decision lies on the existence of a dynamical rule that adjusts domestic assets accumulation, but still allows for consumption through debt accumulation always. Assuming the scaling rule defined in the previous section, $Z_i = \frac{X_i}{K}$, it is possible to reduce this system to three dimensions, without imposing any additional conditions on this dynamical problem. We can redefine the dynamical system in scaled variables and reduce one dimension from this dynamical system, as capital scaled dynamics are always defined by scaled investment dynamics and play no role in the dynamics of the remaining variables. The meaningful equations for this dynamical system are described below:

$$\dot{Z}_1 = Z_1\left(\frac{\rho - r + \left(\delta - Z_4\right)\left(\gamma - 1\right)}{\gamma - 1}\right) \tag{45}$$

$$\dot{Z}_2 = Z_1 + Z_4\left(1 + \frac{h}{2}Z_4\right) + \left(r + \delta - Z_4\right)Z_2 - A \tag{46}$$

$$\dot{Z}_3 = Z_4 - \delta \tag{47}$$

$$\dot{Z}_4 = -\frac{1}{2}\left(Z_4\right)^2 + \left(r + \delta\right)Z_4 + \left(\delta + r - A\right)\frac{1}{h} \tag{48}$$

5.1. Phase Space dynamics and equilibrium

Recall that this setup of convex investment adjustment costs is usually introduced to guarantee steady-state stability by solving the differential system of state and co-state variables. Nevertheless, the system proposed here will allow for a complete bifurcation analysis of the control variable dynamics, following our assumptions described in the previous section. This is important since little is still known in the literature about the adjustment cost parameter h and the full dynamical implication of convex adjustment costs in investment for open economy models. Fortunately, the dynamical system proposed allows for a full qualitative analytical analysis following the usual procedure of linearization in the neighbourhood of the fixed points. We start by defining the fixed point with no economic interpretation obtained by setting scaled consumption to zero:

$$Z_1^* = 0 \Rightarrow \begin{cases} Z_2^* = \dfrac{A - Z_4^*\left(1 + \dfrac{h}{2}Z_4^*\right)}{r + \delta - Z_4^*} \\ Z_4^* = \left(r + \delta\right) \mp \sqrt{\left(r + \delta\right)^2 + \dfrac{2}{h}\left(\delta + r - A\right)} \end{cases} \tag{49}$$

In order to obtain a steady state with an meaningful economic interpretation, Z_i^{**}, we need to impose an endogenous relation that guarantees scaled consumption dynamics to be positive. As scaled investment dynamics are defined independently by the quadratic differential equation (48), we need to impose the following endogenous parameter expression to guarantee feasible scaled consumption outcomes:

$$\bar{Z}_1 \succ 0 \Rightarrow Z_4^{**} = \frac{\rho - r}{\gamma - 1} + \delta = Z_4^* \Rightarrow \frac{\rho - r}{\gamma - 1} = r \mp \sqrt{\left(r + \delta\right)^2 + \frac{2}{h}\left(\delta + r - A\right)} \tag{50}$$

To obtain the steady-state for this system we need to solve first for $\dot{Z}_2 = 0$ and obtain an expression for Z_4^{**} defined in Z_1^{**} and Z_2^{**}:

$$Z_4^{**}\left(Z_1^{**}, Z_2^{**}\right) = \frac{Z_2^{**} - 1 \pm \sqrt{\left(1 - Z_2^{**}\right)^2 - 2h\left(Z_1^{**} + (r+\delta) Z_2^{**} - A\right)}}{h} \tag{51}$$

The steady states for scaled consumption and debt/assets are obtained by solving the system of equations given by $\dot{Z}_1\left(Z_4^{**}\left(Z_1^{**}, Z_2^{**}\right)\right) = 0$ and $\dot{Z}_4\left(Z_4^{**}\left(Z_1^{**}, Z_2^{**}\right)\right) = 0$. This final set of conditions is expressed below:

$$\begin{aligned} \frac{Z_2^{**} - 1 \pm \sqrt{\left(1 - Z_2^{**}\right)^2 - 2h\left(Z_1^{**} + (r+\delta) Z_2^{**} - A\right)}}{h} &= \frac{\rho - r}{\gamma - 1} + \delta \\ \frac{Z_2^{**} - 1 \pm \sqrt{\left(1 - Z_2^{**}\right)^2 - 2h\left(Z_1^{**} + (r+\delta) Z_2^{**} - A\right)}}{h} &= (r+\delta) \mp \sqrt{(r+\delta)^2 + \frac{2}{h}(\delta + r - A)} \end{aligned} \tag{52}$$

Intuitively, we can follow a different path in order to define the dynamics of this system. Assuming that an endogenous equilibrium defined by (50) is feasible, the dynamics of $Z_1(t)$ are given by:

$$Z_1(t) = Z_1(0) e^{\int_0^{\infty}\left(Z_4^{**} - Z_4(t)\right)dt} \tag{53}$$

This result implies that the long run dynamics of scaled consumption and debt/assets are only dependent on initial conditions and on the rate of convergence of scaled investment to equilibrium. Assuming that there are a feasible set of parameters that allow for a positive outcome in $\mathbb{R}^+$ for scaled consumption, we can use this result to evaluate qualitatively the linearized dynamics of this dynamical system.

First, we start by defining the *Jacobian* matrix and characteristic polynomial for this system:

$$J = \begin{bmatrix} \frac{\rho - r}{\gamma - 1} - Z_4 + \delta & 0 & -Z_1 \\ 1 & r + \delta - Z_4 & 1 + hZ_4 - Z_2 \\ 0 & 0 & r + \delta - Z_4 \end{bmatrix}_{Z_i = \bar{Z}_i}, \left(\frac{\rho - r}{\gamma - 1} - \bar{Z}_4 + \delta - \lambda\right)\left(r + \delta - \bar{Z}_4 - \lambda\right)^2 = 0 \tag{54}$$

Conditions for stability of this dynamical system can be given generically by the following pair of expressions:

$$\bar{Z}_4 \succ \frac{\rho - r}{\gamma - 1} + \delta \Rightarrow \frac{\rho - r}{\gamma - 1} \succ r \mp \sqrt{(r+\delta)^2 + \frac{2}{h}(\delta + r - A)} \tag{55}$$

$$\bar{Z}_4 \succ r + \delta \Rightarrow \mp \sqrt{(r+\delta)^2 + \frac{2}{h}(\delta + r - A)} \succ 0 \tag{56}$$

This set of conditions imply that only the positive root of scaled investment, $\bar{Z}_{4,+}$, assuming $\bar{Z}_4 \in \mathbb{R}$, can be a stable fixed point for this system. As there are no eigenvalues with imaginary parts due to the restrictions imposed and we know that only condition (56)[12] matters for the dynamics of the economic meaningful steady state, the dynamics of this system can be described by the following table, taking into account the endogenous parameter restriction (50) for the economic feasible steady state and assuming that domestic net investment is always bigger than the exogenous international interest rate, $\bar{Z}_4 - \delta \succ r$:

[12]Since transitions for scaled consumption are endogenously given by scaled investment dynamics only the *Jacobian* matrix elements describing scaled debt/assets and investment dynamics, $J_{2,2}$ and $J_{3,3}$, are relevant for the solution of the characteristic equation of the reduced two dimensional system.

	$\bar{Z}_{4,+}$	$\bar{Z}_{4,-}$
$\bar{Z}_1 = 0$	Node- $\bar{Z}_4 \succ \frac{\rho + r}{\gamma - 1} + \delta$ Saddle point index 1- $\bar{Z}_4 \prec \frac{\rho + r}{\gamma - 1} + \delta$	Saddle point index 2- $\bar{Z}_4 \succ \frac{\rho + r}{\gamma - 1} + \delta$ Repellor- $\bar{Z}_4 \prec \frac{\rho + r}{\gamma - 1} + \delta$
$\bar{Z}_1 \succ 0$	Node	Repellor

Table 3- Global qualitative dynamics for economy with investment adjustment costs[13]

5.2. Numerical analysis: Investment adjustment costs and endogenous bifurcation

Following the analytical description of this system provided in the previous section, we conclude the analysis of the economy with investment adjustment costs with a numerical exploration of feasible values for the investment costs parameter, h, which allow for endogenous transitions to long-run equilibrium, as described in expression (50), for the economic feasible scaled investment steady state, $Z_{4,+}^{**}$, defined by (56). For this purpose, we decided to evaluate condition (50) by varying the two-dimensional parameter space for exogenous technology, A, and intertemporal elasticity of substitution in consumption, γ. We maintain fixed the parameters r and δ, following the values attributed in the previous experiences for the credit/debit card economy, while presenting in the main text the border case arising when $\rho = 0.03$. Table 4 summarizes the parameter values and intervals for this numerical exploration.

ρ	r	δ	γ	A
0.03	0.05	0.05	[0.01,0.99]	[0.01,1.5]

Table 4- Parameter values for economic feasible regions for the economy with investment adjustment costs

Figures 8 and 9, below, illustrate the feasible outcomes in color for economically feasible values of investment adjustment costs, h^*, and steady state values of $Z_{4,+}^{**}$. The regions with only a black surface representation illustrate non-feasible regions in the parameter space (A, λ), while the colored regions illustrate the feasible outcomes that satisfy the stable endogenous long run solution of this system.

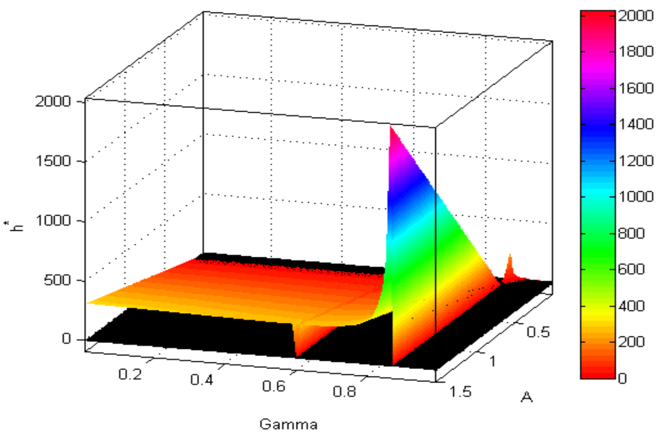

Fig. 8- Economically feasible values for h^* parameter

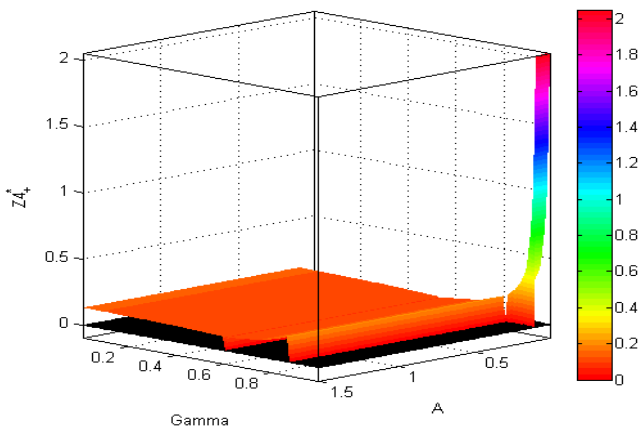

Fig. 9- Economically feasible equilibrium for scaled investment

A quick inspection of these three dimensional images allows us to define the main conclusions regarding this system dynamics. First, only positive values for investment adjustment costs satisfy the feasible endogenous equilibrium. Second, the parameter space is divided in two main regions. For regions where $A \prec r + \delta$, only very large values of γ

[13]By saddle point index 1, we consider the three dimensional linearized dynamics with one eigenvalue with only positive real part and two eigenvalues with only negative real parts. In this case, trajectories approach the saddle point on a surface (the in-set) and diverge along a curve (the out-set). By saddle point index 2 we consider the three dimensional linearized dynamics with two eigenvalues with only positive real parts and one eigenvalue with only negative real part. In this case, trajectories approach the saddle point on a curve (the in-set) and diverge from the saddle point on a surface (the out-set). This description for saddle dynamics in three dimensional systems can be found on pages 126 to 128 in Hilborn (2000) book.

allow for economic feasible outcomes. For regions where $A \succ r+\delta$, only the set of values of γ that did not allow for economic feasible outcomes in the previous discussed region are admissible. The region where $A = r+\delta$ does not allow for the existence of adjustment costs in investment and determines the main frontier regions of this system as expected, following expression (56). The distinctive regions have a simple economic interpretation since the exogenous technology parameter represents the marginal productivity of capital for this economy. The parameter value defining the space frontier region represents outcomes where the net marginal productivity of domestic capital is equal to the international interest rate, which can be interpreted as the net marginal revenue of foreign assets. Following the intuition provided in the previous section for net investment in domestic assets to be always bigger than net investment in foreign assets, this further rule implies that economic feasible equilibrium is most likely to occur in these regions, where we can assume reasonable values for γ and, where the outcomes obtained for scaled investment steady state are not unreasonably high, $Z_4^{**} \succ 0.5$. Finally, we can draw one final conclusion from this numerical experiment, which relates to values of A in the most likely region discussed. As A increases, the feasible parameter values for h^* become increasingly high. This outcome should be considered rather unlikely if we take into account the parameter interpretation when only values for $h^* \in \mathbb{R}^+$ are feasible. In this specific case, higher investment adjustment costs impose an increasing cost on investing in units of domestic capital as opposed to foreign assets. Such outcomes, although plausible if we take into account certain specific macroeconomic conditions, are rather unlikely to occur in real economies, as they suggest that rather successful open economies could only sustain long run growth dynamics if investment adjustment costs in domestic capital are extraordinarily relevant to impose a shift on investment and consumption decisions towards foreign assets/debt[14].

5.3. Discussion

This section served the purpose of introducing investment adjustment costs as a macroeconomic feasible assumption to impose transitions in classical endogenous growth optimal control setups such as ours. By following the methodology of defining consumption indifference dynamical rules and scaling the system in this fashion, we were able to fully describe the dynamics and bifurcations arising from this proposal analytically. Our first conclusion is that transitions to long run feasible outcomes in economies with investment adjustment costs can only occur through endogenous parameter combinations on regions with positive outcomes for investment adjustment costs parameter, meaning that sustained long run growth dynamics can only be achieved if there are costs on domestic capital installation, when an economy faces perfectly open capital markets. Further our numerical analysis suggests that reasonable outcomes are more likely to occur when the net marginal productivity of domestic capital is slightly higher than the net marginal revenue of foreign assets for reasonable values of the intertemporal elasticity of consumption, invalidating the hypothesis of increasing returns on domestic capital through exogenous technology and the hypothesis of adjustment through investment costs in open economies, where domestic capital performance is smaller than the performance of foreign assets. Therefore, it is our conclusion that the introduction of non-linearities in this fashion should be only considered in modeling setups that account for further non-linearities, as the investment adjustment costs hypothesis by itself, is not able to guarantee the existence of feasible dynamic transitions to long run, without the requirement of specific endogenous parameter combinations. We shall discuss this possibility in the following section, where we demonstrate that investment adjustment costs mechanics can play an important role in the macrodynamics of open economies, when considering further non-linearities, such as our risk premia on holdings of foreign assets/debt hypothesis, discussed in section *4.*.

6. Dynamic analysis of an open economy with convex adjustment on foreign debt/assets and investment: Endogenous cycles and global bifurcations

For the complete optimal control problem, (1), described in conditions (2) to (7), we follow the same strategy considered in the previous section to obtain the meaningful dynamical system. Again, we obtain a differential equation for investment arising from the indifference in assets accumulation condition, following the strategy defined in the previous sections. We show in this section how the introduction of both convex adjustment on holdings of foreign assets/debt and

[14]We provide further numerical parameter results in section *2.* of the appendix, by varying intertemporal discount rate parameter, ρ, which has influence on the numerical frontier region for feasible outcomes. Still, these outcomes have no impact on the main conclusions we discussed in this section and serve the purpose of presenting a further in-depth analysis on the parameter space implications arising from this economy dynamics.

domestic capital investment will allow for dynamic adjustment without resorting to state-dependent switching control dynamics, and long-run dynamics will not depend on endogenous parameter combinations. On the other hand, the outcome will be a three-dimensional non-linear dynamical system with limited analytical tractability. This demonstrates that the introduction of both convexities is necessary to allow for full dynamical adjustment without the burden of *a posteriori* rules that are always a reflection of the modeller subjectivity or limited model dynamics.

We start as usual by defining the *Keynes-Ramsey* consumption rule arising from the optimality condition in consumption and taking into account the co-state condition (4), for foreign holdings accumulation. We obtain the same condition as in the credit/debit economy for the credit economy case, (17). Next, we use the relations already defined for the optimal control on investment decisions and its time derivative defined in conditions (37) to (40), but now we use the original co-state condition (5), in order to obtain the second *Keynes-Ramsey* consumption equation:

$$\dot{C}_K = \frac{U_c'(C)}{U_c''(C)}\left(1+hIK^{-1}\right)^{-1}\left[-hK^{-1}\dot{I}+hIK^{-2}\dot{K}+\left(1+hIK^{-1}\right)(\rho+\delta)-\frac{rd}{2}\left(\frac{B}{K}\right)^2-\frac{h}{2}\left(\frac{I}{K}\right)^2-A\right] \quad (57)$$

Again, we can use the indifference condition described in the previous sections, by equalizing (17) and (57) to define the relevant dynamical expression for indifference in holdings accumulation for this economy. After some algebra, we obtain a differential equation for investment decisions:

$$\dot{I} = \frac{1}{2}\frac{I^2}{K}+\left(r+rd\frac{B}{K}\right)I-\frac{rd}{2h}\frac{B^2}{K}+\left(r+rd\frac{B}{K}+\delta-A\right)\frac{K}{h} \quad (58)$$

This system can also be reduced to three dimensions, when assuming the scaling rule, (21), we defined in the previous sections, which renders the capital growth rate dynamics not meaningful for the remaining variables during transitions. We shall proceed with the analysis of phase space dynamics in the final sub sections, taking into account that due to the non-linearities suggested in (59) to (62), much of the results obtained will have to be based on numerical analysis. Following the strategy in the previous sections for scaling economies with convex adjustment costs, we obtain the following dynamical system corresponding to the optimal control problem defined in section *3.*.

$$\dot{Z}_1 = Z_1\left(\frac{\rho-r(1+dZ_2)+(\delta-Z_4)(\gamma-1)}{\gamma-1}\right) \quad (59)$$

$$\dot{Z}_2 = Z_1+Z_4\left(1+\frac{h}{2}Z_4\right)+\left(r+\frac{rd}{2}Z_2+\delta-Z_4\right)Z_2-A \quad (60)$$

$$\dot{Z}_3 = Z_4-\delta \quad (61)$$

$$\dot{Z}_4 = -\frac{1}{2}(Z_4)^2+(r+rdZ_2+\delta)Z_4-\frac{rd}{2h}(Z_2)^2+(\delta+r+rdZ_2-A)\frac{1}{h} \quad (62)$$

6.1. Phase space dynamics and equilibrium

Following the organization of the previous sections, we shall start this subsection by defining the set of fixed points for the economy arising from the complete set of conditions defined in our original optimal control problem. As usual, the sets of fixed points can be divided in two distinct sets, the ones with possible economic meaning and the sets of steady states which do not suggest any meaningful economic intuition at all. Still, one should take into account the likelihood of bifurcations arising from the interactions of local vector fields in the three dimensional field, which may lead to global bifurcations that have important macroeconomic meanings. Also, one should take into account possible local bifurcations leading to endogenous cycles. As our main research proposal lies on finding such global or local dynamics with a limited set of macroeconomic assumptions, we shall use the next paragraphs to define the full set of conditions that will allow for a further analysis of the phase space dynamics. We start by defining the sets of fixed points with no economic meaning arising from setting scaled consumption to zero, $\bar{Z}_1 = 0 \Rightarrow \dot{Z}_1 = 0$. We shall define this set of fixed points as Z_i^* for distinction purposes. Steady states for Z_2 and Z_4 under this conditions are obtained from the following system of equations:

$$Z_4^*\left(1+\frac{h}{2}Z_4^*\right)+\left(r+\frac{rd}{2}Z_2^*+\delta-Z_4^*\right)Z_2^*-A=0 \tag{63}$$

$$-\frac{1}{2}\left(Z_4^*\right)^2+\left(r+rdZ_2^*+\delta\right)Z_4^*-\frac{rd}{2h}\left(Z_2^*\right)^2+\left(\delta+r+rdZ_2^*-A\right)\frac{1}{h}=0 \tag{64}$$

As is straightforwardly observed from expressions (63) and (64), this fixed point can only be easily dealt with numerically. This is also true for the economic meaningful fixed point. The final system defining the solution for Z_2^* and Z_4^*, when $Z_1^*=0$, is given by:

$$Z_2^*=\frac{-\left(r+\delta-Z_4^*\right)\pm\sqrt{\left(r+\delta-Z_4^*\right)^2-2rd\left(Z_4^*\left(1+\frac{h}{2}Z_4^*\right)-A\right)}}{rd} \tag{65}$$

$$Z_4^*=\left(r+rdZ_2^*+\delta\right)\mp\sqrt{\left(r+rdZ_2^*+\delta\right)^2+2\left(-\frac{rd}{2h}\left(Z_2^*\right)^2-\left(r+rdZ_2^*+\delta-A\right)\frac{1}{h}\right)} \tag{66}$$

The sets of fixed points with possible economic meaning are obtained by considering $\bar{Z}_1\in\mathbb{R}^+$. We shall define this set of steady states as Z_i^{**}. This set of fixed points is governed by the relation obtained from $\dot{Z}_1=0$ for Z_2^{**} and Z_4^{**}, which can be found below:

$$Z_4^{**}=\frac{\rho-r\left(1+dZ_2^{**}\right)}{\gamma-1}+\delta \tag{67}$$

The set of stable fixed points for scaled consumption and debt/assets are obtained by substituting this expression in $\dot{Z}_4=0$ and solving for Z_2^{**}.

$$Z_2^{**}=h\frac{-\left(rdZ_4^{**}+\frac{rd}{h}\right)\pm\sqrt{\left(rdZ_4^{**}+\frac{rd}{h}\right)^2+\frac{2rd}{h}\left(-\frac{1}{2}\left(Z_4^{**}\right)^2+\left(r+\delta\right)Z_4^{**}+\left(\delta+r-A\right)\frac{1}{h}\right)}}{-rd} \tag{68}$$

Then, after solving the system given by (67) and (68), we can substitute the pair of fixed points obtained for scaled investment and debt/assets in $\dot{Z}_2=0$ and solve for Z_1^{**}:

$$Z_1^{**}=A-Z_4^{**}\left(1-\frac{h}{2}Z_4^{**}\right)-\left(r-\frac{rd}{2}Z_2^{**}+\delta-Z_4^{**}\right)Z_2^{**} \tag{69}$$

In order to be able to describe the qualitative dynamics of this economy, we will require the use of numerical methods, since both fixed points and the characteristic equation only allow for limited analytical tractability. For this purpose we leave below the generalized *Jacobian* and characteristic polynomial for this system:

$$J=\begin{bmatrix}\frac{\rho-r\left(1+dZ_2\right)+\left(\delta-Z_4\right)\left(\gamma-1\right)}{\gamma-1} & -\frac{rd}{\gamma-1}Z_2 & -Z_1\\ 1 & r+rdZ_2+\delta-Z_4 & 1+hZ_4-Z_2\\ 0 & rdZ_4-\frac{rd}{h}\left(Z_2-1\right) & -Z_4+r+rdZ_2+\delta\end{bmatrix}_{Z_i=\bar{Z}_i} \tag{70}$$

$$\left(J_{1,1}-\lambda\right)\left(J_{2,2}-\lambda\right)\left(J_{3,3}-\lambda\right)+J_{1,3}J_{3,2}-J_{3,2}J_{2,3}\left(J_{1,1}-\lambda\right)-\left(J_{3,3}-\lambda\right)J_{1,2}=0$$

A final simplification may be assumed for each pair of fixed points, since we know $J_{1,3}=0$ for the set of fixed points defined by $Z_1^*=0$ and $J_{1,1}=0$ for the set of economic feasible fixed points defined by relation (67), $\bar{Z}_i=Z_i^{**}$.

As this brief presentation demonstrated, the proposed dynamical system in $\mathbb{R}^3$ does not render straightforward strategies to be tackled analytically in a simple and intuitive fashion. We stop the analytic discussion of this system here, having defined the main conditions for performing numerical computations in order to continue our quest for cycles through local *Hopf* bifurcations. In the next section, we discuss a simple algorithm for this purpose and extend it to define a feasible economic parameter space in (d,h), where this system undergoes local *Hopf* bifurcations, in the vicinity of the economically feasible steady states, defined by both $Z_{i,\pm}^{**}$ solutions to the system of equations described in (67) to (69), for different sets of reasonable parameter values with economic interpretation.

6.2. Numerical analysis: Endogenous Cycles and Global Bifurcations

We finish the analysis of this economy with the discussion, implementation and demonstration of a simple algorithm for determining regions with economic interpretation for the economy with convex adjustment costs in the form of risk premia on holdings of foreign assets/debt and investment decision bias in domestic assets. The definition of *Hopf* bifurcation for systems in $\mathbb{R}^3$ can be summarized by the parameter combinations intervals that allow for the complex conjugate pair of eigenvalues to cross the imaginary axis in the complex plane. The *Hopf* bifurcation is the exact parameter values that correspond to the point where the real part of the complex conjugate pair is equal to zero. We follow this definition and define a algorithm to determine the existence of such regions, in the form of parameter intervals, and extend it to the definition of economic feasibility, $Z_1, Z_4 \in \mathbb{R}^+ \wedge Z_2 \in \mathbb{R}$. The parameter space we are interest in investigating is given by combinations of risk premium and investment adjustment costs parameters, (d,h), for regions in the vicinity of the two economic feasible steady-states, Z_i^{**}, expressed in the system defined by equations (67) to (69). This computation is not straightforward since the calculation of the fixed points and eigenvalues implies the use of non-linear numerical techniques[15], which are always prune to accuracy errors and convergence problems. Nevertheless, the results obtained suggested that the regions determined correspond to *Hopf* bifurcation regions, which lead us to believe that further exploration of this system, can be undertaken using relatively sophisticated numerical techniques and limited computational resources.

We proceed to the description of our routine implementation by describing the specific experiments undertaken to define *Hopf* bifurcation regions in the parameter space for d and h. Preliminary numerical experiments led us in the direction of assuming $h \prec 0$, since we imposed the parameter constraint $d \succ 0$. We discuss the economic implications of this outcome/decision in the end of this section. Following this outcomes, we focused our numerical exploration for the economic reasonable parameter space defined by $d \in [0.001, 10]$ and $h \in [-10, -0.001]$ assuming different combinations of parameters A and γ, while maintaining fixed the remaining parameters given in table 4. We present these results in section *4.1.* of the appendix and leave the remaining of this section to the demonstration of our numerical procedure and the cycle dynamics for this economy. For this purpose, we use a feasible set of parameter values given in table 4 and grid search just parameter d, using a interval equal to 0.001, to determine a numerical interval for d^{hopf} with limits determined by the last value computed before the complex conjugate pair of eigenvalues crosses the imaginary axis and the first value after the crossing of the imaginary axis. We follow this path because it is more intuitive to demonstrate the outcome of our two parameter bifurcation search routine in this fashion, which is demonstrated in section *4.1.* of the appendix, describing numerically the relation between parameters d and h for *Hopf* bifurcation outcomes. Figure 10 shows the output for the eigenvalues in the complex plane near the bifurcation interval and table 5 gives the computed intervals for risk premium and $Z_{i,+}^{**}$ steady-state. In this experiment, we dismiss all computed $Z_{2,-}^{**}$ steady-state values of having a *Hopf* bifurcation region, following our routine restrictions, which were imposed by considering the mean value of the fixed point interval for this purpose. Using the information obtained, we simulated this system dynamics in the vicinity of the *Hopf* bifurcation region, assuming $d = 1.2765$ for $t \in [0, 1000]$. Figures 11 to 15 portray the cycle dynamics for this economy.

ρ	r	δ	h	γ	A	d
0.05	0.05	0.05	-0.001	0.3	0.11	[0.001,10]

Table 5- Parameter values for bifurcation analysis of cycles in $\mathbb{R}^3$

[15]We used *fzero* Matlab routine to compute the steady states and controlled the convergence of the algorithm for determination of possible *Hopf* bifurcation regions. To determine the eigenvalues we used the Matlab routine *eig*.

$Z_1(0)$	$Z_2(0)$	$Z_4(0)$	$Z_{1,+}^{**}$	$Z_{2,+}^{**}$	$Z_{4,+}^{**}$	d^{hopf}
1.1*10^{-5}	1.81	0.21	[-1.1549*10^{-5}, 2.4263*10^{-5}]	[1.8281,1.8283]	[0.2166,0.2168]	[1.276,1.277]

Table 6- Computed intervals for *Hopf* bifurcation and initial values for simulation of cycles in $\mathbb{R}^3$

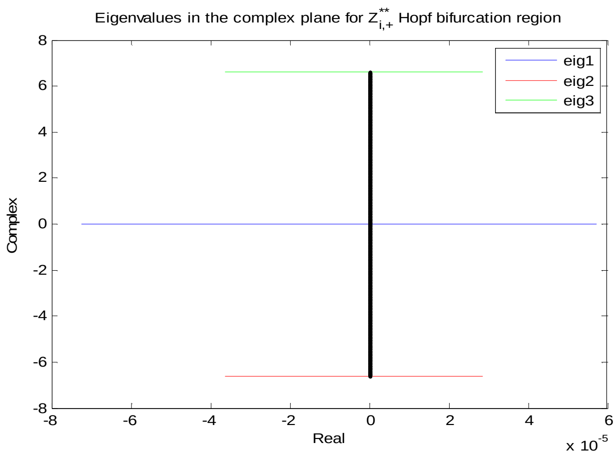

Fig. 10- Eigenvalues in the complex plane for $Z_{i,+}^{**}$ *Hopf* bifurcation region

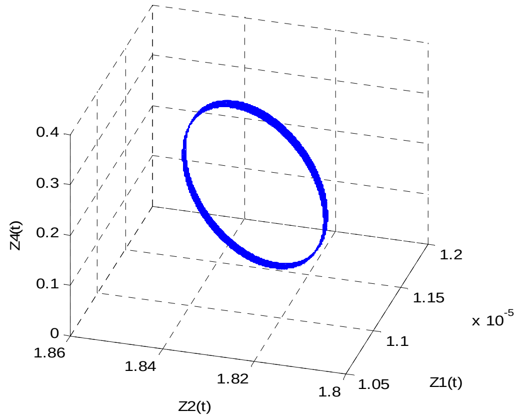

Fig. 11- Phase space dynamics for cycle region

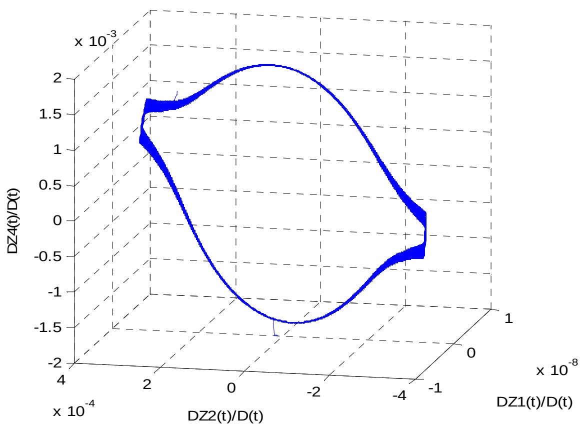

Fig. 12- Phase space dynamics for cycle region

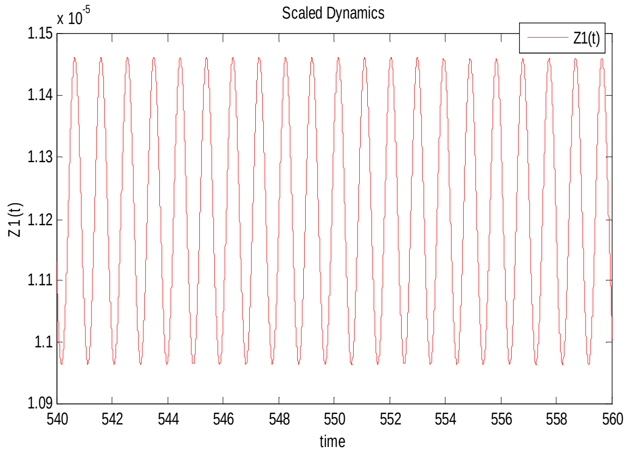

Fig. 13- Cycle dynamics for $Z_1(t)$

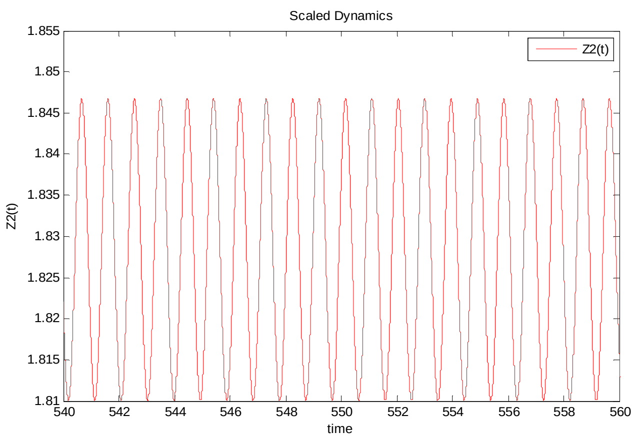

Fig. 14- Cycle dynamics for $Z_2(t)$

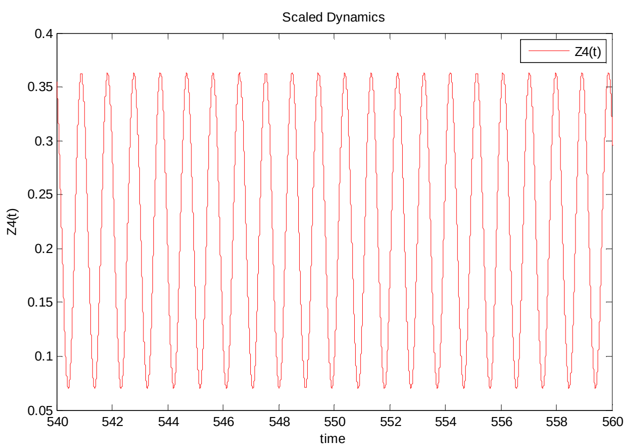

Fig. 15- Cycle dynamics for $Z_4(t)$

This brief description of cycles in $\mathbb{R}^3$ for a growing economy had the main purpose of demonstrating our routine potential to detect cycle regions numerically, rather than proposing a specific applied case for economic reasoning. Nonetheless, we finish this short introduction on cycles for the full optimal control problem by describing some of the implications arising from this specific application. First, the parameters chosen for this experiment describe an economy with a relatively low technology level, facing a relevant risk premium on foreign debt and with almost no home bias on investment in domestic assets. Second, the computed steady states reveal a highly indebted economy with a very low level of consumption relative to domestic assets and a high level of domestic investment. Finally, cycle dynamics suggest small cycles for scaled foreign debt but very wide ones for scaled investment. Scaled consumption cycles are almost negligible. All this evidence suggests that such dynamics match the observed outcomes of economies at the lowest development stage, such as almost all economies from the African continent and many in Latin America and Asia, where institutional arrangements led to an historical outcome governed by foreign indebtedness and slow growth. Still, two puzzles arise from this experiment, the low level of consumption to domestic assets and the cycle amplitude for scaled investment seem to be unrealistic. One explanation could be that such economies have relevant domestic assets, such as natural resources, but as their investment is mainly financed through foreign financial resources the incomes are transferred abroad leading to slow growth and low consumption levels. The large cycle amplitude of investment dynamics, which could help explaining the high volatility of investment time series, could just be a symptom of weak financial position for this open economy, since a quick inspection to $Z_2(t)$ and $Z_4(t)$ cycles portrayed in figures (14) and (15) suggest a negative relation between cycle dynamics of the variables. The economic intuition is straightforward, as the peak for the investment cycle is reached, debt dynamics are driven to the lowest cycle outcome by payments arising from domestic production, at this point the pressure of foreign debt payments drives investment down the cycle because of the low marginal productivity of domestic capital, which does not allow for an improvement on the overall financial position of this economy. As this dynamics dominate the system, few resources can be dedicated to consumption and the remaining majority of resources are used to remunerate foreign financial obligations, in order to maintain the fragile financial position of this economy through foreign financed investment that can only sustain slow economic growth.

6.3. Discussion

We conclude this brief presentation for the open economy facing risk premium and investment adjustment costs, defined by the full optimal control problem introduced in section *3.*, with a short discussion on the main issues that arise from our proposal and numerical outcomes obtained. Although, both our analytical and numerical exploration were much more limited for this model, compared to the extensive discussion dedicated to each particular non-linear case, we were still able to demonstrate the existence of *Hopf* bifurcations for economically feasible regions in this system numerically. The parameter combinations leading to such outcomes are wide and allow for different economic interpretations, still one stable pattern relating risk premium and investment adjustment costs arises from our numerical explorations, which allows for the coexistence of endogenous cycles and long run growth. This pattern implies a negative relation between risk premium and investment adjustment costs, when assuming home bias on investment decisions for domestic capital and risk premia on foreign debt/assets, on the vast majority of the cases we explored numerically, as it becomes obvious after a quick inspection of the graphical outcomes presented in section *4.1.* of the appendix. This result relating the level of financial openness with institutional pressure driving investment bias has a context in economic history, theory and literature, since it was one of the main causes leading highly indebted economies to lower their institutional protectionism, in order to access the benefits of international trade and the global economic expansion. In our proposal, this specific relation drives the system to zones where endogenous growth and cycles coexist and that are feasible for a wide variety of parameter combinations in the form of local *Hopf* bifurcations, when we consider the constant return hypothesis for exogenous technology, $0 \prec A \prec 1$. Of course exceptions always arise, even in our limited parameter space exploration. For the case with the same technology level of the experiment in this section, but with an high intertemporal elasticity of substitution in consumption, $\gamma = 0.9$, the relation changes to positive, as depicted by figure 41. Experiments performed assuming low levels of technology, $A = 0.07$, and high levels of intertemporal substitution rates in consumption, $y = 0.7$ and $\gamma = 0.9$ do not allow for a full definition of the relation involved as the slopes for this case are one in the first case and almost zero in the second case, as portrayed in figures 35 and 36. This issue has to do with the numerical simulation parameters and plotting decisions that still do not allow for a full description of these two cases. One final remark for this specific regions is the possible coexistence of two economically feasible cycle regions arising for each fixed point, $Z^{**}_{i,\pm}$, in the case where $\gamma = 0.9$. Finally, the case with $A = 0.2$ and $A = 0.9$, described in figure 44, suggests the possibility of existence of period doubling for a specific region of the parameters, in the neighbourhood of $Z^{**}_{i,-}$. Both the cases described still require further numerical exploration, which is beyond the scope of this proposal. Nevertheless, this initial numerical exploration on feasible *Hopf* bifurcation region is enough to serve as a departure point for the exploration of regions with more complex dynamics. However, a numerical exploration of the planes arising from the steady states relation is required, in order to deepen the understanding of this economic system and possible mechanics leading to global bifurcations.

These numerical results have important implications for macroeconomic theory since the assumption of institutional arrangements leading to home bias on investment can easily be explained by a wide range of costs arising from investment in foreign based assets. This assumption is also widely supported by economic theory and empirical research on the subject, which relates information, cultural, bureaucratic and installation costs, along with exchange rate uncertainty, as some of the main causes for this outcome. It is reasonable to believe that the continued dismantling of commercial and investment barriers along with the investment banking revolution, arising from information technology innovations and increasing international financial openness that led to relevant reductions on the costs associated with investment in foreign assets, is driving this institutional parameter down and imposing a smaller home bias on investment in domestic assets. Nevertheless, the market share of financial markets for foreign assets investment decisions is still relatively small to accommodate the natural bias towards domestic assets and it is doubtful that such epoch will arise in the near future. Taking this into account, we are able to link this evidence with our outcomes for economically feasible regions that undergo *Hopf* bifurcations and allow for the coexistence of endogenous cycles and growth. Our full proposal is also consistent with the strategy of not imposing *a posteriori* rules for the definition of system dynamics, suggesting that the introduction of further non-linearities on macrodynamic proposals is one of the paths to follow, in order to match the evidence suggested by empirical research, and also to allow for "cleaner" modelling proposals that entail richer dynamical outcomes. Further numerical analysis suggests that this system undergoes global bifurcations, such as the heteroclinic bifurcation portrayed in section *4.2.* of the appendix. These complex outcomes may pave the way for the introduction in the macrodynamics literature of important conceptual themes already discussed in the recent field of macroeconomics phase transitions theory, by taking advantage of the theoretical results provided by the mathematical field of non-linear dynamics. We finish this discussion here with the conviction that our preliminary numerical findings for local *Hopf* bifurcations regions can open the door for the definition of further relevant dynamical outcomes with economic interpretation. Nevertheless, we are fully aware that further results will require a more global perspective of this system and to achieve them we have to aim on improving both the numerical procedures employed and extend the limited initial analytical framework presented.

7. Conclusions and further research

We finish this paper with a brief overview of our main results and future topics of discussion. First, we would like to highlight the role that risk premia on holdings of foreign assets/debt has on the introduction of transitions with endogenous cycles and further interesting dynamics in models of endogenous growth. By assuming always dynamic rules for adjustment in consumption decisions, we were able to propose a meaningful piecewise smooth ODE system[16] in $\mathbb{R}^2$, where *Hopf* bifurcations arise, and to determine that transitions to long run growth in the particular case with investment adjustment costs, are only possible if we consider an additional endogenous rule. Therefore we dismiss the claims that risk premia in the form of convex adjustment on foreign assets/debt is unable to introduce transitions in growth models and that convex investment adjustment costs on its own is a plausible non-linearity to consider in order to introduce transitions to long run growth, since we have to consider both endogenous transition rules and bias on investment on foreign assets, which are unlikely. Finally, we showed how it is possible to obtain a meaningful dynamical system, when both institutional non-linearities are considered, assuming just the dynamic adjustment rule for indifference in accumulation in consumption decisions and no additional *a posteriori* modelling rules. By following this path, we showed that this system is not only able to reproduce the coexistence cycles and growth in the form of local *Hopf* bifurcations, but also we put forward some additional evidence suggesting that more complex dynamics with possible economic interpretation exist in the form of global bifurcations. Finally, we were able to demonstrate numerically that assuming both institutional non-linearities, we are able to solve the issue of bias on investment decisions satisfactorily, which leads us into meaningful economic dynamic outcomes with transitions to coexist with regions undergoing *Hopf* bifurcations in $\mathbb{R}^3$, where there is home bias in investment decisions towards domestic assets. We conclude this presentation by stating that it is our conviction that the path we followed will surely gain its ground on the macrodynamics literature and fill some of the gaps existent in macroeconomic theory of development and growth dynamics. The issues discussed in this paper are still too conceptual to be readily transposed to policy applications,

[16]See Di Bernardo et al (2007) for a detailed introduction to this theme.

nevertheless it is our opinion that the introduction of further non-linearities in macro-models will allow for a better comprehension of the reasons leading to boom/bust dynamics and phase transitions in economic systems. Finally, we would like put a final comment on the issues regarding complexity science theory arising from our proposal. Since we are proposing macro-models based on low dimensional nonlinear dynamics, it is doubtful that our simple proposal in $\mathbb{R}^2$ and even our complete proposal in $\mathbb{R}^3$ can be considered to fit into the contemporary research on complex dynamics. Nevertheless, the testing of the hypotheses forwarded in this paper against empirical data will surely belong to the field of statistical complexity, since the specific applications involved are not straightforward and will surely involve a wide discussion on complex matters arising from economic macro-data and sophisticated methods of statistical estimation, sampling and hypothesis testing.

Appendix

1. Threshold equilibrium dynamics for the economy facing risk premium

Assuming the only equilibrium possibly for the economy with risk premium on foreign assts/debt as the one satisfying the threshold indifference condition $\bar{Z}_2 = Z_2^t$, we can use the system described in (25) to obtain the meaningful expressions that describe long run growth solutions arising from equilibrium in the scaled dynamical system describing this economy. Briefly, the conditions for existence of long run growth on the indifference thresholds of debt/assets and capital accumulation in this system are given by the following algebraic system of equations following (25):

$$\begin{aligned} &\bar{Z}_2 = Z_2^t \Rightarrow \bar{Z}_2 = 1 \pm \frac{\sqrt{(rd)^2 + 2rd\left(r+\delta-A\right)}}{rd} \\ &\dot{Z}_1 = 0 \Rightarrow \bar{Z}_2 = \frac{\rho - r + \left(\delta - \bar{Z}_4\right)\left(\gamma - 1\right)}{rd} \\ &\dot{Z}_2 = 0 \Rightarrow \bar{Z}_1 = A - \bar{Z}_4 - \left(r + rd\bar{Z}_2 + \delta - \bar{Z}_4\right)\bar{Z}_2 \succ 0 \end{aligned} \tag{71}$$

From the algebraic system described in (71) it is straightforward to obtain a parameter constraint for the existence of feasible equilibrium solutions for $\bar{Z}_2$. This expression can be generically given by the following equality:

$$rd \pm \sqrt{(rd)^2 + 2rd\left(r+\delta-A\right)} = \rho - r + \left(\delta - \bar{Z}_4\right)\left(\gamma - 1\right) \tag{72}$$

Since existence of equilibrium in scaled assets/debt depends on a strict restriction given by static parameter combinations, we can rule out the existence of transitional dynamics for this system without going through further analysis on qualitative dynamics. Nevertheless, since there exists two equilibrium solutions arising from indifference in accumulation threshold rule and long run growth solution implies that $\bar{Z}_1 \in \mathbb{R}^+$, $\bar{Z}_2 \in \mathbb{R} \Rightarrow (rd)^2 + 2rd\left(r+\delta-A\right) \geq 0 \Leftrightarrow A \leq r + \delta + \frac{1}{2}rd$ and $\bar{Z}_3 \in \mathbb{R}^+ \Rightarrow \bar{Z}_4 \succ \delta$, we can extend the general expressions for the existence of a BGP given in (71) and (72), and describe the three possible economic feasible outcomes that may arise as long run solutions to the economy with risk premium, assuming indifference on state accumulation as the steady state solution for scaled investment, $\bar{Z}_4$:

$$\bar{Z}_4 = \frac{\rho - r + \delta\left(\gamma-1\right) - rd \mp \sqrt{(rd)^2 + 2rd\left(r+\delta-A\right)}}{\left(\gamma-1\right)} \succ \delta \tag{73}$$

Following (73) we can obtain by substitution the fixed points governing long run growth dynamics for scaled consumption by substituting Z_2^t and $\bar{Z}_4$ in $\bar{Z}_1$.

$$\bar{Z}_1\left(Z_{2,+}^t\right) = A - r - \delta - rd\left(Z_2^t\right)^2 - \left(r + \delta - \bar{Z}_4\left(Z_2^t\right)\right)\left(Z_2^t - 1\right) \succ 0 \tag{74}$$

Assuming these conditions for long run growth equilibrium, we can easily define the parameter restrictions that arise in each of the three possible cases and that are relevant for the solution of this specific hypothesis.

i. Parameter restrictions for economies on credit region- $Z_{2,+}^{t} \geq 1$

$$\begin{aligned} & A \leq r + \delta + \frac{1}{2} rd \\ & A - r - \delta - rd\left(Z_{2,+}^{t}\right)^{2} - \left(r + \delta - \bar{Z}_{4}\left(Z_{2,+}^{t}\right)\right)\left(Z_{2,+}^{t} - 1\right) \succ 0 \\ & \rho - r - rd - \sqrt{(rd)^{2} + 2rd\left(r + \delta - A\right)} \prec 0 \end{aligned} \tag{75}$$

ii. Parameter restrictions for economies on credit region- $0 \leq Z_{2,-}^{t} \leq 1$

$$\begin{aligned} & r + \delta \leq A \leq r + \delta + \frac{1}{2} rd \\ & A - r - \delta - rd\left(Z_{2}^{t}\right)^{2} - \left(r + \delta - \bar{Z}_{4}\left(Z_{2}^{t}\right)\right)\left(Z_{2}^{t} - 1\right) \succ 0 \\ & \rho - r - rd + \sqrt{(rd)^{2} + 2rd\left(r + \delta - A\right)} \prec 0 \end{aligned} \tag{76}$$

iii. Parameter restrictions for economies on debit region- $Z_{2,-}^{t} \prec 0$

$$\begin{aligned} & A \prec r + \delta \\ & A - r - \delta - rd\left(Z_{2}^{t}\right)^{2} - \left(r + \delta - \bar{Z}_{4}\left(Z_{2}^{t}\right)\right)\left(Z_{2}^{t} - 1\right) \succ 0 \\ & \rho - r - rd + \sqrt{(rd)^{2} + 2rd\left(r + \delta - A\right)} \prec 0 \end{aligned} \tag{77}$$

The main economic intuition arising from these restrictions is that the marginal net revenue of capital, $A-\delta$, as to be smaller than the marginal revenue on foreign debt plus the exogenous risk premium, $r+\frac{1}{2}rd$, to allow for the existence of meaningful economic regions. Further, in the cases where the net marginal revenue on domestic capital is small enough, then it is possible to consider as solutions to this problem regions where countries accumulate foreign assets. This can be seen as a substitution effect arising from the differences in net marginal revenue between foreign and domestic assets, which lead this economy to assume a net creditor position, when domestic revenues are structurally low.

In order to evaluate the conditions described for the existence of economic feasible outcomes in this economy, we evaluated numerically the outcomes for scaled consumption and debt/assets steady states assuming the restrictions described in (75) to (77), for each of the three possible situations described. Fixing the parameters $\rho = \delta = 0.5$ and setting for each evaluation feasible values of γ and r, we explored interval regions for parameters A and d, where reasonable outcomes for the steady state values of $\bar{Z}_i$ exist. These results are a mere illustration of some outcomes arising from a broader numerical exploration performed with the purpose of defining which parameter combinations would be more likely to produce reasonable economic outcomes. Figures (16) to (24) below summarize some results from this experiment for each of the possible threshold equilibrium solutions:

Steady- state outcomes for $Z_{2,+}^{t} \geq 1$ *assuming* $r = 0.05$ *and* $\gamma = 0.7$

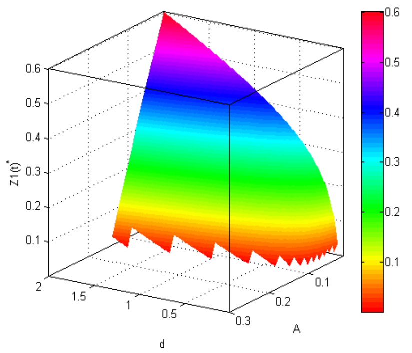

Fig. 16- $\bar{Z}_1$ steady-state surface

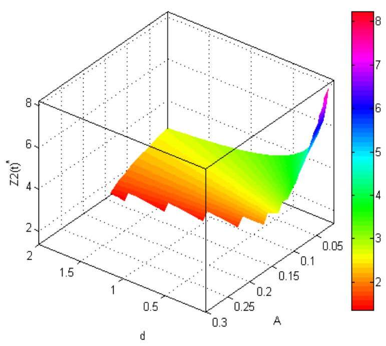

Fig. 17- $\bar{Z}_2$ steady-state surface

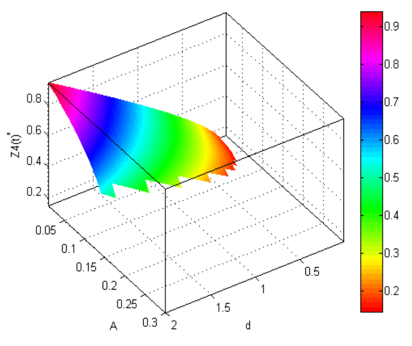

Fig. 18- $\bar{Z}_4$ steady-state surface

Steady- state outcomes for $0 \le Z_{2,-}^{t} \le 1$ *assuming* $r = 0.05$ *and* $\gamma = 0.3$

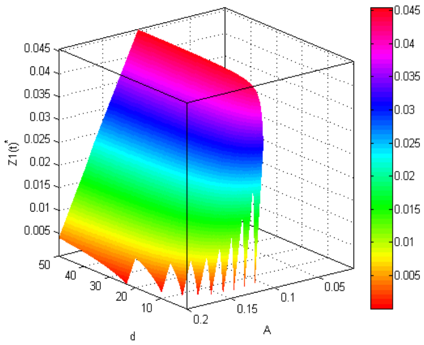

Fig. 19- $\bar{Z}_1$ steady-state surface

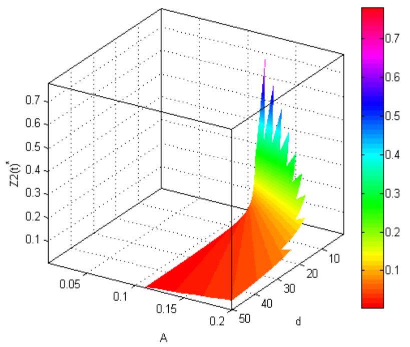

Fig. 20- $\bar{Z}_2$ steady-state surface

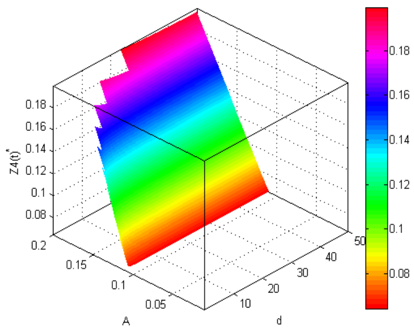

Fig. 21- $\bar{Z}_4$ steady-state surface

Steady- state outcomes for $Z_{2,-}^{t} \prec 0$ *assuming* $r = 0.06$ *and* $\gamma = 0.3$

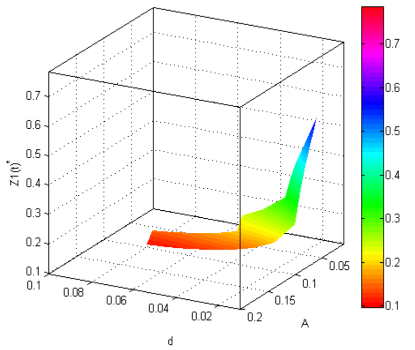

Fig. 22- $\bar{Z}_1$ steady-state surface

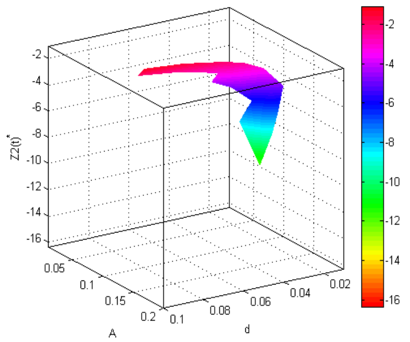

Fig. 23- $\bar{Z}_2$ steady-state surface

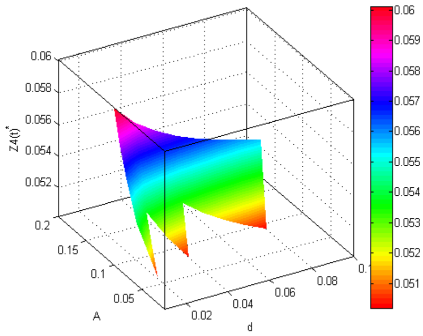

Fig. 24- $\bar{Z}_4$ steady-state surface

2. Linearized dynamics, numerical analysis and experiments for the Credit/Debit Economy

2.1. Credit/Debit Economy linearized dynamic analysis

For the universal fixed point region, both economies will share the second eigenvalues obtained from (31):

$$\lambda_2^{+}\left(Z_{2,+}^{*}\right) = \sqrt{\left(r - \bar{Z}_4 + \delta\right)^2 - 2rd\left(\bar{Z}_4 - A\right)}$$
$$\lambda_2^{-}\left(Z_{2,-}^{*}\right) = -\sqrt{\left(r - \bar{Z}_4 + \delta\right)^2 - 2rd\left(\bar{Z}_4 - A\right)} \qquad (78)$$

Assuming the condition given in expression (79), to guarantee that the universal fixed point has only real parts, it is sufficient to guarantee that there are no eigenvalues with only imaginary parts and this fixed point is hyperbolic. We can assume that the signs for each common eigenvalues depend only on the sign of the root of Z_2^{*} and from here, determine the remainder conditions for obtaining a sketch of the qualitative dynamics involved in this specific region through linearization.

$$d \prec \left(r - \bar{Z}_4 + \delta\right)^2 \left[2r\left(\bar{Z}_4 - A\right)\right]^{-1} \qquad (79)$$

For the credit economy case, the eigenvalues for the universal fixed point dynamics can be simplified to:

$$\lambda_1^{+} = \left(\gamma - 1\right)^{-1}\left[\rho + \left(\delta - \bar{Z}_4\right)\gamma - \sqrt{\left(r - \bar{Z}_4 + \delta\right)^2 - 2rd\left(\bar{Z}_4 - A\right)}\right] \qquad (80)$$

$$\lambda_1^{-} = \left(\gamma - 1\right)^{-1}\left[\rho + \left(\delta - \bar{Z}_4\right)\gamma + \sqrt{\left(r - \bar{Z}_4 + \delta\right)^2 - 2rd\left(\bar{Z}_4 - A\right)}\right] \qquad (81)$$

Assuming now that condition (79) holds and recalling that $\gamma - 1 \prec 0$, we summarize the qualitative dynamics for the credit economy case in the region of the universal fixed point in table 7:

$Z_{2,+}^*$	$Z_{2,-}^*$
Repellor - $\rho+\left(\delta-\bar{Z}_4\right)\gamma-\sqrt{\left(r-\bar{Z}_4+\delta\right)^2-2rd\left(\bar{Z}_4-A\right)}\prec 0$	node- $\rho+\left(\delta-\bar{Z}_4\right)\gamma+\sqrt{\left(r-\bar{Z}_4+\delta\right)^2-2rd\left(\bar{Z}_4-A\right)}\succ 0$
Saddle point - $\rho+\left(\delta-\bar{Z}_4\right)\gamma-\sqrt{\left(r-\bar{Z}_4+\delta\right)^2-2rd\left(\bar{Z}_4-A\right)}\succ 0$	Saddle point- $\rho+\left(\delta-\bar{Z}_4\right)\gamma+\sqrt{\left(r-\bar{Z}_4+\delta\right)^2-2rd\left(\bar{Z}_4-A\right)}\prec 0$

Table 7- Qualitative dynamics of universal fixed point in credit economy region

For the debit economy case, it is also possible to define the qualitative dynamics of this system following the same assumptions given for the credit economy. Still, it is not possible to obtain simplified expressions as given in table 7, for this economy. Nevertheless, we can assume the general expression given in (31) for $\lambda_1^* = \left.\frac{d\dot{Z}_1}{dZ_1}\right|_{Z_2=Z_2^*}$ and describe the qualitative dynamics for this system in the same fashion. Table 8 summarizes the qualitative dynamics for this economy in the vicinity of the universal fixed point, assuming that $\operatorname{Re}\left(\left.\frac{d\dot{Z}_1}{dZ_1}\right|_{Z_2=Z_2^*}\right)\neq 0$ and condition (79) hold, in order to guarantee the universal fixed point is hyperbolic in the debit economy case:

$Z_{2,+}^*$	$Z_{2,-}^*$
Repellor - $\left.\frac{d\dot{Z}_1}{dZ_1}\right|_{Z_2=Z_{2,+}^*}\succ 0$	node- $\left.\frac{d\dot{Z}_1}{dZ_1}\right|_{Z_2=Z_{2,-}^*}\prec 0$
Saddle point - $\left.\frac{d\dot{Z}_1}{dZ_1}\right|_{Z_2=Z_{2,+}^*}\prec 0$	Saddle point- $\left.\frac{d\dot{Z}_1}{dZ_1}\right|_{Z_2=Z_{2,-}^*}\succ 0$

Table 8- Local linearized dynamics for universal fixed point in debit economy region

We conclude this appendix section with the meaningful conditions that describe the specific fixed point dynamics for the credit/debit economy. First, we start by defining the set of conditions for existence of hyperbolic specific fixed points. These conditions can be defined generically for both economies following the eigenvalues expression given in (32) in a intuitive fashion. Expression (82) summarizes these conditions:

$$Z_2^{**}\neq\frac{\bar{Z}_4-\delta-r}{rd}\vee\left.\frac{d\dot{Z}_1}{dZ_2}\right|_{Z_i=Z_i^{**}}\succ 0 \qquad (82)$$

For the credit economy case it is straightforward to observe that the economic feasible region of steady states, $Z_1^{**}\succ 0$ described in (30) is sufficient condition to guarantee both hyperbolicity and saddle path dynamics. This intuition is best observed using the determinant and trace of the general specific *Jacobian* matrix given in (32):

$$Det\left(J^{**}\right)=-\left.\frac{d\dot{Z}_1}{dZ_2}\right|_{Z_i=Z_i^{**}},\quad tr\left(J^{**}\right)=r+rdZ_2^{**}-\bar{Z}_4+\delta \qquad (83)$$

For this specific case the hyperbolicity condition and the determinant of the *Jacobian* matrix expressed below are completely determined by the restriction for the economic meaningful region of steady states, $Z_{2,-}^*\prec Z_2^{B,**}\prec Z_{2,+}^*\Rightarrow Z_1^{B,**}\succ 0$, which is a saddle point under the *Grobman-Hartman* theorem conditions, since $\gamma\prec 1$.

$$\left.\frac{d\dot{Z}_1^B}{dZ_2^B}\right|_{Z_1^B=Z_1^{B,**}}=-\frac{rd}{\gamma-1}Z_1^{b,**}\succ 0\Rightarrow Det\left(J^{B,**}\right)=\frac{rd}{\gamma-1}Z_1^{b,**}\prec 0 \qquad (84)$$

Finally, we describe the linearized dynamics for the debit economy. In this specific case, it is not straightforward to define the conditions to describe the qualitative dynamics following the *Grobman-Hartman* theorem. Assuming that the general conditions for hyperbolicity, given in (82), hold for this case also, the specific determinant for this economy is given by the following expression:

$$Det\left(J^{K,**}\right)=rd\left(\gamma-1\right)^{-1}Z_1^{K,**}Z_2^{K,**} \qquad (85)$$

Under the strict assumptions that $\operatorname{Re}\left(Z_2^{K,**}\right) \neq 0 \Rightarrow 2(rd)^{-1}\left[\rho+\delta-A+\left(\delta-\bar{Z}_4\right)(\gamma-1)\right] \succ 0$ and $Z_1^{K,**} \succ 0 \Rightarrow Z_{2,-}^* \prec Z_2^{K,**} \prec Z_{2,+}^*$, we can easily summarize the dynamic varieties in the vicinity of the debit economy fixed points. These results are summarized in table 9 bellow:

$Z_{2,+}^{K,**}$	$Z_{2,-}^{K,**}$
Saddle point	node- $Z_{2,-}^{K,**} \prec \dfrac{\bar{Z}_4-\delta-r}{rd}$ Repellor- $Z_{2,-}^{K,**} \succ \dfrac{\bar{Z}_4-\delta-r}{rd}$ Hof bifurcation- $Z_{2,-}^{K,**} = \dfrac{\bar{Z}_4-\delta-r}{rd}$

Table 9- Local linearized dynamics for specific fixed points of debit economy

The dynamics near the $Z_{2,-}^{K,**}$ fixed point will be dominated by net flow of capital scaled by the exogenous risk premium combination of parameters, rd . For economies where the net flow of financial capital is positive, $\bar{Z}_4 \succ \delta - r$, this fixed point is always a node. For economies where the net flow of capital is negative this fixed point can be a node or a repellor. Nevertheless, the existence of a *Hopf* bifurcation point means that this specific fixed point might pave the door for an infinite cycle region, through interaction with the universal fixed points. This could mean that within specific regions of the plane arising from specific restrictions on parameters and dominated by debit economy dynamics, might create the conditions for degenerate cycle regions with stability to occur and the existence homoclinic bifurcations. Such dynamics are only possible in economies where the net flow of financial capital is exogenously defined as negative and the specific fixed point for positive foreign assets interacts with the universal fixed points.

2.2. Local Hopf bifurcation feasible economic regions

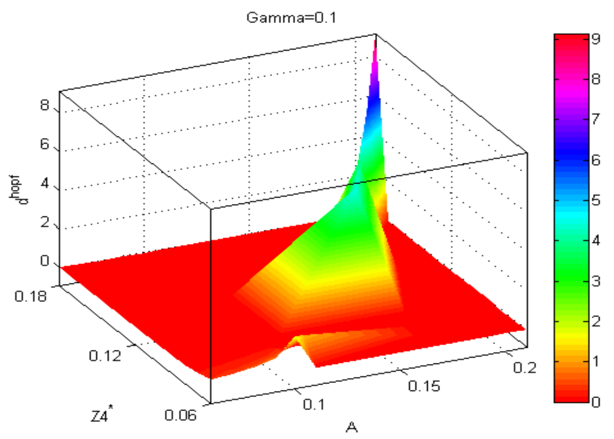

Fig. 25- Feasible parameter space $\gamma = 0.1$

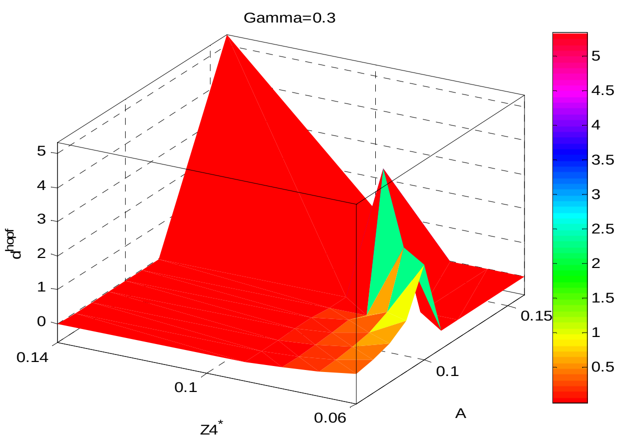

Fig. 26- Feasible parameter space $\gamma = 0.3$

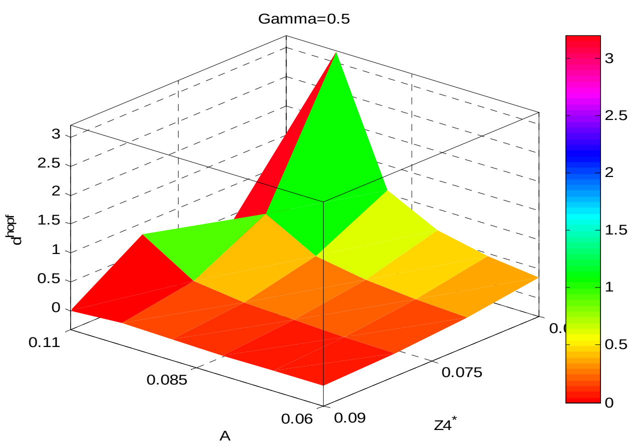

Fig. 27- Feasible parameter space $\gamma = 0.5$

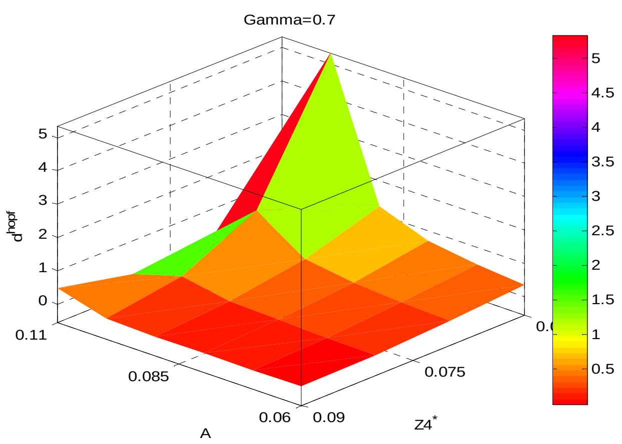

Fig. 28- Feasible parameter space $\gamma = 0.7$

2.3. Further relevant simulations and data reconstruction for Debit/Credit economy

Data reconstruction assuming uncertainty on exogenous risk premium

As discussed in section *4.3.*, we put forward in this short appendix section the two main developments that we consider crucial in future research for the proposed model of a debit/credit card economy. First, we suggest the introduction of risk premium uncertainty of the form, $d(t) \sim N\left(d^{hopf}, 0.01\right)$, where there is no memory from the past moments in each time period. This proposal is reasonable since we already discussed the critical transitions for the *Hopf bifurcation*, so the outcomes all lie in explored regions of the plane. Further, regarding uncertainty in exogenous risk premium in this fashion resembles the information asymmetries faced by the financial sector and the institutional framework that determines risk premium on international debt markets. We experiment on this hypothesis following the same simulation parameters forwarded in table 1 of section *4.3.*. Figures 28 to 31 portrait the outcome of this speculative experiment for $t = 20000$. Although, we didn't follow the usual path of sampling our trials due to time limitations and scope of this paper, this simple experiment shows how uncertainty on risk premium is capable of maintaining cycle stability and reproduce alternating cycle periods with different amplitudes.

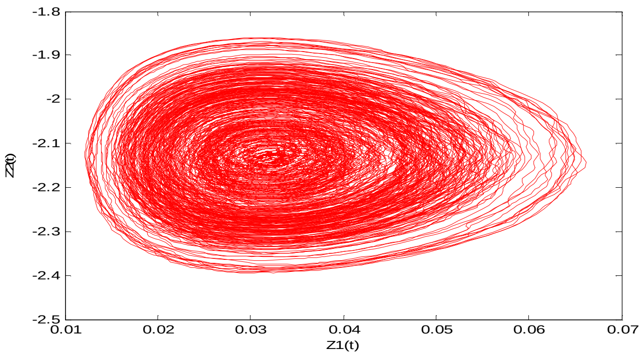

Fig. 28- Two dimensional phase space for risk premium uncertainty

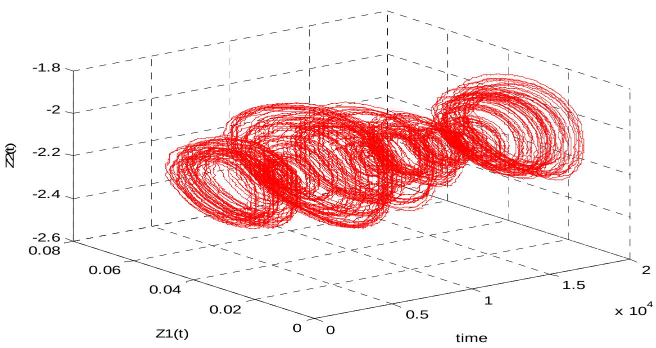

Fig. 29- Three dimensional phase space for risk premium uncertainty

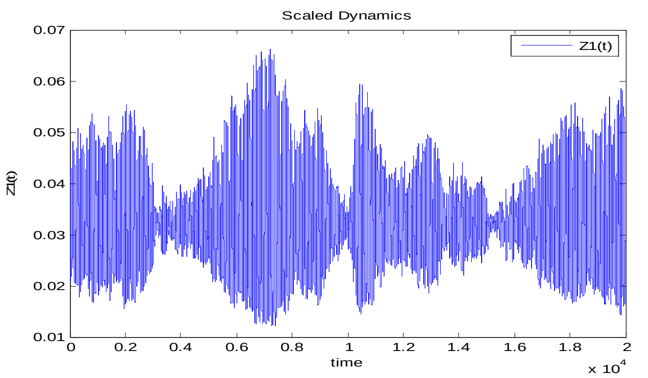

Fig. 30- $Z_1(t)$ time path for risk premium uncertainty

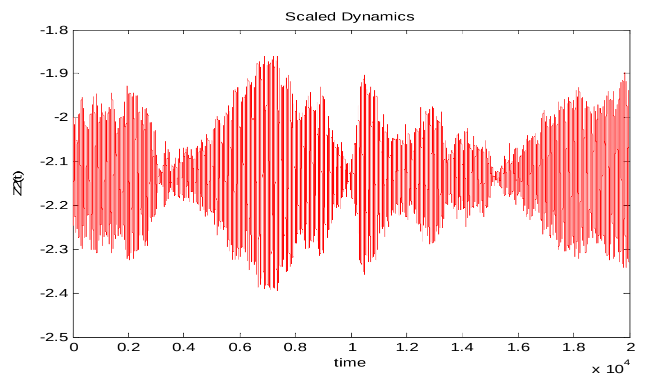

Fig. 31- $Z_2(t)$ time path for risk premium uncertainty

Degenerate infinite period cycles and global switching threshold dynamics for non economic regions

This last experiment on the dynamics of the debit/credit economy deals with the possibility of local fixed point interaction leading to infinite period cycles. This hypothesis was already discussed in the main text and its interest to long run growth macroeconomics argued. The dynamics depicted on this section cannot be directly transposed to economic reasoning since the infinite cycle region portrayed is located in the second and third quadrant of the plane. Nevertheless, we believe that improved intuition on this specific cycle region may allow for further comprehension of the credit/debit economy global dynamics, which in turn will lead to further results with economic meaning. Table 10 summarises the parameters necessary to run this simulation and figures 32 to 35 portray the dynamic outcome of this simulation, which impose four passages on the two control switching state thresholds.

ρ	r	δ	$\bar{Z}_4$	γ	A	d	$Z_1(0)$	$Z_2(0)$
0.05	0.05	0.05	0.15	0.35	0.06	0.5	-0.1	0.5

Table 10- Simulation parameters for negative infinite period cycles

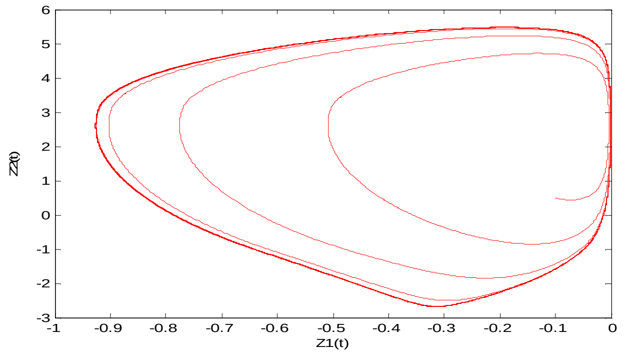

Fig. 32- Two dimensional phase space for negative cycle dynamics

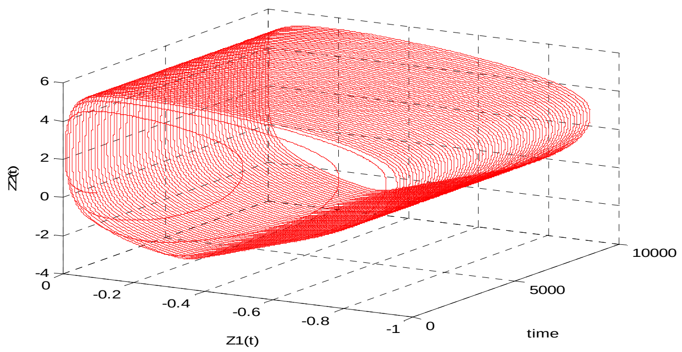

Fig. 33- Three dimensional phase space for negative cycle dynamics

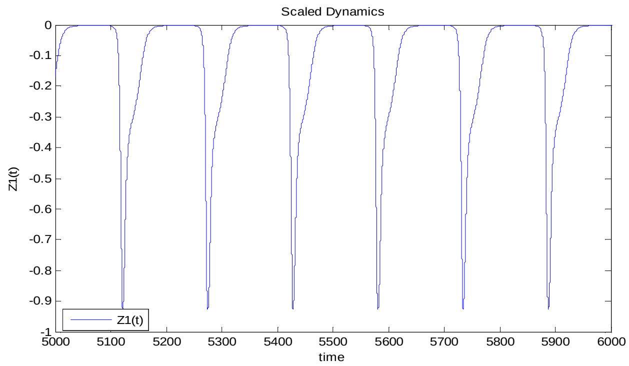

Fig. 34- $Z_1(t)$ time path for negative cycle

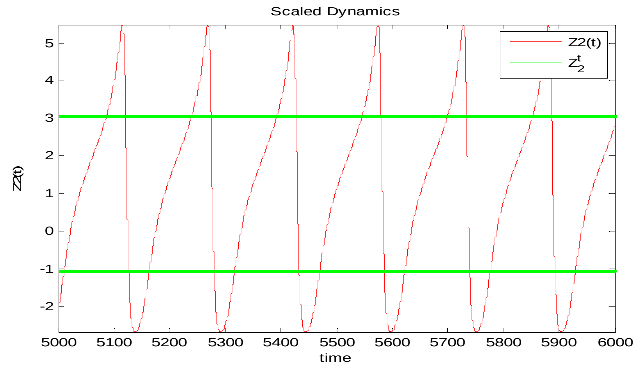

Fig. 35- $Z_2(t)$ time path for negative cycle

3. Numerical results for parameter outcomes in the economy with investment adjustment costs

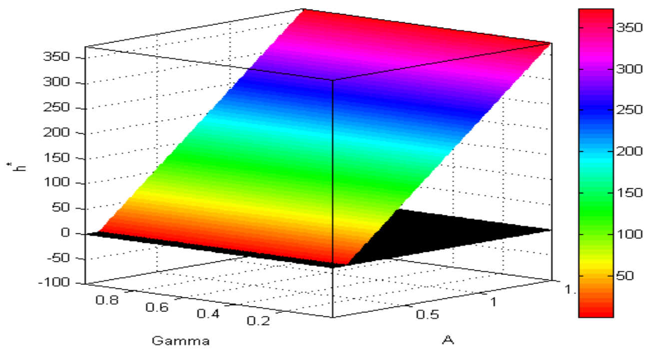

Fig. 36- Economic feasible values for h^* parameter ($\rho = 0.05$)

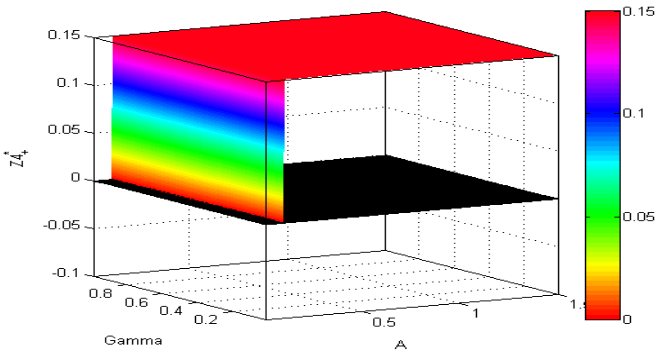

Fig. 37- Economic feasible equilibrium for scaled investment ($Z_{4,+}^{**}$)

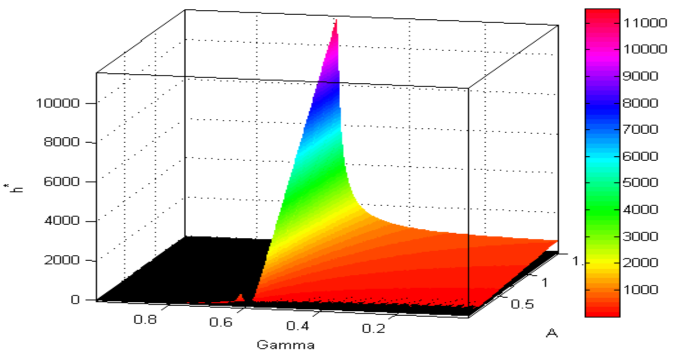

Fig. 38- Economic feasible values for h^* parameter ($\rho = 0.07$)

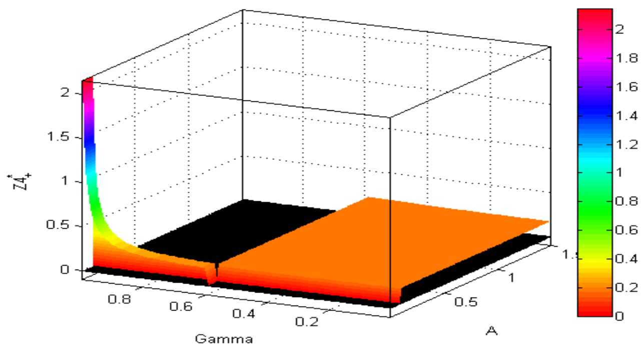

Fig. 39- Economic feasible equilibrium for scaled investment ($Z_{4,+}^{**}$)

4. Numerical analysis for the economy facing risk premium and investment adjustment costs

4.1. Risk premium and investment adjustment costs feasible cycle economic regions in $\mathbb{R}^3$

In this section, we present the computed parameter combinations for economically feasible regions with *Hopf* bifurcations, following our numerical proposal discussed in detail in section *6.3.* of the main text. For different values of exogenous technology, $A \in \{0.07;0.11;0.2;1.05\}$, and intertemporal elasticity of substitution in consumption, $\gamma \in \{0.1;0.3;0.5;0.7;0.9\}$, we loop through $d \in [0.001,10]$ and $h \in [-10,-0.001]$, using a simple grid search procedure with a search interval equal to 0.01, to define parameter combinations where local *Hopf* bifurcations with possible economic interpretation occur in the neighbourhood of both fixed points of the system, depicted by $Z_{i,+}^{**}$(-) and $Z_{i,-}^{**}$(-). The graphics depicting the results obtained for parameter regions that fit into the constraints described in the main text follow below, where the value considered for parameter d is given by the average value of the parameter interval defined when the complex conjugate pair of eigenvalues crosses the imaginary axis[17]. We dismiss for obvious reasons the empty sets obtained from this numerical parameter space exploration, which include all sets with $A = 1.05$:

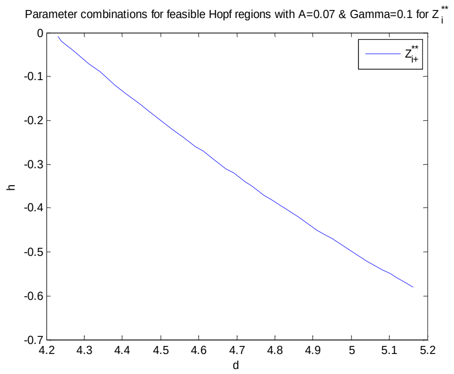

Fig. 40- $A = 0.07$ & $\gamma = 0.1$

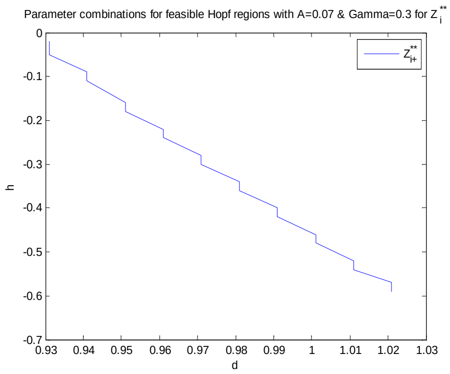

Fig. 41- $A = 0.07$ & $\gamma = 0.3$

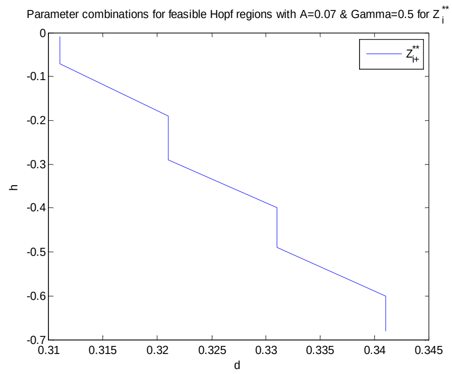

Fig. 42- $A = 0.07$ & $\gamma = 0.5$

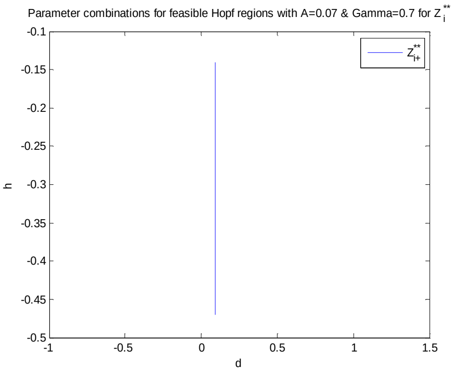

Fig. 43- $A = 0.07$ & $\gamma = 0.7$

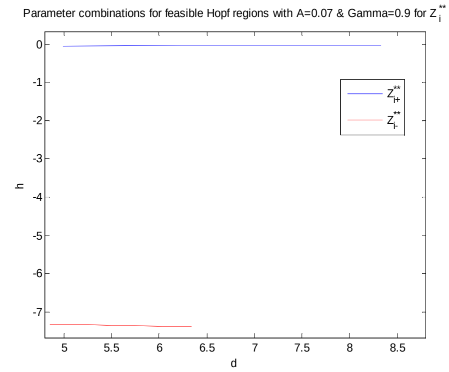

Fig. 44- $A = 0.07$ & $\gamma = 0.9$

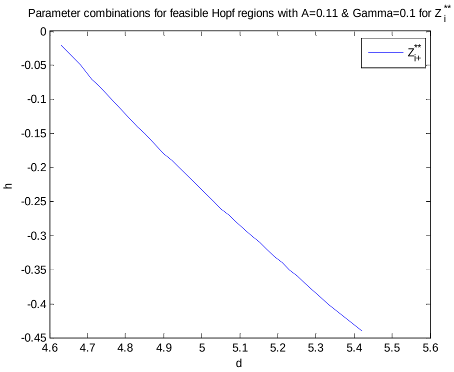

Fig. 45- $A = 0.11$ & $\gamma = 0.1$

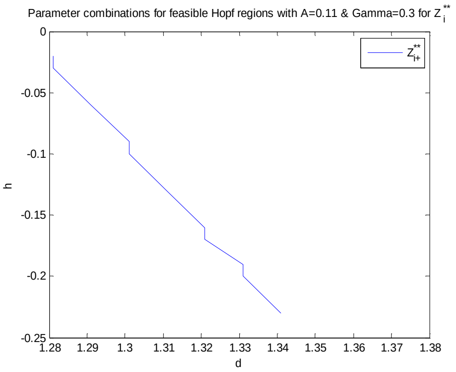

Fig. 46- $A = 0.11$ & $\gamma = 0.3$

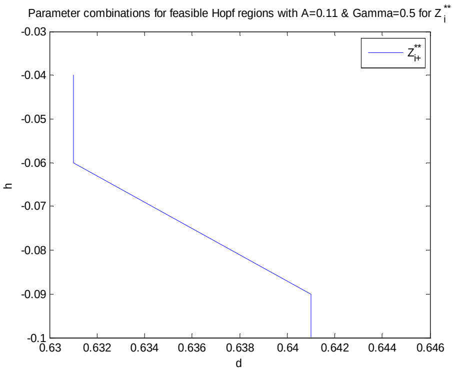

Fig. 47- $A = 0.11$ & $\gamma = 0.5$

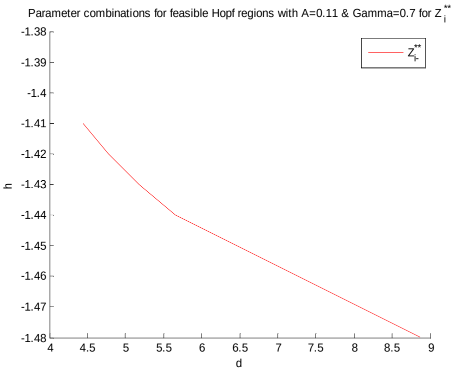

Fig. 48- $A = 0.11$ & $\gamma = 0.7$

[17]By assuming a wider iteration interval, to limit the computation time of our routine and allow for a wider parameter space exploration, some of the figures show discontinuities arising from this decision and the use of the described value for d. Nevertheless, such discontinuities and further numerical errors that may occur in our steady state computations, are already small enough to allow the main patterns governing the adjustment costs parameter relation for regions with *Hopf bifurcations* to arise.

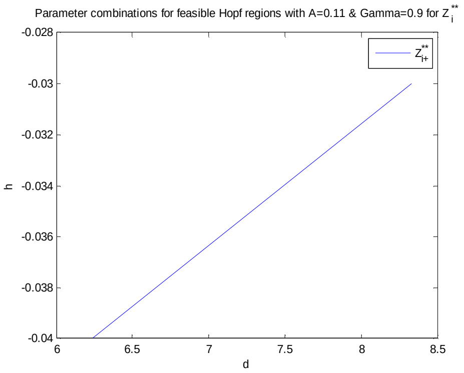

Fig. 49- $A = 0.11$ & $\gamma = 0.9$

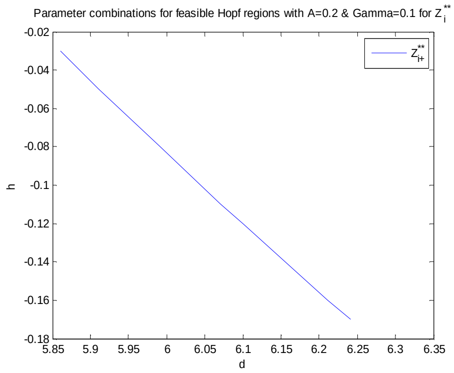

Fig. 50- $A = 0.2$ & $\gamma = 0.1$

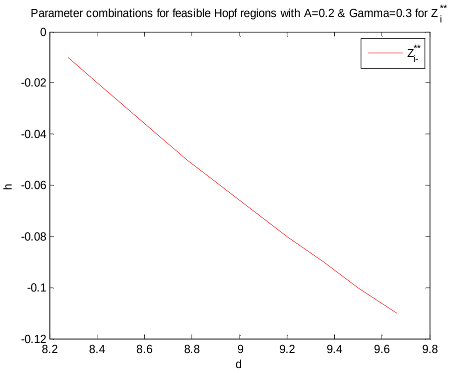

Fig. 51- $A = 0.2$ & $\gamma = 0.3$

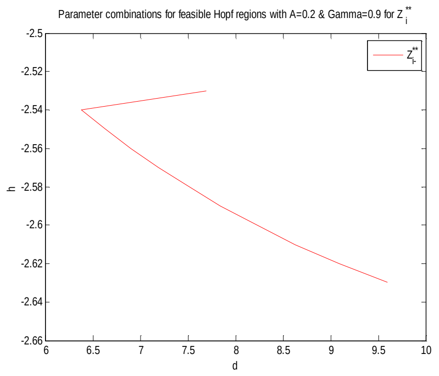

Fig. 52- $A = 0.2$ & $\gamma = 0.9$

4.2. Further numerical experiments and extensions

The last section of this appendix is dedicated to the introduction of further relevant dynamics arising for the economy facing risk premium and investment adjustment costs, which we came about while performing preliminary numerical simulations with this system or from the numerical outcomes obtained for the previous section. This section will not focus on the specific causes leading to the dynamic outcomes we present below, but has the intention of introducing the analysis of global bifurcations for this specific system, in order to promote the introduction of more complex dynamic outcomes in the macrodynamics economic theory.

Global bifurcations, crisis and heteroclinic connections of cycles

ρ	r	δ	h	γ	A	d	$Z_1(0)$	$Z_2(0)$	$Z_4(0)$
0.03	0.05	0.05	-0.01	0.7	0.06	0.4	0.1	-1	0.2596

Table 11-Parameter values for Heteroclinic bifurcation

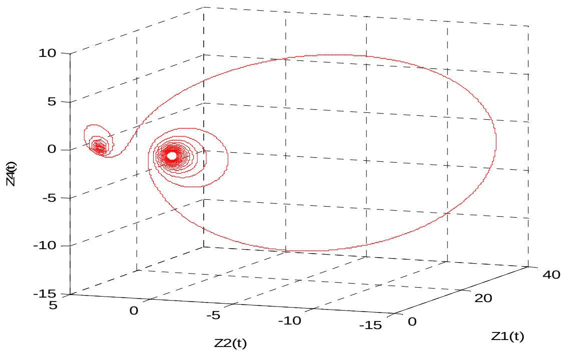

Fig. 53- Phase space for heteroclinic connection of cycles

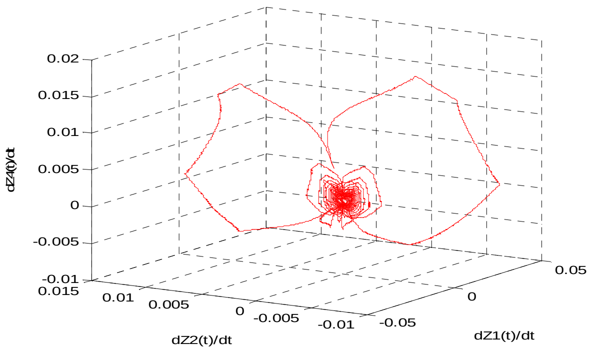

Fig. 54- Phase space for heteroclinic connections of cycles

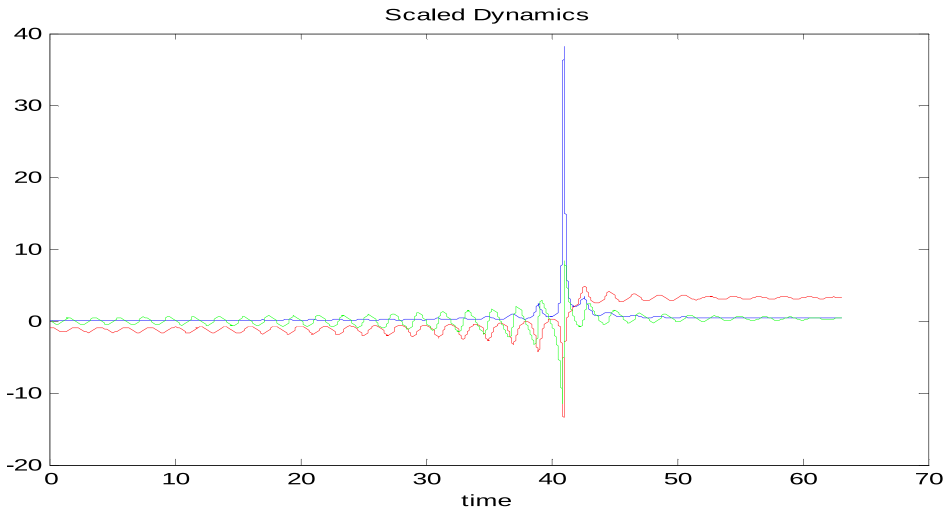

Fig. 55- Scaled dynamics for heteroclinic bifurcation

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