The existence of Nash equilibria in *n*-player LQ-games, with applications to international monetary and trade agreements*

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Abstract. The paper studies the relationship between equilibrium existence in LQ games and the classical theory of economic policy, generalizing some recent results. In particular, by focusing on system controllability instead of the controllability by one or some of the players, we find conditions for the existence of the Nash equilibrium that extend those required by the previous literature. The usefulness of our results is described by some examples in the field of international monetary and trade agreements. The essence of this problem is already well known in the economic literature, at least since Mundell (1968) raised the nth-country problem.

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1. Introduction.

A number of economic problems arise from interactions between different agents. In the literature of the last two decades or so they are usually tackled by using policy games, which are especially useful in dealing with strategic interactions.

Policy interactions among players have raised many issues. One strand of literature concerns problems of coordination and comparisons between the outcomes of decentralized and

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centralized solutions.[1] Only occasionally have these issues been dealt with in terms of controllability of the system and equilibrium existence. However, the essence of this problem is already well known in the economic literature, at least since Mundell (1968) raised the *n*th-country problem. In a world of *n* countries there are *n–1* independent external balances or exchange rates as possible targets, but *n* possible independent policy instruments, such as the interest rates, which could in theory be set by the *n* policymakers. But if each policymaker made independent use of his instrument, a conflict would arise among the *n* countries and no equilibrium would exist because of the adding up constraint that must hold between the *n* external balances or *n* exchange rates.[2]

Conditions for equilibrium existence in strategic games with different information structures have been known for a long time. In particular, Dasgupta and Maskin (1986) have asserted a well known theorem of existence for the Nash equilibrium. More recently, conditions for equilibrium have been an almost incidental object of analysis in linear quadratic (LQ) games. In this context the initial focus, starting with Barro and Gordon (1983) and Gylfason and Lindbeck (1994), was on the effectiveness of monetary policy. But the analysis has lately been conducted at an abstract and more general level, leading to results being stated directly in terms of equilibrium existence when it was found that equilibria typically fail to exist when some of the players are each able, individually, to control a common target variable (see Acocella and Di Bartolomeo, 2006; Acocella, Di Bartolomeo and Hughes Hallett, 2006).

In this paper, by focusing on the entire system, we are able to derive general conditions for equilibrium existence in terms of the controllability of the system. We demonstrate that a decentralized equilibrium cannot exist whenever the total number of independent instruments, summing across players, is greater than the total number of their (independent) targets. Surprisingly perhaps, the availability of instruments more numerous than the targets, which is a virtue in a centralized or “social planner” solution (as it confers some degrees of freedom), is fatal in a decentralized environment as it destroys the possibility of having an equilibrium.

[1] Starting from the pioneering studies of Mundell (1962) or Cooper (1969), and extending to some recent works such as Corsetti and Pesenti (2001) or Pappa (2004).
[2] Robert Mundell was well aware that the existence of a redundancy issue depends on the set of assumptions about targets and instruments as well as on the working of the economic system (see Mundell, 1962; 1968). This was even more evident in von Neumann Whitman (1969).

The paper is organized as follows. In the next section we derive our main proposition, and describe the close relation between our results and Dasgupta and Maskin (1986). We then turn to the implications of our results. Section 3 illustrates our main result with three examples in the area of international monetary and trade agreements. These are important applications because they show that institution design is the crucial factor in such agreements. Section 4 concludes.

2. The basic argument.

We focus on a simple linear-quadratic game, where players aim to minimize losses defined on the same target variables. In this sense we refer to games with overlapping preferences.[3]

Players aim to minimize square losses defined on *n* possible target variables. Formally, player *i* minimizes:

(1) $$U_i = (y - \bar{y}_i)' Q_i (y - \bar{y}_i) + R_i y$$

where Q_i, $\bar{y}_i$ and R_i are parameters. In particular, Q_i is a constant and symmetric matrix of order *n*; and $\bar{y}_i$ and R_i are given vectors of order *n*. We assume that the players' loss function differ at least in one parameter.[4] We denote by *P* the set of players; and by *p* its cardinality, i.e. the number of players.

The variables are related by the following reduced form system:

(2) $$y = \sum_{i \in P} B_i u_i + v$$

where u_i is the vector of the $m(i)$ independent instruments controlled by player *i*. In order to avoid trivial cases of non existence we assume that $m(i)$ is not larger than the rank of Q_i, which is positive definite for all *i*.[5]

[3] The results can be extended to the case of non-overlapping preferences, see Acocella, Di Bartolomeo and Hughes Hallett (2006). Our approach can be also applied to more general cases including finite dynamic games, see Acocella, Di Bartolomeo and Hughes Hallett (2007).

[4] We assume that either R_i or $\bar{y}_i$ are different among players, at least in one element. This rules out the trivial case of equal optimal targets and hence the absence of a conflict of interest between players. Below we discuss the special case where these assumptions do not hold; i.e. where, despite an absence of a conflict of interest among players, a problem of coordinating their instruments nevertheless occurs.

[5] Assuming that Q_i is a full rank matrix implies that each player's *first best* is finite (i.e. he has a satiation point). When a loss is linearly decreasing (or increasing) in an argument, a finite optimal policy may not exist independently of the strategic context. In particular, it would not exist if the number of the policymaker's instruments is greater than the number of his quadratic targets.

Although we do not take account of the instrument costs in the players' losses, our representation is rather general. We do not explicitly consider instrument costs to keep the targets and instruments formally separate. However, the cost of any instrument can easily be introduced by adding an auxiliary target variable into equation (1); and an equality constraint between it and the instrument in question into equation (2).[6]

Eliminato: (1)

Eliminato: (2)

Theorem. The Nash equilibrium exists only if the sum of the number of the players' independent instruments does not exceed the number of independent targets.

Proof. Each player's first order condition can be written in the target space as

(3) $$B_i'Q_i\left(y-\bar{y}_i\right)+\tfrac{1}{2}R_i=0 \qquad \forall i\in P$$

Putting all the first order conditions together we obtain the following system of equations:

(4) $$\begin{pmatrix} B_1'Q_1 & \varnothing & \dots & \varnothing \\ \varnothing & B_2'Q_2 & \dots & \varnothing \\ \dots & \dots & \dots & \dots \\ \varnothing & \varnothing & \dots & B_p'Q_p \end{pmatrix}\begin{pmatrix} y \\ y \\ \dots \\ y \end{pmatrix}=\begin{pmatrix} B_1'Q_1\bar{y}_1-\frac{1}{2}R_1 \\ B_2'Q_2\bar{y}_2-\frac{1}{2}R_2 \\ \dots \\ B_p'Q_p\bar{y}_p-\frac{1}{2}R_p \end{pmatrix} \text{ or } KY=Z$$

where K is of order $\sum_{i\in P} m(i)\times pn$ and Y and Z are vectors of order $\sum_{i\in P} m(i)\times 1$.Now define $V=\left[I_n : I_n : \dots : I_n\right]'$, which is of order $pn\times n$. Then equation (4) can be rewritten as:

Eliminato: (4)

(5) $$KVy=Z$$

where KV is a rectangular $\sum_{i\in P} m(i)\times n$ matrix. Equation (5) can be solved if $\sum_{i\in P} m(i)\leq n$

Eliminato: (5)

and the solution is:

(6) $$y=(KV)^{+}Z\,. \square$$

This theorem derives a necessary condition for the existence of the Nash equilibrium in terms of instrumental and objective variables. In contrast to the earlier literature, this condition is defined in terms of the total number of instruments and targets, rather than in terms of controllability by one or more players. Thus the decentralized solution does not exist if the condition specified in the theorem is not met; and decentralization itself may lead to

[6] See e.g. Acocella and Di Bartolomeo (2006).

instability if, as seems likely, players (policymakers) start to use randomized strategies in an attempt to overcome the non-existence of the Nash equilibrium.[7]

The over-determination of an economic system in LQ games has no consequences for equilibrium existence in a centralized solution however. In fact, for a social planner minimizing a convex combination of the players' losses, infinite solutions exist if the economic system (2) is over-determined.[8] They are simply obtained by fixing the values of the instruments in excess of the targets. Arbitrary values, and any group of *m(i)-n* instruments, may be taken to make up the excess set. But the equilibrium will still exist.

Eliminato: (2)

From the above theorem, three corollaries can be easily derived.

Corollary 1. Controllability by at least one player implies that the equilibrium does not exist.

Corollary 2. The introduction of instrument costs implies that the theorem is never satisfied, and therefore guarantees that the equilibrium exists.

Hence:

Corollary 3. Under the conditions of this theorem, there cannot be more than n–1 active players if there is to be a non-cooperative policy equilibrium.

The first corollary relates our results to those in earlier work by Acocella and Di Bartolomeo (2006). Although Acocella and Di Bartolomeo analyzed the existence of a solution in terms of controllability by single players, they did not find our result. In their case, only controllability by two or more players could imply non-existence. Here we have found that, by looking at the controllability of the entire system, a more stringent condition for existence emerges.

The second corollary relates our results to a well-known existence theorem for Nash equilibria due to Dasgupta and Maskin (1986), which relates the existence of an equilibrium to the costs of using the instruments. We have likewise expressed the necessary condition for existence, in a similar setup, in terms of an instruments/targets counting rule.

Dasgupta and Maskin (1986) also show that a sufficient condition for the Nash equilibrium existence is that the space of strategies of each player is convex and compact. If the players'

[7] By assuming mixed strategies the equilibrium might exist also in the decentralized case even if the equilibrium in pure strategies does not (Osborne and Rubinstein, 1994: proposition 20.3), but it will be Pareto inferior with respect to the centralized solution because of an excessive volatility in both the targets and the instruments.

[8] In the terminology of the classical theory of the economic policy this means that the total number of instruments exceeds the total number of targets.

controls are unbounded, then the Nash equilibrium may not exist. However, in static linear quadratic games, the introduction of quadratic instrument costs would automatically make them bounded, thus assuring the existence of an equilibrium in our case too. In fact, the introduction of quadratic instrument costs imply that the dimensions of matrices Q_i become $(n(i)+m(i))\times m(i)$. The number of instruments would then always be less than that of targets, the system would not be controllable and the equilibrium would exist. It is worth noticing, however, that our theorem is more general than the Dasgupta-Maskin result since a situation with instrument costs is then just a special case of our theorem.

The corollary 3 stresses that, differently from the case of a single player, if there are too many instruments available, in a game a problem of coordination arises: there might be too many degrees of freedom, and it might become impossible for each player to make conjectures about the policy of the opponents, and a non cooperative equilibrium could fail to exist.[9] In this sense an interesting case may arise when all players have the same target values and only quadratic preferences: $\bar{y}_i = \bar{y}$ and $R_i = 0$ for all i in (1). In such a case, the Nash equilibrium in the space of outcomes exists and it is clearly $y_i = \bar{y}$ since no player has any incentive to deviate from his/her first best. However our theorem also applies in this case. In fact, even if the Nash equilibrium exists in outcome space, a unique equilibrium in instrument space will not exist, instruments remain indeterminate. Instead, an infinite number of Nash equilibria in instrument space support $\bar{y}_i = \bar{y}$.[10]

3. Global Public Goods and Indeterminacy in the International Monetary System.

Our theorem has many practical applications throughout the literature on policy games. In this section we demonstrate the lack of coordination that arises through decentralized policy-making, and the consequent non-existence of equilibrium, with reference to a simple policy game between *P* countries in the context of the international monetary system. For the sake of brevity, we indicate both the set of countries and its cardinality by *P*.

[9] For the sake of simplicity, in the discussion, we implicitly interpret the Nash equilibrium as the result of mutual rational expectations of the players about the policy of the opponents.
[10] This case is illustrated by the example 1 in the next section.

Example 1: Global Trade Imbalances

Consider the case where each country tries to stabilize its balance of payments, but is also concerned to preserve the stability of the international monetary system. The latter is then the overlapping target of the *P* countries, which plays the role of a global public good.

Formally, we can write the preference of country *i* as:

(7) $$U_i = -\frac{\beta_i}{2} b_i^2 - \frac{1}{2} s \quad i \in P$$

where *b* is the balance of payments and *s* is an index of the international monetary system's stability or sustainability. More precisely, the latter is defined as follows:

(8) $$s = \sum_{i \in P} b_i^2$$

Each player controls an instrument, indicated by *m* (it might be an exchange rate, interest rate or domestic expenditures), and the model reduced form can then be expressed as follows:

(9) $$b_i = \alpha_i m_i + \sum_{j \in P/i} \delta_j m_j \qquad i \in P$$

where α_i measures the effectiveness of country *i*'s instrument and δ_j measures the spillover effects from actions taken in other countries.

An additional constraint is represented by Walras's law (or adding up constraint):

(10) $$\sum_{i \in P} b_i = 0$$

The model can be rewritten in an LQ form that allows us to apply our theorem. From (10):

(11) $$b_j = -\sum_{i \in P/j} b_i$$

Thus the stability index then becomes:

(12) $$s = \sum_{i \in P/j} b_i^2 + \left(\sum_{i \in P/j} b_i \right)^2$$

By using equations (11) and (12), the country utilities can be rewritten as:

Eliminato: (11)

Eliminato: (12)

(13) $$U_i = -\frac{1+\beta_i}{2} b_i^2 - \frac{1}{2} \sum_{z \in P/j,i} b_z^2 - \frac{1}{2} \left(\sum_{i \in P/j} b_i \right)^2 \qquad i \in P/j$$

(14) $$U_j = -\frac{1}{2} \sum_{z \in P/j} b_z^2 - \frac{\beta_j + 1}{2} \left(\sum_{z \in P/j} b_z \right)^2$$

and the Nash equilibrium can be found by minimizing equations (13) and (14) under the following constraint:

(15) $$b_i = \alpha_i m_i + \sum_{j \in P/i} \delta_j m_j \qquad i \in P/j$$

Eliminato: (13)

Eliminato: (14)

By looking closely at equations (13), (14) and (15), we have *P* players who aim to minimize a function of *P–1* overlapping targets; hence the overall number of targets is *P–1*. Since each player can use an independent instrument, the available instruments are *P*. From our theorem it follows that the Nash equilibrium of this game does not exist, at least not in pure strategies.

Eliminato: (13)

Eliminato: (14)

Eliminato: (15)

By contrast the cooperative or the social planner solution can exist, and it leads to the first best outcome:

(16) $$b_i = 0 \qquad i \in P$$

which is the Pareto optimal solution. More surprisingly perhaps, suppose that one country exits the game by fixing its instrument in advance, i.e. country *z* fixes $m_z = \bar{m}$. It is trivial to show that a solution of the game still exists and it is also the Pareto outcome.

Even in a model where the differences between countries are at a minimum (indeed where there is no conflict at all), but where all of them are interested in stabilizing the international monetary system, we can see that there is a need for either an international institution to coordinate the countries' policies; or for a country-leader (hegemon) *à la* Kindleberger (see Kindleberger, 1973) that acts to impose such a solution by picking arbitrary values for its instrument, and thereby acts as an anchor for the other countries' expectations. Both solutions highlight the need for international coordination in this situation. However the nature of this coordination is very specific. It is not a problem of coordination arising from the existence of externalities of conflicting targets, as is usual in policy games.[11] The problem instead is related to the formation of expectations by rational players. It is easy to understand the point by considering the Nash equilibrium as the outcome of a self-fulfilling process of rational expectations. Even if there is no differences between countries, if they play simultaneously but non-cooperatively they are not able to predict the behavior of the other (because there is no equilibrium to their game) and the rational expectations equilibrium will no longer exist.[12]

[11] Setting $\delta_j = 0$ in (9) and (15) above changes nothing and our results still hold.

[12] A similar problem, with less dramatic consequences for the existence of the Nash equilibrium, emerges in Dixit and Lambertini (2003).

Example 2: Negotiated Tariff Reductions

Now consider the case where *P* countries meet to negotiate a series of tariff reductions under WTO auspices, as in the Doha and earlier rounds of world tariff reductions. In this case there will be a target reduction value for the average tariff level after the negotiations: from its existing level to say $P^{-1}\sum_{i\in P} b_i = \bar{b}$. We can define national deviations from the desired average tariff rate to be $\tilde{b}_i = b_i - \bar{b}$, for each $i \in P$. The adding up constraint now becomes

(17) $$\sum_{i\in P} \tilde{b}_i = 0$$

in place of (10); and the desire to keep the system as a whole in place without undue strains, as evidenced by the WTO's dispute resolution procedure which all participants have signed up to, will be represented by

(18) $$s = \sum_{i\in P} \tilde{b}_i^2$$

and the presence of this definition of *s* in each country's objective function (7). Our results go through again, as before, in this example. There are more instruments (national tariffs) than there are free and independent targets. The decentralized equilibrium does not exist therefore; and an equilibrium outcome without randomized behavior and tariff wars is only possible with either a hegemon/social planner, or cooperation in the form of a common institution with agreed rules for resolving disputes when $\tilde{b}_i > 0$ gets too large (protection) or when $\tilde{b}_i < 0$ gets too large (dumping). Otherwise there will be no equilibrium, policy instability and periodic trade wars. History bears this result out. Trade policies were stable in the 1950s and 1960s under US hegemony, and arguably progressive in the 1980s with the US and European Union as social planners; but fell into chaos in the 1930s in the absence of such conditions, and threatened to do so in the 1970s until the need for explicit coordination was accepted by the G7 policymakers at the time of the Bonn summit in 1978. The possibility that the same could happen again in the 1990s, as the EU and China emerged to rival the US, produced the WTO as a mutual coordinating device.

Example 3: Exchange Rate Targeting and Exchange Rate Regimes

Exchange rate targeting mechanisms fit straight into our framework. A bilateral peg implies

(19) $e_i = e_j^{-1}$ for any two countries *i, j*, and *P=2,*

where e_i is the nominal exchange rate of country *i* in terms of country *j*'s currency. That supplies the adding up constraint which can be substituted straight into (7) with a sufficiently large value of β to represent the degree of commitment being given to the fixed parity values underlying (19). The financial stability index *s,* defined in terms of the exchange rates, may also be incorporated if exchange rate stability in an absolute sense is thought to be important.

Similarly, multilateral exchange rate mechanisms, like the European monetary system (EMS), may also be included in our framework. In this case, the adding up constraint is

(20) $w_1 DM / \$ + w_2 FF / \$ + ... - Ecu / \$ = 0$

where w_i are the country weights; and DM/\$, FF/\$ etc represent the component exchange rates expressed against the dollar as an outside currency. This is just an alternative version of (19) with a larger value of *P*. In each of these cases our results go through as before. They demonstrate that an exchange rate regime requires a hegemon, or some explicitly coordinating mechanism or institution to produce an equilibrium. Otherwise one can expect erratic policies and periods of financial instability. Again history bears this result out. The Bretton Words system had the IMF as a coordinating institution, and US trade deficits (hegemony). The EMS had the European Commission (actually the monetary policy committee) as social planner in its initial phases; but later on that role fell to the Bundesbank. The Euro now has the European Central Bank as coordinating institution. But at other times there has been little tendency for exchange rates to find equilibrium values, and evident periods of financial stress.

4. Conclusions.

This paper offers a new angle to study some properties of non-cooperative games among several policymakers. In particular, we concentrate on the Nash equilibrium in multi-player LQ games with overlapping preferences, which is a very common situation in studying the international economic interactions among national policymakers, even if it is not exclusive of this context.

Abstracting from controllability by one or more players, and concentrating on controllability of the (decentralized) system as a whole, we find that the necessary condition for the existence of the Nash equilibrium in a LQ policy game with overlapping preferences is that the total number of instruments available for all the players should not exceed the total number of targets of all the players.

This implies the existence of a fundamental asymmetry in institutional design: if an economic system is over-determined, it can be solved by some 'social planner' (hegemon) who sets the excess of instruments over targets at suitably chosen but arbitrary values before the other players choose their instruments. The decentralized solution, by contrast, will not exist. The advantages of centralization, due to an oversupply of available instruments, are lost while decentralization leads to indeterminacy and a destabilization (if not collapse) of the system as players randomize their policies in response. The only way out, in the absence of a social planner, is explicit institution based coordination.

Our results extend previous literature focused on the controllability of an economic system by some players. In looking for the necessary conditions for the existence of a Nash equilibrium, we have in fact assumed that no players can control the system (because of overlapping preferences), so the necessary condition for existence stated in Acocella and Di Bartolomeo (2006) and Acocella, Di Bartolomeo and Hughes Hallett (2006) no longer applies. Our necessary conditions for the existence of the Nash equilibrium in LQ games are thus additional to those required by previous studies. That is evidence of the complexity of this issue.

Our proposition also explains the role played by instruments costs, which are often a device to avoid non-existence of the equilibrium. The importance of our proposition lies in the fact that similar conditions appear to be essential for both model building and institution design in many areas of economics: for example, the international monetary system, trade policy, global public goods, natural resource games, all of which have the features of the general game analyzed here.

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